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Solvability of the Solution of Superlinear Hyperbolic Dirichlet Problem

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Abstract. In this paper, we aim to study the solutions of superlinear hyperbolic problems with boundary condition of Dirichlet type where we show the existence and the uniqueness of the strong solutions for the superlinear problems by the method of energy inequality.

1. Introduction and position of the problem

The partial differential equations were probably formulated for the first time during the birth of rational mechanics in the 17th century [1–3]. Then the catalog of Partial Differential Equations (PDEs) have been enriched as the science developed and in particular physics [4–7]. If we only have to remember a few names, we must cite that of Euler, then those of Navier and Stokes, for the equations of fluid mechanics, those of Fourier in the heat equation, Maxwell for those of electromagnetism, Schrodinger and Heisenberg for the equations of quantum mechanics, and of course that of Einstein for the PDEs of the theory of relativity. A giant leap was made by L. Schwartz when he gave birth to the theory of distributions (around the 1950s), and at least comparable progress is due to L. Hormander for the development of pseudo differential calculus (in the early 1970s). The complexity of nonlinearity and challenges in their theoretical study in have attracted a lot of interest from many mathematicians and scientists see [8–11].

Many natural phenomena and modern problems of physics, mechanics, biology, and technology can be modeled by nonlinear hyperbolic equations. The method used here is one of the most efficient

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functional analysis methods in solving partial differential equations, it is called a priori estimate method or the energy-integral method, see [10]. In this work, we study the solutions to hyperbolic problems with boundary conditions of Dirichlet type where we show the existence and uniqueness of the strong solutions for semilinear problems by the method of energy inequality, where we found a difficulty in the choice of the multiplier, and the uniqueness which is emanating from a priori estimate. Let $T > 0, \Omega \subset \mathbb{R}^n$ and

$$Q = \Omega \times (0,T) = \left\{(x,t) \in \mathbb{R}^{n+1} : x \in \Omega, 0 < t < T\right\}.$$

We consider the nonlinear parabolic problem

$$\begin{cases} u_{tt} - a\Delta u + b(x,t)u_t + u^q = f(x,t) \\ u(x,0) = \varphi(x), \\ u_t(x,0) = \psi(x), \\ u(x,t) \mid_{\Gamma} = 0 \end{cases}, \tag{P$_1$}$$

in which the nonlinear parabolic equation is given as follows

$$\mathcal{L}u = u_{tt} - a\Delta u + b(x,t)u_t + u^q = f(x,t), \tag{1.1}$$

with the initial condition

$$lu = u(x,0) = \varphi(x), \tag{1.2}$$

and the Dirichlet boundary conditions

$$u(x,t) \mid_{\Gamma} = 0, \ \forall t \in (0,T), \tag{1.3}$$

where a, q are positive odd integers, $p \geq 1$, and where $f(x,t)$, $\varphi(x)$ and $\psi(x)$ are given functions and $b(x,t)$ satisfies the following assumption:

A1. $b_1 \leq b(x,t) \leq b_0, \ (x,t) \in \bar{Q}$.

We establish a priori bound and prove the existence of a solution of problem (1.1)-(1.3). To this aim, let $Lu = F$, where $L = (\mathcal{L}F, l_1, l_2)$, and $F = (f, \varphi, \psi)$ be the operator equation corresponding to problem (1.1)-(1.3). The operator L acts from E to F, and the Banach space E consists of all functions $u(x,t)$ with the finite norm

$$\|u\|_E^2 = \max_{0\leq\tau\leq T} \|u_\tau(x,\tau)\|_{L^2(\Omega)}^2 + \max_{0\leq\tau\leq T} \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(Q)}^2 + \max_{0\leq\tau\leq T} \|u(x,\tau)\|_{L^{q+1}(\Omega)}^{q+1}. \tag{1.4}$$

The Hilbert space F consists of the vector valued functions $F = (f, u_0)$ with the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|\varphi_x\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^{q+1}(\Omega)}^{q+1}. \tag{1.5}$$

The associated inner product is given as

$$(\mathcal{F}, G)_F = (f,g)_{L^2(Q)} + \left(\varphi_x, (g_0)_x\right)_{L^2(\Omega)} + (\psi, g_1)_{L^2(\Omega)}. \tag{1.6}$$

We assume that the data functions φ and ψ satisfy the conditions of the form (1.3), i.e.,

$$\varphi\mid_{\Gamma}=\psi\mid_{\Gamma}=0.$$

At the upcoming section, we intend to establish a priori estimate for the solution of problem (1.1)-(1.3).

