Int. J. Anal. Appl. (2022), 20:64

International Journal of Analysis and Applications

Inversion Formula for the Wavelet Transform on Abelian Group

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Abstract. In this paper a reconstruction and inversion formula of the continuous wavelet transform on abelian group for band-limited function is defined. This formula possesses a more explicit expression than the well-known result. Also, Parseval and other interesting results on abelian group are obtained.

1. Introduction

A set S defines a group if an operator, $+$, holds the following properties:

- $x + (y + z) = (x + y) + z \ \forall x, y, z \in S$
- There exists an element 0 , such that $x + 0 = 0 + x = x \ \forall x \in S$
- For each $\forall x \in S$ there exists an inverse element $x^{-1} = -x$, such that $x+(-x) = (-x)+x = 0$.

S is a topological group if it has a group operation and a topology such that the maps $\alpha : G \times G \to G$ and $\beta : G \times G \to G$ are continuous, where $\alpha(x, y) = x + y$ and $\beta(x) = x^{-1}$.
If S is locally compact, that is every point in S is contained in a compact neighborhood, and its group operation is commutative, then it is called locally compact abelian (LCA) group.
In order to define the Fourier transform on LCA groups, we should introduce the concept of integral over these groups. Let $M(X)$ be the space of all complex-valued regular measures on X where $||\mu|| = |\mu(S)|$ is finite. A Haar measure is a measure which is non negative, regular and invariant. The corresponding integral is called the Haar integral, which is translation invariant.

Received: Oct. 2, 2022.

2010 *Mathematics Subject Classification.* 65T60, 43A25, 11F85.

Key words and phrases. wavelet transform; abelian group; Fourier transform.

https://doi.org/10.28924/2291-8639-20-2022-64
ISSN: 2291-8639

Let G be LCA group , we define an $L^P(G)$ space to be the space of all complex valued functions f on G such that the integral $\int_G |f|^p d\mu$ exists with respect to the Haar measure.

Definition 1.1. *A complex function ω on a LCA group G [1] is called a character of G if $|\omega(x)| = 1$ for all $x \in G$ and if the functional equation $\omega(x + y) = \omega(x)\omega(y)$ for all $(x, y) \in G$ is satisfied. The set of all continuous characters of G form a group Ω , the dual group of G . Now it is customary to write $(x, \omega) = \omega(x)\omega(x)$ satisfy the following properties [1, 5]*

- $(0, \omega) = (x, 0) = 1$
- $(-x, \omega) = (x, -\omega) = (x, \omega)^{-1} = \overline{(x, \omega)}$
- $(x + y, \omega) = (x, \omega)(y, \omega)$
- $(x, \omega_1 + \omega_2) = (x, \omega_1)(x, \omega_2)$

Definition 1.2. *The Fourier transform [2] of $f \in L^1(G)$ is denoted by $\hat{f}(\omega)$ defined by $\hat{f}(\omega) = \int_G f(x)(-x, \omega)dx$, and its inverse Fourier transform is defined [1,5] by $f(x) = \int_G \hat{f}(\omega)(x, \omega)d\omega, x \in G$ The Fourier transform holds the following properties [4]:*

- $\|\hat{f}\|_{L^\infty(G)} \leqslant \|f\|_{L^1(G)}$
- *If $f \in L^1(G) \cap L^2(G)$, then $\|\hat{f}\|_{L^2(G)} = \|f\|_{L^2(G)}$*
- *If the convolution of f and g is defined as*
  $(f * g)(x) = \int_G f(x - y)g(y)dy$ then $F((f * g)) = F(f)F(g)$

For $f(x) \in L^2(G)$, denote $f_{b,a}(x) = \frac{1}{\sqrt{|a|}} f\left(\frac{x-b}{a}\right)$ and $suppf = clos\{x \in G : f(x) \neq 0\}$. If $supp\hat{f}$ is a bounded set, then we say f is band-limited.

The characteristic function on a set E is denoted by $X_E(x)$.

In 1984, Morlet introduced first wavelet transform [7] that is defined as follows:

Let $\psi \in L^2(G)$, the transform:

$$(W_\psi)(b, a) = \int_G f(x)\overline{\psi_{b,a}(x)}dx \qquad \text{for any} \quad f \in L^2(G) \tag{1.1}$$

is said to be a wavelet transform.

