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Generalization of Fixed Point Approximation of Contraction and Suzuki Generalized Non-Expansive Mappings in Banach Domain

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Abstract. By principal motivation from the results of the new iterative scheme that produces faster results than K-iteration. In this article, we study generalized results by a new iteration scheme to approximate fixed points of generalized contraction and Suzuki non-expansive mappings. We establish strong convergence results of generalized contraction mappings of closed convex Banach space and also deduce data dependent results. Furthermore, we prove some weak and strong convergence theorems in the sense of generalized Suzuki non-expansive mapping by applying condition (C).

1. **Introduction**

Mappings play a vital role in the field of inequalities (see of example [17–20]). The mappings which have Lipschitzís constant equal to 1 are called as non-expansive mappings. Let Z be a non empty bounded closed convex subset of k. A Banach space Z has the fixed point property (FPP) for non expansive mapping if for every non-empty bounded closed convex subset of Z contains a fixed point. Meanwhile, in 1965 major struggle has been proposed to study the theory of fixed point of non-expansive mappings in the setting of reflexive and non-reflexive Banach domain. Since then, a number of generalizations and extensions of non-expansive mappings and their results have been obtained by many authors. We can say that FPP provides basis of physical appearance of the Banach space. When K is a weakly compact convex subset of Z, a non-expansive self-mapping of K requires not have fixed point. However, if the norm of Z has suitable ordered properties (i.e., uniform convex and some others) each non-expansive self-mapping of every weakly closed convex subset of Z has a fixed point. In this case, K is called a weak fixed point property. All over in this article we assume that

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K is a non-empty subset of a Banach space Z and $Q(T)$, the set of all fixed points of the mapping T over K. A mapping $T : K \to K$ is called to be a non-expansive if $\|Tx_o - Ty_o\| \leq \|x_o - y_o\|$, of all x_o, $y_o \in K$. This is also called quasi non-expansive if $Q(T) \neq \phi$ and $\|Tx_o - p\| \leq \|x_o - p\|$, of all $x_o \in K$ and of all $p \in Q(T)$. It is known as $Q(T)$ is non-empty while Z is uniformly convex, K be a bounded closed convex subset of X and T be a non-expansive mapping [2]. In 2008, Japanese mathematician Suzuki [3] presented idea of generalized non-expansive mappings which is also called condition (C) and defined as A self-mapping T on K is said to be condition (C) that,

$$\frac{1}{2}\|x_o - Tx_o\| \leq \|x_o - y_o\| \Longrightarrow \|Tx_o - Ty_o\| \leq \|x_o - y_o\|, \forall x_o, y_o \in K$$

of such mappings, Suzuki also obtained the existence of fixed point and convergence results. In [4], he proved that condition(C) have faster results as compared to non-expansive mappings. For a self-mapping T be defined over $[0,3]$

$$T(x_o) = [0, \text{ if } x_o \neq 0 \qquad 1, \text{ if } x_o = 0].$$

By this, we claim that T satisfy the condition (C), but T is not a non-expansive mapping. In extension, Picard's iterative scheme is approximate the fixed point of contraction mappings in the Banach contraction principle. Over time, many mathematicians [6–10] played a fundamental role in the development of the current literature. Sahu V. K [5] and many other tried their best as compared to the previous one and added outstanding work. Inspired by the above, now we generalize some results by a new iteration scheme to approximate fixed point of generalized contraction and Suzuki's non-expansive mappings. Also we discuss strong convergence theorems of generalized contraction mappings with closed convex Banach space and some data dependence results are also deduce. Moreover, we prove some weak and strong convergence results in type of generalized Suzuki non-expansive mappings by using condition (C).

2. **Preliminaries**

(2.1) [12] Opial property if for each sequence $\{\varpi_n\}$ in X, (where X be a Banach space) converging

weakly to $x_o \in X$ take

$$\lim_{n\to\infty} \sup \|\varpi_n - x_o\| < \lim_{n\to\infty} \sup \|\varpi_n - y_o\| \,\forall\, y_o \in X$$

such that $y_o \neq x_o$.

(2.2) Let K be a non-empty bounded sequence convex subset of a Banach space Z and consider $\{\varpi_n\}$ be in Z of $x_0 \in Z$, that

$$d(z, \{\varpi_n\} = \lim_{n\to\infty} \sup \|\varpi_n - x_o\|.$$

The asymptotic radius of $\{\varpi_n\}$ comparative to K is given that

$$d(K, \{\varpi_n\}) = \inf \{d(x_o, \{\varpi_n\}) : x_o \in K\}.$$

The asymptotic center of $\{\varpi_n\}$ relative to K is the set

$$B(K, \varpi_n\}) = \{x_o \in K : d(x_o, \{\varpi_n\}) = d(K, \{\varpi_n\})\}.$$

(2.3) A uniformly convex Banach space, X and $\{\varpi_n\}$ be a real sequence such that $0 < s \leq \varpi_n \leq t < 1$,$\forall$ $n \geq 1$. Consider $\{\varpi_n\}$ and $\{\omega_n\}$ be two sequences of K given that $lim_{n\to\infty} \sup \|\varpi_n\| \leq d$, $\lim_{n\to\infty} \sup \|\omega_n\| \leq d$ and $lim_{n\to\infty} \sup \|\varpi_n + (1-\varpi_n)\omega_n\| = d$ holds of some $d \geq 0$. Then, $lim_{n\to\infty} \|\varpi_n - \omega_n\| = 0$. It is called a uniformly convex Banach space, $B(K, \{\varpi_n\})$ contains exactly one point.

(2.4) A mapping $T : K \to K$ is called demi-closed with respect to $y_o \in K$ if of each sequence $\{\varpi_n\}$ in K and K be a closed, convex and non-empty subset of a Banach space K of each $x_o \in K, \{\varpi n\}$ converges weakly at x_o and $\{T\varpi_n\}$ converges strongly at $y_o \Longrightarrow Tx_o = y_o$.

(2.5) [16] Let$\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two fixed points iteration sequences that converges to the same fixed point q. If $\|u_n - q\| \leq a_n$ and $\|v_n - q\| \leq b_n$, of all $n \geq 0$, wherever $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real convergent sequences. Then we say that $\{u_n\}_{n=0}^{\infty}$ converge faster than $\{v_n\}_{n=0}^{\infty}$ to q if $\{a_n\}_{n=0}^{\infty}$ converges faster as compare to $\{b_n\}_{n=0}^{\infty}$.