2. A priori bound

In the theory of PDEs, an a priori estimate (also called an apriori estimate or a priori bound) is an estimate for the size of a solution or its derivatives of a PDE. A priori is Latin for "from before" and refers to the fact that the estimate for the solution is derived before the solution is known to exist. One reason for their importance is that if one can prove an a priori estimate for solutions of a differential equation, then it is often possible to prove that solutions exist using the continuity method or a fixed point theorem. Some important definitions and theorems will be next listed in this section.

Theorem 2.1. *If assumption* **A1** *is satisfied, then for any function $u \in D(L)$, there exists a positive constant c independent of u such that*

$$\begin{aligned}\max_{0\leq\tau\leq T}\|u_\tau(x,\tau)\|^2_{L^2(\Omega)}+\max_{0\leq\tau\leq T}\|\nabla u\|^2_{L^2(\Omega)}+\|u_t\|^2_{L^2(Q)}+\max_{0\leq\tau\leq T}\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)}\\ \leq c\left(\|f\|^2_{L^2(Q)}+\|\psi\|^2_{L^2(\Omega)}+\|\varphi_x\|^2_{L^2(\Omega)}+\|\varphi\|^{q+1}_{L^{q+1}(\Omega)}\right),\end{aligned} \tag{2.1}$$

and $D(L)$ is the domain of definition of the operator L defined by

$$D(L)=\{u:\ u\in L^\infty\left(0,T,L^{q+1}(\Omega)\right),\ u_t\in L^\infty\left(0,T,L^2\left(\Omega\right)\right)\}$$

satisfying condition (1.3).

Proof. Taking the scalar product in $L^2(Q)$ of Eq. (1.1) and the operator $Mu=u_t$, where $Q^\tau=\Omega\times(0,T)$, yields

$$\begin{aligned}(\mathcal{L}u,Mu)_{L^2(Q^\tau)}&=(u_{tt},u_t)_{L^2(Q^\tau)}-a\left(\Delta u,u_t\right)_{L^2(Q^\tau)}+(bu_t,u_t)_{L^2(Q^\tau)}+(u^q,u_t)_{L^2(Q^\tau)}\\ &=(f,u_t)_{L^2(Q^\tau)}.\end{aligned} \tag{2.2}$$

The successive integration by parts of integrals on the right-hand side of (2.2) gives

$$\begin{aligned}(u_{tt},u_t)_{L^2(Q^\tau)}&=\int_{Q^\tau}u_{tt}\cdot u_t dxdt\\ &=\frac{1}{2}\int_\Omega u_t^2dx-\frac{1}{2}\int_\Omega\psi^2dx\\ &=\frac{1}{2}\|u_t(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}\|\psi\|^2_{L^2(\Omega)},\end{aligned} \tag{2.3}$$

besides we have

$$\begin{aligned}
-a\,(\Delta u, u)_{L^2(Q^\tau)} &= -a\int_{Q^\tau} \Delta u\cdot u_t\,dxdt \\
&= a\int_{\Omega}\nabla u^2 dx - \frac{1}{2}\int_{\Omega}\varphi_x^2 dx \\
&= a\,\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2}\,\|\varphi_x\|^2_{L^2(\Omega)}\,, \qquad (2.4)
\end{aligned}$$

and

$$(bu, u)_{L^2(Q^\tau)} = \int_{Q^\tau} b(x,t)u_t^2\,dxdt. \qquad (2.5)$$

In this regard, we have

$$\begin{aligned}
(cu^q, u_t)_{L^2(Q^\tau)} &= \frac{1}{q+1}\int_{\Omega} u^{q+1}dx - \frac{1}{q+1}\int_{\Omega}\varphi^{q+1}dx \\
&= \frac{1}{q+1}\,\|u_t(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} - \frac{1}{q+1}\,\|\varphi\|^{q+1}_{L^{q+1}(\Omega)}\,.
\end{aligned} \qquad (2.6)$$

