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Prime Bi-Ideals in Ordered Ternary Semigroups

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Abstract. In this work, we introduce the concepts of prime, semiprime, strongly prime irreducible, and strongly irreducible bi-ideals of ordered ternary semigroups. We characterize those ordered ternary semigroups for which every bi-ideal is strongly prime. We also prove that the space of all strongly prime bi-ideals forms a topological space.

1. Introduction

The concept of ternary algebraic structures was introduced by Lehmer [11]. He investigated certain triple systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was first introduced by Banach [12] who showed by an example that a ternary semigroup does not necessary reduce to an ordinary semigroup. Semigroups can be regarded as ternary semigroups, but ternary semigroups do not necessarily reduce to semigroups. The concepts of ideals and radicals of ternary semigroups were studied by Sioson [16]. Later, Dixit and Dewan [3,4] introduced and studied the concepts of quasi-ideals and bi-ideals of ternary semigroups. Ternary semigroups have been studied by many authors, for example, see [5–7,10,16,17].

The notion of ordered ternary semigroups was defined by Iampan [8]. He characterized the minimality and maximality of lateral ideals in ordered ternary semigroups. The same author [9] also introduced the concept of ideal extensions in ordered ternary semigroups. In 2012, Daddi and Pawar [2] defined the concepts of quasi-ideals and bi-ideals in ordered ternary semigroups and studied their properties. Changphas [1] characterized left and right simple ordered ternary semigroups. Sanborisoot and Changphas [13] introduced and studied the concepts of pure ideals in ordered ternary semigroups.

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The concepts of prime, semiprime, strongly prime, irreducible, strongly irreducible bi-ideals of semigroups were introduced by Shabir and Kanwal [14]. Later, M. Shabir and M. Bano [15] introduced the concepts of prime, semiprime, strongly prime, irreducible, strongly irreducible bi-ideals of ternary semigroups. The authors characterized different classes of ternary semigroups by the properties of their quasi-ideals and bi-ideals; moreover, the space of all strongly prime bi-ideals of ternary semigroups is a topological space.

In this work, we define the concepts of prime, semiprime, strongly prime irreducible, and strongly irreducible bi-ideals of ordered ternary semigroups. Characterizations of some classes of ordered ternary semigroups by the properties of their quasi-ideals and bi-ideals are presented. We prove that the space of all strongly prime bi-ideals is a topological space.

2. Preliminaries

A nonempty set T together with a ternary operation $[\ ]$ defined on T is called a *ternary semigroup* if $[\ ]$ satisfies the following associative law:

$$[[abc]de] = [a[bcd]e] = [ab[cde]]$$

for all $a, b, c, d, e \in T$.

Example 2.1. *Let $T = \{-i, 0, i, \}$. Then T is a ternary semigroup under multiplication over the complex numbers while T is not a semigroup under the same multiplication.*

For nonempty subsets A, B and C of a ternary semigroup T, let

$$[ABC] = \{[abc] \mid a \in A, b \in B, c \in C\}.$$

For $x \in T$, let $[xAB] = [\{x\}AB]$. The other cases are defined analogously. For simplicity, we write [abc] as abc and [ABC] as ABC.

A ternary semigroup T is called an *ordered ternary semigroup* if there exists a partially ordered relation $\leq$ where for each $a, b, c, d \in T$,

$$c \leq d \Rightarrow cab \leq dab, acb \leq adb, abc \leq abd.$$

Let T be an ordered ternary semigroup. For $A \subseteq T$, we let

$$(A] = \{x \in T \mid x \leq a \text{ for some } a \in A\}.$$

Let A, B and C be any subsets of T. Then we have (1) $A \subseteq (A]$; (2) $A \subseteq B$ implies $(A] \subseteq (B]$; (3) $((A]] = (A]$; (4) $(A](B](C] \subseteq (ABC]$; (5) $(A \cap B] \subseteq [A) \cap [B)$; and (6) $(A \cup B] = [A) \cup [B)$. A nonempty subset A of T is called a *ternary subsemigroup* if and only if $AAA \subseteq A$. An element x of T is called a *zero element* if $xab = axb = abx = x$ for all $a, b \in T$, and $x \leq a$ for all $a \in T$. If $x \in T$ is a zero element, it is denoted by 0. A nonempty subset A of T is called *left ideal* (resp. *right ideal, lateral ideal*)) of T if $TTA \subseteq A$ (resp. ATT, TAT) and $(A] \subseteq A$. If A is a left, a right and a lateral ideal of T then A is called an *ideal* of T. If A is a left and a right ideal of T, then A is called a *two-sided ideal* of T. A subset A of T is called *idempotent* of T if $A^3 = AAA = A$(cf. [2, 10, 13]).

A nonempty subset Q of an ordered ternary semigroup T is called a *quasi-ideal* [10] of T if

(1) $(TTQ] \cap (TQT] \cap (QTT] \subseteq Q$;
(2) $(TTQ] \cap (TTQTT] \cap (QTT] \subseteq Q$;
(3) $(Q] = Q$.

A nonempty subset B of an ordered ternary semigroup T is called a *bi-ideal* [10] of T if

(1) $BTBTB \subseteq B$;
(2) $(B] = B$.