(2.6) Let K be a non-empty subset of a Banach space Z. Consider that a mapping $T : K \to K$ is said to be generalized contraction when $\exists$ $0 \leq h \leq 1$ such that $\|Ts - Tt\| \leq h \max[\|s-t\|, \|s-Ts\|, \|t-Tt\|, \|s-Tt\| + \|t-Ts\|]$ $\forall$ $s, t \in K$.

(2.7) A Banach space K is known as uniformly convex if of each ϵ belongs to $(0, 2]$ there is a $\delta > 0$ such that of $s, t \in K$

$$\begin{aligned} \|s\| &\leq 1, \\ \|t\| &\leq 1, \\ \|s-t\| &> \epsilon, \end{aligned}$$

Implies that

$$\frac{\|s+t\|}{2} \leq \delta.$$

[1] Let X be a non-empty set and φ is collection of X, then

(1) X belongs to τ.

(2) Absolute union of number of τ belongs to τ.

(3) Limited intersection of τ belongs to τ.

Than τ be a topology over X so, (X, τ) is called topological space. Topology also helpful in different properties like convergence, existence, convex and many other.

3. **Some Basic Results**

Proposition (3.1) ([3]) Let Z be a non-empty subset K of a Banach space K and T be a self mappings.

(a_1) If T be non-expansive mapping then T satisfies condition (C).

(a_2) Every mapping satisfying condition (C) with a fixed point is quasi non-expansive.

(a_3) If T satisfies condition (C)

$$\|x_o - Ty_o\| \leq 3\|Tx_o - y_o\| \Longrightarrow \|Tx_o - Ty_o\| + \|x_o - y_o\| \, \forall x_o, y_o \in C.$$

Lemma (3.2) Let $\{\lambda_m\}_{m=0}^{\infty}$ and $\{\mu_m\}_{m=0}^{\infty}$ be a non negative real sequences satisfying the given inequality $\lambda_{m+1} \leq (1-\xi_m)\lambda_m + \mu_m$, also $\xi_m \in (0,1)$ $\forall$ $m \in N$, $\Sigma_{m=0}^{\infty}$ $\xi_m = \infty$ and $\frac{\mu_m}{\xi_m} \to 0$ as $m \to \infty$, then $lim_{m\to\infty}\lambda_m = 0$.

Lemma (3.3) ([15]) Let$\{\lambda_n\}_{n=0}^{\infty}$ be a non-negative real sequence for which assume that $\exists$ $n_0 \in N$ such that $\forall$ $n \geq n_0$, the given inequalities satisfies $\lambda_{n+1} \leq (1-\nu_n)\lambda_n + \nu_n\mu_n$, also $\nu_n \in (0,1)$ $\forall$ $n \in N$, $\Sigma_{n=0}^{\infty}\nu_n = \infty$ and $\mu_n \geq 0$ $\forall$, $n \in N$, so $0 \leq lim_{n\to\infty} \sup \lambda_n \leq lim_{n\to\infty} \sup \mu_n$.

Lemma (3.4) [13] Let K be a uniformly convex Banach space and T be a self-mapping over a weakly compact convex subset K. Consider that T fulfil condition (C), then T has a fixed point.

Lemma (3.5) Suppose that X be a subset K of a Banach space with the Opial's property [12] and T be a self mapping over X. Suppose T satisfies the criteria of condition (C). If $\{\varpi_n\}$ converges weakly to τ and $lim_{n\to\infty} \|\varpi n - T\varpi n = 0\|$, then $T\tau = \tau$. It is $I - T$ demi-closed at 0.

Here we define our new iterative process, it has better approximations and have faster rate of convergence then previous all (for further details see [11]). Now we generalized our results by this faster iterative scheme

$$\begin{aligned} u_0 &\in K \\ z_n &= T[(1-\delta_n)u_n + \delta_n T u_n] \\ y_n &= T[(1-\alpha_n)Tu_n + \alpha_n T z_n] \\ u_{n+1} &= Ty_n. \end{aligned} \tag{1}$$

4. **Main Results**

In this section we generalize results via new faster iterative scheme to approximate fixed point of generalized contraction and Suzuki's non-expansive mappings. We generalize strong convergence results in closed convex Banach space and some data dependence results are also deduce.

Theorem 4.1. *Suppose K be a non-empty closed convex subset of a Banach space Z and $T : K \to K$ be generalized contraction mapping. Assume $\{h_n\}_{n=0}^{\infty}$ be an iterative sequence which is generated by* (1)*, with the real sequence $\{\eta_n\}_{n=0}^{\infty}$ and $\{\kappa_n\}_{n=0}^{\infty}$ in* $[0,1]$ *satisfying $\Sigma_{n=0}^{\infty}\eta_n\gamma_n = 0$ then, $\{h_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point of T.*

Proof. The well-known Banach principle has guarantees of existence and uniqueness of fixed point g. We prove that $\{h_n\}$ converges to a fixed point g, by using (1) we get

$$\begin{aligned}
\|z_n - g\| &= \|T\left[(1-\gamma_n) h_n + \gamma_n T h_n\right] - g\| \\
&\leq h \ \max[\|(1-\gamma_n)h + \gamma_n T h_n - g\|, \\
&\quad \|((1-\gamma_n)h_n + \gamma_n T h_n) - T\{(1-\gamma_n)h_n + h_n\gamma_n\}\|, \\
&\quad \|g - Tg\|, \|(1-\gamma_n)h_n + \gamma_n T h_n - Tg\| + \\
&\quad \|g - T((1-\gamma_n)h_n + \gamma_n h_n)\|] \\
&\leq h \max[\|(1-\gamma_n) h_n + \gamma_n T h_n - Tg\|, \|((1-\gamma_n)h_n + \gamma_n T h_n) - z_n\|, \\
&\quad \|(1-\gamma_n)h_n + \gamma_n T h_n - g\| + \|g - z_n\|]. \qquad (2)
\end{aligned}$$