When $\psi \in L^1 \cap L^2(G)$ and $C_\psi = 2\pi \int_G \frac{|\hat{\psi}(\omega)|^2}{|\omega|} < \infty$, the known inversion formula [8] is stated as follows:

$$f(x) = \frac{1}{C_\psi} \int_G \int_G (W_\psi f)(b, a)\psi_{b,a}(x)\frac{dadb}{|a|^2} \tag{1.2}$$

The above equality holds in $L^2(G)$ sense.

The aim of this paper is that for band-limited function we give another kind of inversion formula of wavelet transform.

Theorem 1.1. *Let $\psi(x) \in L^1 \cap L^2(G)$. Take $\phi(x) \in L^1 \cap L^2(G)$ satisfying $\hat{\phi}(\omega) = O(|\omega|^{-2})$. Then for any $f \in L^1 \cap L^2(G)$ and $supp\hat{f} \subseteq [-\Omega, \Omega]$, the following inversion formula holds:*

$$f(x) = \frac{1}{(2\pi)^{\frac{3}{2}}(\varphi, \psi)} \int_H \int_G (W_\psi f)(b, a)(\varphi_{b,a} * h)(x)\frac{dada}{|a|} \tag{1.3}$$

where $h(x)$ satisfies $\hat{h}(\omega) = |\omega| X_{[-\Omega,\Omega]}(\omega)$ and the above equality holds in L^2- sense.

2. Lemma

To prove theorem, we first give the following

Lemma: Let $\psi(x), \varphi(x)$ and $f(x)$ be stated in theorem. Then for any $g \in L^2(G)$ the following formula is valid:

$$\frac{1}{2\pi}\int_G\int_G (W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}\frac{dbda}{|a|} = (\varphi,\psi)(f,g), \quad \text{where } (D_\varphi g)(b,a) = \frac{1}{\sqrt{2\pi}}(g, \varphi_{(b,a)} * h) \tag{2.1}$$

Proof: By Parseval identity of Fourier transform, we have

$$(W_\psi f)(b,a) = |a|^{\frac{1}{2}}\int_G \hat{f}(\omega)\overline{\hat{\psi}(a\omega)}(b,\omega)d\omega \tag{2.2}$$

Using the convolution formula [3] and Parseval identity, we also obtain from (2.1) that

$$(D_\varphi g)(b,a) = |a|^{\frac{1}{2}}\int_G \hat{g}(\omega)\overline{\hat{\varphi}(a\omega)\hat{h}(\omega)}(b,\omega)d\omega \tag{2.3}$$

Applying the inversion formula of Fourier transform, it follows from (2.2) and (2.3) that

$$\frac{1}{\sqrt{2\pi}|a|^{\frac{1}{2}}}(W_\psi f)(b,a) = (\hat{f}(\omega)\overline{\hat{\psi}(a\omega)})^{\vee}(b) \tag{2.4}$$

and

$$\frac{1}{\sqrt{2\pi}|a|^{\frac{1}{2}}}(D_\varphi g)(b,a) = (\hat{g}(\omega)\overline{\hat{\varphi}(a\omega)\hat{h}(\omega)})^{\vee}(b) \tag{2.5}$$

Finally, again using Parseval identity, we get

$$\frac{1}{2\pi|a|}\int_G (W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}db = \int_G \hat{f}(\omega)\overline{\hat{g}(\omega)}\hat{h}(\omega)\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)d\omega$$

Since supp$\hat{f} \subseteq [-\Omega,\Omega] =$ supp $\hat{h}(\omega)$ and $\hat{h}(\omega) = |\omega|X_{[-\Omega,\Omega]}(\omega)$, we know that $\hat{f}(\omega)\hat{h}(\omega) = |\omega|\hat{f}(\omega)$, $\omega \in G$.