By substituting (2.3)-(2.6) into (2.2), we obtain

$$\begin{aligned}
&\frac{1}{2}\,\|u_t(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\,\|\psi\|^2_{L^2(\Omega)} + a\,\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2}\,\|\varphi_x\|^2_{L^2(\Omega)} \\
&+\int_{Q^\tau} b(x,t)u_t^2\,dxdt + \frac{1}{q+1}\,\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} - \frac{1}{q+1}\,\|\varphi\|^{q+1}_{L^{q+1}(\Omega)} = (f, u_t)\,.
\end{aligned} \qquad (2.7)$$

By applying Cauchy inequality with ε, $\left(\text{i.e., } |ab| \le \dfrac{a^2}{2\varepsilon} + \dfrac{\varepsilon b^2}{2}\right)$, we can estimate the last term on the right-hand side of (2.7) and get

$$\begin{aligned}
&\frac{1}{2}\,\|u_t(x,\tau)\|^2_{L^2(\Omega)} + a\,\|\nabla u\|^2_{L^2(\Omega)} + \int_{Q^\tau} b(x,t)u_t^2\,dxdt + \frac{1}{q+1}\,\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} \\
&\le \frac{1}{2\varepsilon}\,\|f\|^2_{L^2(Q^\tau)} + \frac{\varepsilon}{2}\,\|u\|^2_{L^2(Q^\tau)} + \frac{1}{2}\,\|\psi\|^2_{L^2(\Omega)} + \frac{1}{2}\,\|\varphi_x\|^2_{L^2(\Omega)} + \frac{1}{q+1}\,\|\varphi\|^{q+1}_{L^{q+1}(\Omega)}\,.
\end{aligned}$$

By using assumptions **A1** and using the Gronwall's Lemma, the estimate (2.8) becomes

$$\begin{aligned}
&\|u_t(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \int_{Q^\tau} u_t^2\,dxdt + \|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} \\
&\le \frac{\max\left\{\frac{1}{2}, \frac{1}{2\varepsilon}, b_0, \frac{1}{q+1}\right\}}{\min\left\{\frac{1}{2}, a, b_1, \frac{1}{q+1}\right\}}\exp\left(\frac{\varepsilon}{2}T\right) \\
&\times\left[\|f\|^2_{L^2(Q)} + \|\psi\|^2_{L^2(\Omega)} + \|\varphi_x\|^2_{L^2(\Omega)} + \|\varphi\|^{q+1}_{L^{q+1}(\Omega)}\right].
\end{aligned}$$

Then, by passing to the maximum, we get

$$\begin{aligned}
&\max_{0\le\tau\le T}\|u_\tau(x,\tau)\|^2_{L^2(\Omega)} + \max_{0\le\tau\le T}\|\nabla u\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(Q)} + \max_{0\le\tau\le T}\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} \\
&\le c\left[\|f\|^2_{L^2(Q)} + \|\psi\|^2_{L^2(\Omega)} + \|\varphi_x\|^2_{L^2(\Omega)} + \|\varphi\|^{q+1}_{L^{q+1}(\Omega)}\right],
\end{aligned}$$

where

$$c=\frac{\max\left\{\frac{1}{2},\frac{1}{2\varepsilon},b_{0},\frac{1}{q+1}\right\}}{\min\left\{\frac{1}{2},a,b_{1},\frac{1}{q+1}\right\}}\exp\left(\frac{\varepsilon}{2}T\right).$$

So, we have

$$\|u\|_{E}\leq\sqrt{c}\,\|Lu\|_{F}\,. \tag{2.8}$$

□

Now, we let $R(L)$ be the range of the operator L. Since we do not have any information about $R(L)$, except that $R(L)\subset F$, we must extend L so that estimate (1.6) holds for this extension and its range represents the whole space F. For this purpose, we present the next proposition.