Case#1 Let

$$\|z_n - g\| \leq h[\|(1-\gamma_n) h_n + \gamma_n T h_n - g\|]. \qquad (3)$$

Case#2 Let

$$\begin{aligned}
\|z_n - g\| &\leq h[\|(1-\gamma_n) h_n + \gamma_n T h_n - z_n\|] \\
&= h[\|(1-\gamma_n) h_n + \gamma_n T h_n - g + g - z_n\|] \\
&\leq [\|(1-\gamma_n) h_n + \gamma_n T h_n - g\| + \|z_n - g\|] \\
\|z_n - g\| &\leq \frac{h}{1-h}[\|(1-\gamma_n) h_n + \gamma_n h_n - g\|]. \qquad (4)
\end{aligned}$$

Case#3 Let

$$\begin{aligned}
\|z_n - g\| &\leq h[\|(1-\gamma_n) h_n + \gamma_n T h_n - g\| + \|z_n - g\|] \\
\|z_n - g\| &\leq \frac{h}{1-h}[\|(1-\gamma_n) h_n + \gamma_n T h_n - g\|].
\end{aligned}$$

Let $\eta = \max[h, \frac{h}{1-h}] \in (0,1)$

$$\begin{aligned}
\|z_n - g\| &\leq \eta \|(1-\gamma_n) h_n + \gamma_n h_n - g\| \\
&\leq \eta \|(1-\gamma_n) h_n - (1-\gamma_n)g + \gamma_n T h_n - \gamma_n g\| \\
&\leq \eta \ [(1-\gamma_n) \|h_n - g\| + \gamma_n \|T h_n - g\|.
\end{aligned}$$

Now

$$\begin{aligned}
\|T h_n - g\| &\leq h \max[\|h_n - g\|, \|Th - g\|, \|Tg - g\| + \|g - T h_n\|] \\
&= h \max[\|h_n - g\|, \|T h_n - g\|, \|T h_n - g\| + \|h_n - g\|] \\
&= h \max[\|h_n - g\|, \|T h_n - g\| + \|h_n - g\|] \\
&\leq \eta \|h_n - g\|.
\end{aligned}$$

$$\begin{aligned}
\|z_n - g\| &\leq \eta\ [(1-\gamma_n)\|h_n - g\| + \gamma_n\eta\|h_n - g\| \\
&\leq \eta\ [(1-\gamma_n + \gamma_n\eta)]\|h_n - g\| \\
&\leq \eta\ [(1-\gamma_n)(1-\alpha)]\|h_n - g\|. \qquad (5)
\end{aligned}$$

Similarly

$$\begin{aligned}
\left\|h_n' - g\right\| &= \|T(1-\eta_n)Th_n + \eta_n Tz_n - Tg\| \\
&\leq h\max[\|(1-\eta_n)Th_n + \eta_n Tz_n - g\|, \\
&\quad \|(1-\eta_n)Th_n + \eta_n Tz_n - T((1-\eta_n)Th_n + \eta_n Tz_n)\|, \\
&\quad \|g - Tg\|, \|((1-\eta_n)Th_n + \eta_n Tz_n) - Tg\| + \\
&\quad \|g - T((1-\eta_n)Th_n + \eta_n Tz_n)\|.
\end{aligned}$$

Case#1

$$\left\|h_n' - g\right\| \leq h[\|(1-\eta_n)Th_n + \eta_n Tz_n - g\|]. \qquad (6)$$

Case#2

$$\begin{aligned}
\left\|h_n' - g\right\| &\leq h[\left\|(1-\eta_n)Th_n + \eta_n Tz_n - h_n'\right\|] \\
&= h[\left\|(1-\eta_n)Th_n + \eta_n Tz_n - g + g - h_n'\right\|] \\
\left\|h_n' - g\right\| &\leq \frac{h}{1-h}[\|(1-\eta_n)Th_n + \eta_n Tz_n - g\|]. \qquad (7)
\end{aligned}$$

Case#3

$$\begin{aligned}
\left\|h_n' - g\right\| &\leq h[\|(1-\eta_n)Tq_n + \eta_n Tz_n - q\| + \left\|q - h_n'\right\|] \\
\left\|h_n' - g\right\| &\leq \frac{h}{1-h}[\|(1-\eta_n)Th_n + \eta_n Tz_n - q\|].
\end{aligned}$$

Let $\eta = \max\{h, \frac{h}{1-h}\} \in (0,1)$

$$\begin{aligned}
\left\|h_n' - g\right\| &\leq \eta\ [\|(1-\eta_n)Th_n + \eta_n Tz_n - g\|] \\
&= \eta\ [(1-\eta_n)\|Th_n - g\| + \eta_n\|Tz_n - g\|] \\
&\leq \eta\ [(1-\eta_n)\|Th_n - g\| + \eta\eta_n\|z_n - g\|] \\
&\leq \eta\ [\eta\ (1-\eta_n)\|h_n - g\| + \eta^2\eta_n(1-\gamma_n(1-\eta))\|h_n - g\|] \\
&\leq \eta^2[(1-\eta_n)\|h_n - g\| + \eta\eta_n(1-\gamma_n(1-\eta))\|h_n - g\|] \\
&\leq \eta^2[(1-\eta_n)\|h_n - g\| + \eta\eta_n(1-\gamma_n(1-\eta))\|h_n - g\|] \\
&\leq \eta^2[(1-\eta_n + \eta\eta_n - \eta\eta_n\gamma_n(1-\eta)]\|h_n - g\| \\
&\leq \eta^2[(1-\eta_n(1-\eta) - \eta\eta_n\gamma_n(1-\eta_n))]\|h_n - g\|
\end{aligned}$$

$$\left\|h_n' - g\right\| \leq \eta^2[(1-(1-\eta)\,\eta_n(1+\eta\gamma_n)]\|h_n - g\| \qquad (8)$$

$$
\begin{aligned}
\|h_{n+1}-g\| &= \left\|Th'_n - g\right\| \\
&= \|T\left[T\left(1-\eta_n\right)Th_n + \eta_n T z_n\right] - g\| \\
&\leq \max[\left\|h'_n - g\right\|, \left\|h'_n - Th'_n\right\|, \|g-Tg\|, \left\|h'_n - Tg\right\| + \left\|g - Th'_n\right\| \\
&\leq h\max\|T\left(1-\eta_n\right)Th_n + \eta_n T z_n - g\|, \\
&\quad \|T((1-\eta_n)Th_n + \eta_n T z_n) - T(T(1-\eta_n)Th_n + \eta_n T z_n))\|, \\
&\quad \|Tg-g\|, \|T(T(1-\eta_n)Th_n - \eta_n T z_n)) - g\| \\
&\quad + \|T((1-\eta_n)Th_n + \eta_n T z_n) - g\|] \\
&= h\max[\|h_n - g\|, \left\|h'_n - Th'_n\right\|, 0, \left\|Th'_n - g\right\| + \left\|h'_n - g\right\| \\
&= h\max[\left\|h'_n - g\right\|, \left\|h_{n+1} - h'_n\right\|, \|h_{n+1}-g\| + \left\|h'_n - g\right\|.
\end{aligned}
$$