Further,

$$\frac{1}{2\pi|a|}\int_G (W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}db = \int_G \hat{f}(\omega)\overline{\hat{g}(\omega)}|\omega|\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)d\omega$$

In view of

$$\begin{aligned}\int_G\int_G |\hat{f}(\omega)\overline{\hat{g}(\omega)}\omega\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)|d\omega da &= \int_G |\hat{f}(\omega)\overline{\hat{g}(\omega)}|\left(|\omega|\int_G |\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)|da\right)d\omega \\ &= \left(\int_G |\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)|d\omega\right)\left(\int_G |\hat{f}(\omega)\overline{\hat{g}(\omega)}|d\omega\right) \\ &\leqslant \|\varphi\|_2\|\psi\|_2\|f\|_2\|g\|_2\end{aligned}$$

By Fubini theorem, we have

$$\frac{1}{2\pi}\int_G\left(\int_G (W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}\right)\frac{da}{|a|} = \int_G\left(\int_G \hat{f}(\omega)\overline{\hat{g}(\omega)}|\omega|\overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)d\omega\right)da$$
$$= \int_G \hat{f}(\omega)\overline{\hat{g}(\omega}\left(|\omega|\int_G \overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)da\right)d\omega$$

Again, noticing that

$$|\omega|\int_G \overline{\hat{\psi}(a\omega)}\hat{\varphi}(a\omega)da = (\hat{\varphi},\hat{\psi}) = (\varphi,\psi),$$

For repeated integral, we get

$$\frac{1}{2\pi}\int_G\left(\int_G (W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}\right)\frac{da}{|a|} = (\varphi,\psi)(f,g).$$

In order to complete the proof of lemma, by Fubini theorem, we only need to prove that

$$K = \int_G\int_G |(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}|\frac{dadb}{|a|} < \infty \tag{2.6}$$

we split the above integral into two parts, namely,

$$\begin{aligned} K &= \left(\int_H + \int_{G-H}\right)\left(\int |(W_\psi f)(b,a)\overline{(D_\varphi g)(b,a)}|\frac{db}{|a|}\right)da \\ &= K_1 + K_2 \end{aligned}\tag{2.7}$$

Where H is a subgroup of G. First, we estimate K_1.

Using Cauchy inequality, we get

$$\begin{aligned} K_1^2 &\leqslant \int_H\left(\int_G |(W_\psi f)(b,a)|^2\frac{db}{|a|}\right)da \cdot \int_H\left(\int_G |(D_\varphi g)(b,a)|^2\frac{db}{|a|}\right)da \\ &= K_{11}\cdot K_{12} \end{aligned}$$

Applying (2.4) and Parseval identity, we have

$$K_{11} = 2\pi\int_H\left(\int_G |\hat{f}(\omega)|^2|\hat{\psi}(a\omega)|^2 d\omega\right)da$$

By $\psi \in L^1(G)$ we know that there is a $M > 0$ such that $|\hat{\psi}(\omega)| \leqslant M$, so $K_{11} \leqslant 4\pi M^2\|f\|_2^2$.

On the other hand, applying (2.5) and Parseval identity, we also have

$$K_{12} = 2\pi\int_H\left(\int_G |\hat{g}(\omega)|^2|\hat{\varphi}(a\omega)|^2|\hat{h}(\omega)|^2 d\omega\right)da \tag{2.8}$$

By $\varphi \in L^1(G)$ we know that there is an $N > 0$ such that $|\hat{\varphi}(\omega)| \leqslant N$. Again noticing that $|\hat{h}(\omega)| \leqslant \Omega$, we have $K_{12} \leqslant 4\pi N^2\Omega^2\|g\|_2^2$. so $K_1 < \infty$.

Next we estminate K_2,

From (2.7), we know that for any given $0 < \epsilon < \frac{1}{2}$,

$$K_2 = \int_{G-H}\left(\int_G |a|^{-1+\frac{1}{2}}|(W_\psi f)(b,a)|\cdot|a|^{-\frac{1}{2}}|(D_\varphi g)(b,a)|db\right)da$$

Using Cauchy inequality, we get

$$\begin{aligned} K_2^2 &\leqslant \int_{G-H}\left(\int_G |(W_\psi f)(b,a)|^2 \frac{db}{|a|^{2-\epsilon}}\right) da \cdot \int_{G-H}\left(\int_G |(D_\varphi g)(b,a)|^2 \frac{db}{|a|^{\epsilon}}\right) da \\ &= K_{21}\cdot K_{22} \end{aligned}$$