Proposition 2.1. *The operator $L:E\longrightarrow F$ has a closure.*

Proof. Let $(u_n)_{n\in\mathbb{N}}\subset D\left(L\right)$ be a sequence where

$$u_n\longrightarrow 0\quad \text{in } E,$$

and

$$Lu_n\longrightarrow\left(f;\varphi_x,\psi\right)\qquad\text{in } F. \tag{2.9}$$

Now, we must prove that

$$f\equiv 0 \text{ and } \left(\varphi,\psi\right)\equiv\left(0,0\right).$$

The convergence of u_n to 0 in E drives:

$$u_n\longrightarrow 0\qquad\text{in } D'\left(Q\right)\ . \tag{2.10}$$

According to the continuity of the derivation of $D'\left(Q\right)$ in $D'\left(Q\right)$ and the continuity the distribution of the function u^q, the relation (2.10) involve

$$\mathcal{L}u_n\longrightarrow 0\qquad\text{in } D'\left(Q\right). \tag{2.11}$$

Moreover, the convergence of $\mathcal{L}u_n$ to f in $L^2\left(Q\right)$ gives:

$$\mathcal{L}u_n\longrightarrow f\qquad\text{in } D'\left(Q\right). \tag{2.12}$$

As we have the uniqueness of the limit in $D'\left(Q\right)$, we conclude from (2.11) and (2.12) that $f=0$. Then it is generated from (2.9) that

$$l_1u_n\longrightarrow\varphi_x \text{ and } l_2u_n\longrightarrow\psi\qquad\text{in } L^2\left(\Omega\right).$$

On the other hand, we have

$$
\begin{aligned}
&\|u\|_E^2 \\
&= \max_{0\le\tau\le T} \|u_\tau(x,\tau)\|_{L^2(\Omega)}^2 + \max_{0\le\tau\le T} \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(Q)}^2 + \max_{0\le\tau\le T} \|u(x,\tau)\|_{L^{q+1}(\Omega)}^{q+1} \\
&\ge \|u_x(x,0)\|_{L^2(\Omega)}^2 + \|u_t(x,0)\|_{L^2(\Omega)}^2 \\
&\ge \|\varphi_x\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Now, due to $u_n \longrightarrow 0$ in E, then $\|u\|_E^2 \longrightarrow 0$ in $\mathbb{R}$. Consequently, we get

$$0 \ge \|\varphi_x\|_{L^2(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 .$$

Then, we obtain

$$\varphi_x = 0 \quad \text{and } \psi = 0.$$

Let L be the closure of this operator with the domain of definition $D(L)$, and hence the result holds. □

Definition 2.1. *A solution of the operator equation $\bar{L}u = \mathcal{F}$ is called a strong solution to problem (1.1)-(1.3).*

The priori estimate (2.1) can be then extended to strong solution, i.e., we have the estimate

$$
\begin{aligned}
&\max_{0\le\tau\le T} \|u_\tau(x,\tau)\|_{L^2(\Omega)}^2 + \max_{0\le\tau\le T} \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(Q)}^2 + \max_{0\le\tau\le T} \|u(x,\tau)\|_{L^{q+1}(\Omega)}^{q+1} \\
&\qquad \le c\left(\|f\|_{L^2(Q)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|\varphi_x\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^{q+1}(\Omega)}^{q+1}\right), \ \forall u \in D(\bar{L}).
\end{aligned} \tag{2.13}
$$

In light of the estimate given in (2.13), we can infer the next theoretical results.

Corollary 2.1. *The range $R(\bar{L})$ of the operator $\bar{L}$ is closed in F and is equal to the closure $\overline{R(L)}$ of $R(L)$, i.e. $R(\bar{L}) = \overline{R(L)}$.*

Proof. Let $z \in \overline{R(L)}$ such that there is a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituted of the elements of the set $R(L)$ such as

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

There is then a corresponding sequence $u_n \in D(L)$ such as $z_n = Lu_n$. Immediately, the estimate (2.8) becomes:

$$\|u_p - u_q\|_E \le C\,\|Lu_p - Lu_q\|_F \to 0,$$

where p and q tend towards infinity. We can consequently deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. So like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \quad \text{in } E.$$

By virtue of the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} u_n = u$ in E, if $\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} z_n = z$, and then $\lim_{n\longrightarrow+\infty} \bar{L}u_n = z$ as $\bar{L}$ is closed, and so $\bar{L}u = z$), the function u satisfies:

$$u \in D\left(\bar{L}\right), \ \bar{L}u = z.$$

Then $z \in R(\bar{L})$, and so $\overline{R(L)} \subset R(\bar{L})$. Also, we conclude here that $R(\bar{L})$ is closed because it is Banach (any complete subspace of a metric space, not necessarily complete, is closed). Thus, it remains to show the reverse inclusion either $z \in R(\bar{L})$, and then it exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in F constituted of the elements of the set $R(\bar{L})$ such that $\lim\limits_{n\longrightarrow+\infty} z_n = z$, or $z \in R(\bar{L})$ because $R(\bar{L})$ is closed subset. So $R(\bar{L})$ is complete. There is then a corresponding sequence $u_n \in D(\bar{L})$ such that $\bar{L}u_n = z_n$. Consequently from (2.8), we get

$$\|u_p - u_q\|_E \leq C \left\|\bar{L}u_p - \bar{L}u_q\right\|_F \to 0,$$

where p and q tend towards infinity. We can immediately deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, and so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \text{ in } E.$$