Case#1

$$\|h_{n+1}-g\| \leq h\left\|h'_n - g\right\|.$$

Case#2

$$\|h_{n+1}-g\| \leq \frac{h}{1-h}\left\|h'_n - g\right\|.$$

Case#3

$$\|h_{n+1}-g\| \leq \frac{h}{1-h}\left\|h'_n - g\right\|$$

$\eta = \max\{h, \frac{h}{1-h}\} \in (0,1)$

$$
\begin{aligned}
\|h_{n+1}-g\| &\leq \eta\left\|h'_n - g\right\| \\
&\leq \eta[\eta^2(1-\eta_n(1+\eta\gamma_n)(1-\eta))\|h_n - g\|] \\
&\leq \eta^3(1-\eta_n(1+\eta\gamma_n)(1-\eta))\|h_n - g\|]. \qquad (9)
\end{aligned}
$$

Repetition of above scheme gives the following inequalities

$$
\begin{aligned}
\|h_{n+1}-g\| &\leq \eta^3(1-\eta_n(1+\eta\gamma_n)(1-\eta))\|h_n - g\| \\
\|h_n - g\| &\leq \eta^3(1-\eta_{n-1}(1+\eta\gamma_{n-1})(1-\eta))\|h_{n-1} - g\| \\
\|h_{n-1} - g\| &\leq \eta^3(1-\eta_{n-2}(1+\eta\gamma_{n-2})(1-\eta))\|h_{n-2} - g\| \\
&\vdots \\
\|h_1 - g\| &\leq \eta^3(1-\eta_0(1+\eta\gamma_0)(1-\eta))\|h_0 - g\|. \qquad (10)
\end{aligned}
$$

From (10) we can easily derive

$$\|h_{n+1}-g\| \leq \|h_0 - g\|\,\eta^{3(n+1)}\prod_{k=0}^{n} 1 - \eta_k(1+\eta\gamma_k)(1-\eta) \qquad (11)$$

Where $1-\eta_k(1+\eta\gamma_k)(1-\eta)<1$ because $\eta\in(0,1)$ and $\eta_n\gamma_n\in(0,1)\ \forall\ n\in N$. We know that $1-h\leq \varrho^{-h}\ \forall\ x\in(0,1)$ then by (11), we have

$$\|h_{n+1}-g\|\leq \frac{\|h_0-q\|\,\eta^{3(n+1)}}{\varrho^{(1-\eta)}\Sigma_{k=0}^{n}\eta_k(1+\eta\gamma_k)} \tag{12}$$

Taking limit of both sides in (12), we get $\lim_{n\to\infty}\|h_n-g\|$ i.e $h_n\to g$ of $n\to\infty$ as required.

□

Theorem 4.2. *Suppose that K be a non-empty closed convex subset of a Banach space Z and $T:K\to K$ be a generalized contraction mappings. Consider $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence that is generated from (1) with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=o}^{\infty}$ in $[0,1]$ satisfying the criteria of $\Sigma_{n=0}^{\infty}\alpha_n\beta_n=\infty$. Then, iteration scheme (1) be T- stable.*

Proof. Let $\{u_n\}_{n=0}^{\infty}\subset Z$ be arbitrary sequence in K. Suppose that the sequence generated from (1) be $x_{n+1}=f(T,x_n)$ converging to a unique fixed point q (by theorem 4.1) and $\epsilon_n=||u_{n+1}-f(T,u_n)||$ we prove that $lim_{n\to\infty}\ \epsilon_n=0\iff lim_{n\to\infty}u_n=q$.

Assume $\lim_{n\to\infty}\epsilon_n=0$ we take

$$\begin{aligned}\|u_{n+1}-q\| &\leq \|u_{n+1}-f\ (T,u_n)\|+\|f(T,u_n)-q\|\\ &= \epsilon_n+\|T(T((1-\beta_n)Tu_n+\beta_nT((1-\alpha_n)u_n+\alpha_nTu_n)))-q\|\\ &\leq \alpha^3(1-(1-\alpha))\alpha_n(1+\beta_n\alpha)\,\|u_n-q\|+\epsilon_n.\end{aligned}$$

Since $\alpha\in(0,1)$ and $\alpha_n,\beta_n\in[0,1]\ \forall n\in N$ and $\lim_{n\to\infty}\epsilon_n=0$. So, by above inequality and lemma 3.2 which leads $\lim_{n\to\infty}\|u_n-q\|=0$. Hence $\lim_{n\to\infty}u_n=q$.

Conversely

Consider that $\lim_{n\to\infty}u_n=q$ we get

$$\begin{aligned}\epsilon_n &= \|u_{n+1}-f\ (T,u_n)\|\\ &\leq \|u_{n+1}-q\|+\|f(T,u_n)-q\|\\ &\leq \|u_{n+1}-q\|+\alpha^3(1-(1-\alpha)\alpha_n(1+\alpha\beta_n)\,\|u_n-q\|\end{aligned}$$

$\iff lim_{n\to\infty}\epsilon_n=0$. Hence, (1) is stable w.r.t T.

□

Theorem 4.3. *Suppose that K be a non-empty closed convex subset of a Banach space Z and $T:K\to K$ be a generalized contraction mapping of a fixed point p . It is given that $u_0=x_0\in C$, let $\{u_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ be iterative sequences generated by (1) respectively, with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying*

(S_1) $\alpha\leq\alpha_n<1$ and $\beta\leq\beta_n<1$, for some results like $\alpha,\beta>0$ and $\forall\ n\in N$. Then, $\{x_n\}_{n=0}^{\infty}$ converges to p faster as than $\{u_n\}_{n=0}^{\infty}$ does.