Since

$$|(W_\psi f)(b,a)| = \left|\frac{1}{\sqrt{|a|}}\int_G f(x)\overline{\psi\left(\frac{x-b}{a}\right)}dx\right| \leqslant \|f\|_2\|\psi\|_2$$

And

$$\frac{1}{\sqrt{a}}\int_G |(W_\psi f)(b,a)|db \leqslant \int_G\int_G \left|f(x)\psi\left(\frac{x-b}{a}\right)\right|\frac{dxdb}{a} \leqslant \|f\|_1\|\psi\|_1 \tag{2.9}$$

We get

$$\begin{aligned} K_{21} &\leqslant \|f\|_2\|\psi\|_2 \int_{G-H}\left(\int_G |(W_\psi f)(b,a)|^2 db\right)\frac{1}{|a|^{2-\epsilon}}da \\ &\leqslant \|f\|_2\|\psi\|_2\|f\|_1\|\psi\|_1 \int_{G-H}\frac{1}{|a|^{\frac{3}{2}-\epsilon}}da \\ &= \frac{4}{1-2\epsilon}\|f\|_2\|\psi\|_2\|f\|_1\|\psi\|_1 \end{aligned}$$

Similar to the argument of (2.8), we have

$$K_{22} = 2\pi\int_{G-H}\left(\int_G |\hat{g}(\omega)|^2|\hat{\varphi}(a\omega)|^2|\hat{h}(\omega)|^2|a|^{1-\epsilon}\right) da$$

Further, by the definition of $h(x)$,

$$K_{22} = 2\pi\int_{G-H}|a|^{-1-\epsilon}\left(\int_G |a\omega|^2|\hat{g}(\omega)|^2|\hat{\varphi}(a\omega)|^2 d\omega\right) da$$

From $\hat{\varphi}(\omega) = O(|\omega|^{-2})$, we have $|\hat{\varphi}(a\omega)|^2 \leqslant M_1$($M_1$ is an absolute constant).
Further $K_{22} \leqslant \frac{4\pi M_1}{\epsilon}\|g\|_2^2$.
So $K_2 < \infty$.
We finally obtain (2.6). The proof of lemma is completed.

3. Proof of theorem

From $|(\varphi_{b,a} * h)(x)| \leqslant \|\varphi_{b,a}\|_2\|h\|_2 = \|\varphi\|_2\|h\|_2$ and (2.9), we have

$$\begin{aligned} \int_H\int_G |(W_\psi f)(b,a)(\varphi_{b,a} * h)(x)|\frac{dadb}{|a|} &\leqslant \|\varphi\|_2\|h\|_2\int_H \frac{1}{|a|^{\frac{1}{2}}}\left(\frac{1}{|a|^{\frac{1}{2}}}\int_G |(W_\psi f)(b,a)|db\right) da \\ &\leqslant 4\|\varphi\|_2\|h\|_2\|\psi\|_1\|f\|_1 \end{aligned} \tag{3.1}$$

So, for all $x \in G$, we know that $(W_\psi f)(b,a)(\varphi_{b,a} * h)(x)\frac{1}{|a|} \in L^1(H\times G)$.
Set

$$\triangle_H(x) = \frac{1}{(2\pi)^{\frac{3}{2}}(\varphi,\psi)}\int_H\int_G (W_\psi f)(b,a)(\varphi_{b,a} * h)(x)\frac{dadb}{|a|}.$$

By the known result in theory of Hilbert space, we know that

$$\|f(x) - \triangle H(x)\|_2 = \sup_{\|g\|_2=1} |(f, g) - (\triangle_H, g)|. \tag{3.2}$$

Again by (2.1), we get

$$\begin{aligned}(\triangle_H, g) &= \frac{1}{(2\pi)^{\frac{3}{2}}(\varphi, \psi)} \int_G \left(\int_H \int_G (W_\psi f)(b, a)(\varphi_{b,a} * h)(x)\overline{g(x)} \frac{dadb}{|a|} \right) dx \\ &= \frac{1}{(2\pi)(\varphi, \psi)} \int_H \int_G (W_\psi f)(b, a)\overline{(D_\varphi g)(b, a)} \frac{dbda}{|a|}.\end{aligned} \tag{3.3}$$