Once again, there is a corresponding sequel $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such as

$$\bar{L}u_n = Lu_n \text{ on } R(L), \forall n \in \mathbb{N}.$$

So we have $\lim\limits_{n\longrightarrow+\infty} Lu_n = z$ and consequently $z \in \overline{R(L)}$, which implies $R\left(\bar{L}\right) \subset \overline{R(L)}$. □

3. Existence and uniqueness of solution

In this section, additional results are listed below, which are related to the existence and uniqueness of strong solution for the main Problem (P1).

Theorem 3.1. *Let assumption* **A1** *be satisfied. Then for all $F = (f, \varphi) \in F$, there exists a unique strong solution $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ of problem (1.1)-(1.3).*

Proof. To prove this result, we should note that we first have

$$(Lu, W)_F = \int_Q \mathcal{L}u.wdxdt + \int_\Omega l_1u.w_0dx + \int_\Omega l_2u.w_1dx, \tag{3.1}$$

where $W = (w, w_0, w1)$. So for $w \in L^2(Q)$ and for all

$$u \in D_0(L) = \{u,\ u \in D(L) : l_1u = 0, l_2u = 0\},$$

we have

$$\int_Q \mathcal{L}u.wdxdt = 0.$$

By putting $w = u_t$, we obtain

$$\begin{aligned}\int_{Q^\tau} u_{tt}u_t + \int_{Q^\tau} b(x,t)u_t^2dxdt + \int_{Q^\tau} u^{q+1}dxdt &= a\int_{Q^\tau} \Delta u.u_t \\ \frac{1}{2}\|u_t(x,t)\|^2_{L^2(\Omega)} + \int_{Q^\tau} b(x,t)u_t^2dxdt + \frac{1}{q+1}\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} &= -a\|\nabla u\|^2_{L^2(\Omega)}.\end{aligned}$$

This gives

$$\begin{aligned} \frac{1}{2}\|u_t(x,t)\|^2_{L^2(\Omega)} + \int_{Q^\tau} b(x,t)u_t^2 dxdt + \frac{1}{q+1}\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} &\leq 0, \\ \max_{0\leq\tau\leq T}\|u_\tau(x,\tau)\|^2_{L^2(\Omega)} + b_1\int_{Q^\tau} u_t^2 dxdt + \frac{1}{q+1}\|u(x,\tau)\|^{q+1}_{L^{q+1}(\Omega)} &\leq 0. \end{aligned}$$

Therefore, we have $u_t = w = 0$. Since the range of the trace operators is everywhere dense in the Hilbert space F with the associate norms $\|\varphi_x\|_{L^2(\Omega)}$ and $\|\psi\|_{L^2(\Omega)}$, then the equality (3.1) implies that $\omega_0 = 0$ and $\omega_1 = 0$. Hence $W = 0$ implies $\overline{R(L)} = F$. □

Corollary 3.1. *If for any function $u \in D(L)$, we have the following estimate:*

$$\|u\|_E \leq \sqrt{c}\,\|\mathcal{F}\|_F\,,$$

Then the solution of the problem (P1), if it exists, is unique.

Proof. Let u_1 and u_2 be two solutions of problem (P1), i.e.,

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0.$$

As L is linear, we then obtain

$$L(u_1 - u_2) = 0.$$

According to (2.8), we obtain

$$\|u_1 - u_2\|^2_E \leq c\,\|0\|^2_F = 0,$$

which gives $u_1 = u_2$. □

4. Conclusion

We have used the method of energy inequality for the super liner problems to show the existence and the uniqueness of the solution. In addition, we have studied the solution of superlinear hyperbolic problems with boundary condition of Dirichlet type.

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