Proof. By (11) we get

$$\|x_{n+1}-p\| \leq \|x_0-p\|\,\alpha^{3(n+1)}\Pi_{k=0}^{n}(1-(1-\alpha))\alpha_k(1+\alpha\beta_k). \tag{13}$$

The following inequality is due to (9) and Lemma (3.2) which is obtained from (1), also converges to a unique fixed point p.

$$\|u_{n+1}-p\| \leq \|u_0-p\|\,\alpha^{2(n+1)}\Pi_{k=0}^{n}(1-(1-\alpha))\alpha_k(1+\alpha\beta_k) \tag{14}$$

Together with supposition (S_1) and (13) implies that

$$\begin{aligned}\|x_{n+1}-p\| &\leq \|x_0-p\|\,\alpha^{3(n+1)}\Pi_{k=0}^{n}[(1-(1-\alpha))\alpha(1+\alpha\beta)]\\ &= \|x_0-p\|\,\alpha^{3(n+1)}[(1-(1-\alpha))\alpha(1+\alpha\beta)]^{n+1}. \end{aligned}\tag{15}$$

Similarly (15) and supposition (S_1)

$$\begin{aligned}\|u_{n+1}-p\| &= \|u_0-p\|\,\alpha^{2(n+1)}\Pi_{k=0}^{n}(1-(1-\alpha))\alpha(1+\beta\alpha)\\ &= \|u_0-p\|\,\alpha^{2(n+1)}[(1-(1-\alpha))\alpha(1+\beta\alpha)]^{n+1} \end{aligned}\tag{16}$$

Define

$$\begin{aligned}a_n &= \|x_0-p\|\,\alpha^{3(n+1)}[(1-(1-\alpha))\alpha(1+\alpha\beta)]^{n+1}\\ b_n &= \|u_0-p\|\,\alpha^{2(n+1)}[(1-(1-\alpha))\alpha(1+\alpha\beta)]^{n+1}\end{aligned}$$

Then

$$\begin{aligned}\Psi_n &= \frac{a_n}{b_n}\\ &= \frac{\|x_0-p\|\,\alpha^{3(n+1)}[(1-(1-\alpha))\alpha(1+\alpha\beta)]^{n+1}}{\|u_0-p\|\,\alpha^{2(n+1)}[(1-(1-\alpha))\alpha(1+\alpha\beta)]^{n+1}}\\ &= \left\|\alpha^{n+1}\right\| \end{aligned}\tag{17}$$

Since

$$\lim_{n\to\infty}\frac{\Psi_{n+1}}{\Psi_n}=\lim_{n\to\infty}\frac{\alpha^{n+2}}{\alpha^{n+1}}=\alpha<1$$

By applying the ratio test we get

$$\Sigma_{n=0}^{\infty}\Psi_n<\infty$$

Hence from (17), we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\Psi_n=0$$

Implies that $\{x_n\}_{n=0}^{\infty}$ is faster than $\{u_n\}_{n=0}^{\infty}$.

Now we prove following data dependence results.

□

Theorem 4.4. *Suppose that $\widetilde{T}$ be an approximate operator of a generalized contraction mapping T. Consider that $\{h_n\}_{n=0}^{\infty}$ be an iterative sequence generated from equation (1) for T and we define an iterative sequence $\{\widetilde{h}_n\}_{n=0}^{\infty}$ which is given as*

$$\begin{aligned} \widetilde{h}_0 &\in K \\ \widetilde{z}_n &= T[(1-\gamma_n)\widetilde{h}_n + \gamma_n \widetilde{T}\widetilde{h}_n] \\ \widetilde{h'}_n &= \widetilde{T}[(1-\alpha_n)\widetilde{T}\widetilde{h}_n + \alpha_n \widetilde{T}\widetilde{z}_n] \\ \widetilde{h}_{n+1} &= \widetilde{T}\widetilde{h'}_n \end{aligned} \tag{18}$$

With real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$ which satisfying the
(I) $\frac{1}{2} \leq \alpha_n \gamma_n \ \forall \ n \in N$
(II) $\Sigma_{n=0}^{\infty} \alpha_n \gamma_n = \infty$ if $Tq = q$ and $\widetilde{T}\widetilde{q} = \widetilde{q}$
such that $\lim_{n\to\infty} \widetilde{h}_n = \widetilde{q}$, then we get

$$\|q - \widetilde{q}\| \leq \frac{7\epsilon}{1-\alpha}$$

Where $\epsilon \geq 0$ is a fixed number.

Proof. It follows from (1) and (18)

$$\begin{aligned} \|z_n - \widetilde{z}_n\| &= \left\| T(1-\gamma_n)h_n + \gamma_n T h_n) - \widetilde{T}((1-\gamma_n)\widetilde{h}_n - \gamma_n \widetilde{T}\widetilde{h}_n) \right\| \\ &\leq \left\| T(1-\gamma_n)h_n + \gamma_n T h_n) - T(1-\gamma_n)\widetilde{h}_n + \gamma_n \widetilde{T}\widetilde{h}_n) \right\| \\ &\quad + \left\| T((1-\gamma_n)\widetilde{h}_n + \gamma_n \widetilde{T}\widetilde{h}_n) - \widetilde{T}((1-\gamma_n)\widetilde{h}_n + \gamma_n \widetilde{T}\widetilde{h}_n) \right\| \\ &\leq \alpha \left\| (1-\gamma_n)h_n + \gamma_n T h_n - (1-\gamma_n)\widetilde{h}_n - \gamma_n \widetilde{T}\widetilde{h}_n) \right\| + \epsilon \\ &\leq \alpha[(1-\gamma_n) \left\| h_n + \widetilde{h}_n \right\| + \gamma_n \left\| T h_n - \widetilde{T}\widetilde{h}_n \right\| + \epsilon \\ &\leq \alpha[(1-\gamma_n) \left\| h_n - \widetilde{h}_n \right\| + \gamma_n \{ \left\| T h_n - \widetilde{T}\widetilde{h}_n \right\| + \left\| T\widetilde{h}_n - \widetilde{T}\widetilde{h}_n \right\| \} + \epsilon \\ &\leq \alpha[(1-\gamma_n) \left\| h_n - \widetilde{h}_n \right\| + \gamma_n \alpha \left\| h_n - \widetilde{h}_n \right\| + \gamma_n \epsilon] + \epsilon \\ &\leq \alpha[1 - \gamma_n (1-\alpha) \left\| h_n - \widetilde{h}_n \right\| + \gamma_n \epsilon] + \epsilon. \end{aligned} \tag{19}$$