The reason for interchanging the order of the above integrals is stated as follows.
By (2.9) and

$$\begin{aligned}\int_G |(\varphi_{b,a} * h)(x)\overline{g(x)}| \frac{1}{\sqrt{|a|}} dx &\leqslant \| \frac{1}{\sqrt{|a|}} (\varphi_{b,a} * h)(x)\|_2 \|g\|_2 \\ &\leqslant \|\varphi\|_1 \|h\|_2 \|g\|_2\end{aligned}$$

We get
$$\begin{aligned}&\int_G \left(\int_H \int_G (W_\psi f)(b, a)(\varphi_{b,a} * h)(x)\overline{g(x)} \frac{dadb}{|a|} \right) dx \\ &= \int_H \left(\int_G |(W_\psi f)(b, a)| \left(\int_G |(\varphi_{b,a} * h)(x)\overline{g(x)}| \frac{1}{\sqrt{|a|}} dx \right) \frac{db}{\sqrt{|a|}} \right) da \\ &\leqslant 2\|\phi\|_1 \|h\|_2 \|g\|_2 \|f\|_1 \|\psi\|_1\end{aligned}$$

So, the order of integrals in (3.3) can be interchanged.

Using lemma and (3.3),

$$(f, g) - (\triangle_H, g) = \frac{1}{(2\pi)(\varphi, \psi)} \int_{G-H} \int_G (W_\psi f)(b, a)\overline{(D_\varphi g)(b, a)} \frac{dbda}{|a|}$$

Further we get from (3.2)

$$\begin{aligned}\|(f, g) - (\triangle_H, g)\|_2 &\leqslant \sup_{\|g\|_2=1} \left(\frac{1}{(2\pi)|(\varphi, \psi)|} \int_{G-H} \int_G (W_\psi f)(b, a)\overline{(D_\varphi g)(b, a)} \frac{dbda}{|a|} \right) \\ &= \sup_{\|g\|_2=1} \left(\frac{1}{(2\pi)|(\varphi, \psi)|} I(\overline{h}) \right).\end{aligned} \tag{3.4}$$

Where $I(\overline{h}) = \int_{G-H} \int_G |(W_\psi f)(b, a)\overline{(D_\varphi g)(b, a)}| \frac{dbda}{|a|}, \forall \overline{h} \in H$
Using cauchy inequality, we can see that

$$\begin{aligned}I^2(\overline{h}) &\leqslant \int_{G-H} \left(\int_G |(W_\psi f)(b, a)|^2 \frac{db}{|a|^{2-\epsilon}} \right) da \cdot \int_{G-H} \left(\int_G |(D_\varphi g)(b, a)|^2 \frac{db}{|a|^\epsilon} \right) da \\ &= I_1(\overline{h}) I_2(\overline{h})\end{aligned} \tag{3.5}$$

Imitating the estimates of K_{21} and K_{22} in lemma, we can get

$$I_1(\overline{h}) \leqslant \frac{4}{1 - 2\epsilon} \overline{h}^{-\frac{1}{2}+\epsilon} \|f\|_2 \|\psi\|_2 \|f\|_1 \|\psi\|_1$$

and

$$I_2(\overline{h}) \leqslant \frac{4\pi M_1}{\epsilon}\overline{h}^{-\epsilon}\|g\|_2^2.$$

From this and (3.4),(3.5), we know that

$$\begin{aligned}\|f(x) - \triangle_H(x)\|_2 &\leqslant \sup_{\|g\|_2=1}\left(\frac{1}{(2\pi)|(\varphi,\psi)|}I(\overline{h})\right)\\ &\leqslant \frac{1}{(2\pi)|(\varphi,\psi)|}\left(\frac{16\pi M_1}{(1-2\epsilon)\epsilon}\overline{h}^{-\frac{1}{2}}\|f\|_2\|\psi\|_2\|f\|_1\|\psi\|_1\right)^{\frac{1}{2}}.\end{aligned}$$

So,

$$\lim_{\overline{h}\to+\infty}\|f(x) - \triangle_H(x)\|_2 = 0.$$

This proof of Theorem is completed.

Acknowledgements: This work is supported by UGC grant No. 16-9(June 2018)/2019(NET/CSIR) and CSIR grant No. 09/725(014)/2019-EMR-1.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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