Using (19), we have

$$\begin{aligned} \left\| h'_n - \widetilde{h'}_n \right\| &= \left\| T((1-\alpha_n) T h_n + \alpha_n T z_n) - \widetilde{T}((1-\alpha_n)\widetilde{T}\widetilde{h}_n + \widetilde{T}\widetilde{z}_n) \right\| \\ &\leq T((1-\alpha_n) T h_n + \alpha_n T z_n) - T((1-\alpha_n)\widetilde{T}\widetilde{h}_n + \alpha_n \widetilde{T}\widetilde{z}_n) \\ &\quad + T((1-\alpha_n)\widetilde{T}\widetilde{h}_n + \alpha_n \widetilde{T}\widetilde{z}_n) - \widetilde{T}((1-\alpha_n)\widetilde{T}\widetilde{h}_n + \alpha_n \widetilde{T}\widetilde{z}_n) \\ &\leq \alpha \left\| (1-\alpha_n) T h_n + \alpha_n T z_n - (1-\alpha_n)\widetilde{T}\widetilde{h}_n - \alpha_n \widetilde{T}\widetilde{z}_n) \right\| + \epsilon \\ &\leq \alpha[(1-\alpha_n) \left\| T h_n - \widetilde{T}\widetilde{h}_n \right\| + \alpha_n \left\| T z_n - \widetilde{T}\widetilde{z}_n \right\| + \epsilon \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha[(1-\alpha_n)\left\|Th_n - T\widetilde{h}_n\right\| + \left\|T\widetilde{h}_n - \widetilde{T}\widetilde{h}_n\right\| \\
&\quad +\alpha_n[\|Tz_n - T\widetilde{z}_n\| + \left\|T\widetilde{z}_n - \widetilde{T}\widetilde{z}_n\right\|] + \epsilon \\
&\leq \alpha[(1-\alpha_n)\alpha\left\|x_n - \widetilde{h}_n\right\| \\
&\quad +\alpha_n\alpha[\alpha(1-\gamma_n)(1-\alpha)||h_n - \widetilde{h}_n|| + \gamma_n\epsilon + \epsilon] + \epsilon \\
&\leq \alpha^2[(1-\alpha_n)\left\|x_n - \widetilde{h}_n\right\| + \alpha^3\alpha_n[1-\gamma_n(1-\alpha)\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha^3\alpha_n\gamma_n + \alpha^2\alpha_n]\epsilon \\
&\leq \alpha^2[1-\alpha_n+\alpha_n\alpha+\alpha(1-\alpha)\alpha_n\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha\epsilon(1+\alpha\alpha_n\gamma_n) + \epsilon \\
&\leq \alpha^2[1-(1-\alpha)\alpha_n - \alpha(1-\alpha)\alpha_n\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha\epsilon(1+\alpha\alpha_n\gamma_n) + \epsilon \\
&\leq \alpha^2[1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha\epsilon(1+\alpha\alpha_n\gamma_n) + \epsilon. \qquad (20)
\end{aligned}
$$

By using (20), we have

$$
\begin{aligned}
\left\|h'_{n+1} - \widetilde{h'}_{n-1}\right\| &= \left\|Th'_n - \widetilde{T}\widetilde{h'}_n\right\| \\
&\leq \left\|h'_n - \widetilde{h'}_n\right\| + \epsilon \\
&\leq \alpha^3[1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha^2\epsilon(1+\alpha\alpha_n\gamma_n)] + \epsilon\alpha + \epsilon \\
&\leq [1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\epsilon(1+\alpha\alpha_n\gamma_n)] + \epsilon + \epsilon \\
&\leq [1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha_n\gamma_n\epsilon + 3\epsilon \\
&\leq [1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha_n\gamma_n\epsilon + 3(1-\alpha_n\gamma_n+\alpha_n\gamma_n)\epsilon \qquad (21)
\end{aligned}
$$

By supposition (I) we have $1-\alpha_n\gamma_n \leq \alpha_n\gamma_n$

$$
\begin{aligned}
\left\|h_{n+1} - \widetilde{h}_{n+1}\right\| &\leq [1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +7\alpha_n\gamma_n\epsilon \\
&= [1-(1-\alpha)\alpha_n(1+\alpha\gamma_n)]\left\|h_n - \widetilde{h}_n\right\| \\
&\quad +\alpha_n\gamma_n(1-\alpha)\frac{7\epsilon}{1-\alpha}. \qquad (22)
\end{aligned}
$$

Let $\Psi_n = \left\| h_n - \widetilde{h}_n \right\|$, $\phi_n = (1-\alpha)\,\alpha_n(1+\alpha\gamma_n)$, $\phi_n = \frac{7\epsilon}{1-\alpha}$, then from lemma 3.2 together with (22) we get

$$0 \leq \lim_{n\to\infty} \sup \left\| h_n - \widetilde{h}_n \right\| \leq \lim_{n\to\infty} \sup \frac{7\epsilon}{1-\alpha} \tag{23}$$

Since by theorem 4.1 we have $lim_{n\to\infty} h_n = q$ and from supposition I and II we get $lim_{n\to\infty} \widetilde{h}_n = \widetilde{q}$ by using these together with (23) and we get $\|q - \widetilde{q}\| \leq \frac{7\epsilon}{1-\alpha}$ as required. □

5. **Convergence Results of Suzuki Generalized Non-Expansive Mappings of Condition** (C)

In this section, we prove some weak and strong convergence theorems of a sequence generated from new iterative scheme of Suzuki generalized non-expansive mappings with condition (C) by uniformly convex Banach spaces.

Lemma 5.1. *Suppose that K be a non-empty uniformly closed convex subset of a Banach space Z. Let $T : K \to K$ be a mapping satisfying condition (C) with $Q(T) \neq 0$. For arbitrary chosen $h_0 \in K$, Consider that a sequence $\{h_n\}$ is generated from (1), then $lim_{n\to\infty} \|h_n - q\|$ exists for any $g \in Q(T)$.*

Proof. Consider that $g \in Q(T)$ and $z \in K$. So T satisfies condition $(C) \leq 0$

$$\frac{1}{2}\|g - Tg\| = 0 \leq \|g - z\| \Rightarrow \|Tg - Tz\| \leq \|g - z\|$$

so by proposition (a_2) we get

$$\begin{aligned}
\|z_n - g\| &= \|T[(1-\gamma_n)h_n + \gamma_{n\delta}T_n] - g\| \\
&= \|T[(1-\gamma_n)h_n + \gamma_n T_n] - Tg\| \\
&\leq ||(1-\gamma_n)h_n + \gamma_n T_n - g|| \\
&\leq (1-\gamma_n)\|h_n - g\| + \gamma_n \|Tx_n - g\| \\
&\leq (1-\gamma_n)\|h_n - g\| + \gamma_n \|h_n - g\| \\
&\leq \|h_n - g\|. \qquad (24)
\end{aligned}$$

By using (24) we have

$$\begin{aligned}
\left\|h'_n - g\right\| &= \|T((1-\delta_n)Th_n + \delta_n Tz_n) - g\| \\
&\leq \|(1-\delta_n)Th_n + \delta_n Tz_n - g\| \\
&\leq (1-\delta_n)\|Tx_n - g\| + \delta_n \|Tz_n - g\| \\
&\leq (1-\delta_n)\|h_n - g\| + \delta_n \|z_n - g\| \\
&\leq (1-\delta_n)\|h_n - g\| + \delta_n \|h_n - g\| \\
&= \|h_n - g\|. \qquad (25)
\end{aligned}$$

Similarly by using (25) we have

$$\begin{aligned} \|h_{n+1} - g\| &= \left\|Th'_n - g\right\| \\ &\leq \left\|h'_n - g\right\| \\ &\leq \|h_n - g\| \end{aligned} \tag{26}$$

$\Rightarrow \|h_n - g\|$ be a bounded and decreasing $\forall\ g \in Q\,(T)$ So, $\lim_{n\to\infty} \|h_n - g\|$ exist as required.

□

Theorem 5.1. *Suppose that K is a non-empty and uniformly closed convex of subset of a Banach space Z. and let $T : K \to K$ be a mapping satisfying condition (C). For arbitrary chosen $h_0 \in K$, consider that the sequence $\{h_n\}$ be generated from (1) $\forall\ n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real numbers in $[u, v]$ for some u, v with $0 < u \leq v < 1$. So, $Q(T) \neq \theta \iff \{h_n\}$ is bounded sequence and $lim_{n\to\infty} \|Th_n - h_n\| = 0$.*

Proof. Suppose that $Q(T) \neq \phi$ and let $g \in Q(T)$. Then, from Lemma 5.1, $lim_{n\to\infty} \|h_n - g\|$ exists and $\{h_n\}$ is bounded.

$$\lim_{n\to\infty} \|h_n - g\| = r \tag{27}$$

From (24) and (27), we have

$$\lim_{n\to\infty} \sup \|z_n - g\| \leq \lim_{n\to\infty} \sup \|h_n - g\| = r \tag{28}$$

So by proposition 3.1 (a_2)

$$\lim_{n\to\infty} \sup \|Th_n - g\| \leq \lim_{n\to\infty} \sup \|h_n - g\| = r \tag{29}$$

Also

$$\begin{aligned} \|h_{n+1} - g\| &= \left\|Th'_n - g\right\| \\ &\leq \left\|h'_n - g\right\| \\ &= \|T((1-\alpha_n)Th_n + \alpha_n Tz_n) - g\| \\ &\leq \|(1-\alpha_n)Th_n + \alpha_n Tz_n - g\| \\ &\leq (1-\alpha_n)\|Th_n - g\| + \alpha_n \|Tz_n - g\| \\ &\leq (1-\alpha_n)\|h_n - g\| + \alpha_n \|z_n - g\| \\ &\leq \|h_n - g\| - \alpha_n \|h_n - g\| + \alpha_n \|z_n - g\| \end{aligned} \tag{30}$$

This implies

$$\frac{\|h_{n+1} - g\| - \|h_n - g\|}{\alpha_n} \leq \|z_n - g\| - \|y_n - g\|$$

$$\begin{aligned} \|h_{n+1} - g\| - \|h_n - g\| &\leq \frac{\|h_{n+1} - g\| - \|h_n - g\|}{\alpha_n} \\ &\leq \|z_n - g\| - \|h_n - g\| \end{aligned}$$

$$\implies \|x_{n+1} - p\| \leq \|z_n - p\|$$

Therefore $r \leq \lim_{n\to\infty} \inf \|z_n - g\|$ from (28) and (30), we have

$$\begin{aligned} r &= ||z_n - g|| \\ &= \lim_{n\to\infty} \|T((1-\gamma_n)h_n + \gamma_n T_n)g\| \\ &\leq \lim_{n\to\infty} \|\gamma_n(Th_n - g) + (1-\gamma_n)h_n - g\| \end{aligned} \tag{31}$$

From (27), (29) and (31) together with Lemma 3.3 we get $\lim_{n\to\infty} \|Th_n - h_n\| = 0$.

Conversely

Suppose that $\{h_n\}$ is bounded and

$$\lim_{n\to\infty} \|Th_n - h_n\| = 0$$

Consider that $g \in (c, \{h_n\})$ by proposition 3.1 we get

$$\begin{aligned} r(Tg, \{h_n\}) &= \lim_{n\to\infty} \sup \|h_n - Tg\| \\ &\leq \lim_{n\to\infty} \sup 3\|h_n - Tg\| + \|h_n - g\| \\ &\leq \lim_{n\to\infty} \sup \|h_n - g\| \\ &= r(g, \{h_n\}). \end{aligned}$$

This implies that $Tg \in B(K, \{h_n\})$. Since Z is uniformly convex, $B(K, \{h_n\})$ singleton and we get $Tg = g$. Hence $Q(T) \neq \phi$.

We are able to prove weak convergence theorem.

□

Theorem 5.2. *Let K be a non-empty closed convex subset of a uniformly convex Banach space Z, with Opial property, and consider that $T : K \to K$ be a mapping satisfying condition (C). For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ is generated from (1) of all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of a real numbers in $[l, m]$ for some l, m with $0 < l \leq m < 1$ such that $Q(T) \neq \phi$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. From Theorem 5.1 we get $\{x_m\}$ be bounded and $\lim_{n\to\infty} \|Tx_m - x_m\| = 0$. Since, X be a uniformly convex and reflexive thereof, from Eberlin's theorem $\exists$ a subsequence $\{x_{mu}\}$ of $\{x_m\}$ which converges weakly to some points $q_1 \in Z$. Since, K is closed and convex from Mazur's theorem $q_1 \in K$. From lemma 3.4, $q_1 \in Q(T)$. Now, we prove that $\{x_m\}$ converges weakly to q_1. In fact if this is false than there must exist a subsequence $\{x_{mv}\}$ of $\{x_m\}$ such that $\{x_{mv}\}$ converges weakly to

$q_2 \in K$ and $q_2 \neq q_1$. From lemma 3.5 $q_2 \in Q(T)$. Since, $lim_{n\to\infty} \|x_m - p\|$ exists $\exists$, $p \in Q(T)$. By Theorem 5.1 and by Opial's property, we get

$$\begin{aligned}
\lim_{m\to\infty} \inf \|x_m - q_1\| &= \lim_{u\to\infty} \inf \|x_{mu} - q_1\| \\
&< \lim_{u\to\infty} \inf \|x_{mu} - q_2\| \\
&= \lim_{m\to\infty} \inf \|x_m - q_2\| \\
&= \lim_{v\to\infty} \inf \|x_{mv} - q_2\| \\
&< \lim_{v\to\infty} \inf \|x_{mv} - q_1\| \\
&= \lim_{m\to\infty} \inf \|x_m - q_1\|
\end{aligned}$$

which is contradiction so $q_1 = q_2$. This implies $\{x_m\}$ converges weakly to a fixed point of T. Now we establish strong convergence results.

□

Theorem 5.3. *Suppose that K be a non-empty compact convex subset of a uniformly convex Banach space Z. Let $T : K \to K$ be a mapping satisfying condition (C). For arbitrary chosen $l_0 \in K$, consider that a sequence $\{l_n\}$ is generated from* (1) $\forall$, $n \geq 1$, *where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real numbers in $[u, v]$ for some u, v with $0 < u \leq v < 1$. Then $\{x_n\}$ converges strongly to a fixed point for T.*

Proof. By lemma 3.4, $Q(T) \neq \phi$ and by theorem 5.1 we have $lim_{n\to\infty} \|Tl_n - l_n\| = 0$. Since K is compact and $\exists$ a subsequence $\{l_{nk}\}$ of $\{l_n\}$ such as $\{l_{nk}\}$ converges strongly to p for some $p \in K$. From proposition a_3 we get

$$\|l_{nk} - Tp\| \leq 3\,\|Tl_{nk} - l_{nk}\| + \|l_{nk} - p\|$$

For all $n \geq 1$. Suppose that $k \to \infty$, than we get $Tp = p$, *i.e.*, $p \in Q(T)$. Since, from lemma 5.1, $lim_{n\to\infty} \|l_n - p\|$ exists for all $p \in Q(T)$, since x_n converges strongly to p. Senter and Dotson [14] both mathematicians discovered notion of mapping which satisfying condition (I) as. A mapping $T : K \to K$ is called to satisfy condition (I), if $\exists$ an increasing function $f : [0, \infty) \to [0, \infty)$ in $f(0) = 0$ and $f(r') > 0$ for all $r' > 0$ such as $\|l - Tl\| \geq f(d(l, Q(T)))$ for all $l \in K$, and $d(l, Q(T)) = \inf_p \in Q(T) \|l - p\|$.

□

Theorem 5.4. *Suppose that K be a non-empty uniformly closed convex subset of a Banach space Z, and consider that $T : K \to K$ be a mapping which satisfying condition (C). For arbitrary chosen $y_0 \in K$, let the sequence $\{y_n\}$ be generated from* (1) *for all $n \geq 1$. Since $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[u, v]$ for some u, v with $0 < u \leq v < 1$ such that $Q(T) \neq \phi$. If T fulfil condition (I), so $\{y_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 5.1, we get $lim_{n\to\infty} \|y_n - p\|$ exist $\forall\ p \in Q(T)$ and $lim_{n\to\infty} d(y_n, Q(T))$ exists. Consider that $lim_{n\to\infty} \|y_n - Q\| = s'$ for some $s' \geq 0$. If $s' = 0$ then results follows. Suppose that $s' > 0$, from proposition (3.1) and condition (I),

$$f(d(y_n, Q(T))) \leq \|Ty_n - y_n\| \tag{32}$$

Since, $Q(T) \neq \phi$, from theorem 5.2 with (32) $\iff \lim_{n\to\infty} \|Ty_n - y_n\| = 0$

$$\lim_{n\to\infty} f(d(y_n, Q(T))) = 0 \tag{33}$$

Since f is an increasing function and by (33), we have $\lim_{n\to\infty} d(y_n, Q(T)) = 0$. Thus we get a subsequence $\{y_{nk}\}$ of $\{y_n\}$ and a sequence $\{y'_k\} \subset Q(T)$ such that

$$\left\|y_{nk} - y'_k\right\| < \frac{1}{2^k}$$

$\forall$, $k \in N$ than by applying (26), we have

$$\begin{aligned}
\left\|y_{nk+1} - y'_k\right\| &\leq \left\|y_{nk} - y'_k\right\| < \frac{1}{2^k} \\
&\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\
\frac{1}{2^{k-1}} &\to 0 \text{ as } k \to \infty
\end{aligned}$$

This shows that $\{y'_k\}$ is a Cauchy sequence with $Q(T)$ and so it converges to a point p. Since $Q(T)$ is closed, therefore $p \in Q(T)$ and then $\{y_{nk}\}$ converges strongly to p. Since $lim_{n\to\infty} \|y_n - p\|$ exists, we get $y_n \to p \in Q(T)$. □

6. Conclusion

In this article we discussed generalized results by using new iterative scheme to approximate fixed point of generalized contraction and Suzuki non-expansive mappings. Here we developed new strongly convergence results of generalized contraction mappings of closed convex Banach space and also produced some new data dependence results. In addition, we proved some weak and strong convergence results in sense of generalized Suzuki non expansive mapping by applying condition (C).

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