Every quasi-ideal of an ordered ternary semigroup T is a bi-ideal of T. Moreover, every left, right, and lateral ideal of T is a quasi-ideal, but the converse does not hold in general. Every quasi-ideal of T is a ternary subsemigroup of T. The intersection of a left, a right and a lateral ideal of T is a quasi-ideal of T (see, [10]).

Lemma 2.1. *[2] If B_1, B_2, B_3 are bi-ideals of an ordered ternary semigroup T, then $(B_1B_2B_3]$ is a bi-ideal of T.*

The nonempty intersection of a family of quasi-ideals of T is a quasi-ideal of T. Similarly, the nonempty intersection of a family of bi-ideals of T is a bi-ideal of T.

Lemma 2.2. *Let T be an ordered ternary semigroup. If R is a right, M a middle, and L a left ideal of T then $L \cap M \cap R$ is a quasi-ideal of T.*

Proof. Let R be a right, M a middle, and L a left ideal of T. Then we have

$$\begin{aligned}(TT(L\cap M\cap R)] \cap (T(L\cap M\cap R)T] \cap ((L\cap M\cap R)TT] &\subseteq (TTL] \cap (TMT] \cap (RTT] \\ &\subseteq L\cap M\cap R.\end{aligned}$$

Also, $(TT(L\cap M\cap R)] \cap (TT(L\cap M\cap R)TT] \cap ((L\cap M\cap R)TT] \subseteq (TTL] \cap (TTMTT] \cap (TTL] \subseteq L\cap M\cap R$.

Finally, we have $(L\cap M\cap R] \subseteq (L] \cap (M] \cap (R] = L\cap M\cap R$. This shows that $L\cap M\cap R$ is a quasi-ideal of T. □

Let T be an ordered ternary semigroup. For a nonempty subset A of T, the intersection of all left ideals of T containing A, denoted by $(A)_l$, is the smallest left ideal of T containing A such that $(A)_l = (A \cup TTA]$. Similarly,

$$\begin{aligned}(A)_r &= (A \cup ATT] \\ (A)_m &= (A \cup TAT \cup TTATT]\end{aligned}$$

are the right ideal, the lateral ideal of T generated by A, respectively.

3. Regular Ordered Ternary Semigroups

Definition 3.1. *[2,13] An element a of T is called* regular *if there exists $x \in T$ such that $a \le axa$ and T is called* regular *if every element of T is regular.*

This means that T is regular if and only if $a \in (aTa]$ for all $a \in T$.

Theorem 3.1. *Let T be an ordered ternary semigroup. If every quasi-ideal Q of T is idempotent, then T is regular.*

Proof. Let $a \in T$. Since $(a)_l$ is a left ideal, $(a)_m$ is a middle ideal, and $(a)_r$ is a right ideal of T, by Lemma 2.2, $(a)_l \cap (a)_m \cap (a)_r$ is a quasi-ideal of T. By hypothesis,

$$(a)_l \cap (a)_m \cap (a)_r = ((a)_l \cap (a)_m \cap (a)_r)^3 \subseteq (a)_r(a)_m(a)_l.$$

Consider

$$\begin{aligned} (a)_l \cap (a)_m \cap (a)_r &\subseteq (a)_r(a)_m(a)_l \\ &\subseteq (a \cup aTT]T(a \cup TTa] \\ &\subseteq (aTa \cup aTTTa \cup aTTTa \cup aTTTTTa] \\ &\subseteq (aTa]. \end{aligned}$$

Clearly, $a \in (a)_l \cap (a)_m \cap (a)_r$, whence $a \in (aTa]$. □

Theorem 3.2. *Let T be an ordered ternary semigroup. The following assertions are equivalent:*

(1) *T is regular;*
(2) *$R \cap L = (RTL]$ for every left ideal L and every right ideal R of T;*
(3) *for any $a, b \in T$, $(a)_r \cap (b)_l = ((a)_r T(b)_l]$;*
(4) *for any $a \in T$, $(a)_r \cap (a)_l = ((a)_r T(a)_l]$.*

Proof. (1) ⇒ (2) Let L be a left and R a right ideal of T. To show that $R \cap L = (RTL]$. Let $a \in R \cap L$. Then, by (1), there exists $x \in T$ where $a \leq axa$. So that $a \in (RTL]$. For the converse, we have $(RTL] \subseteq (RTT] \subseteq (R] = R$ and $(RTL] \subseteq (TTL] \subseteq (L] = L$. This shows that $(RTL] \subseteq R \cap L$.

(2) ⇒ (3) and (3) ⇒ (4) are trivial.

(4) ⇒ (1) Let $a \in T$. By (4), we get

$$\begin{aligned} a \in (a)_r \cap (a)_l &= ((a)_r T(a)_l] \\ &\subseteq (a \cup aTT]T(a \cup TTa] \\ &\subseteq (aTa \cup aTTTa \cup aTTTa \cup aTTTTTa] \\ &\subseteq (aTa]. \end{aligned}$$

Thus T is regular. □

Theorem 3.3. *Let T be an ordered ternary semigroup. The following assertions are equivalent:*

(1) *T is regular;*
(2) *$B = (BTB]$ for every bi-ideal B of T;*
(3) *$Q = (QTQ]$ for every quasi-ideal Q of T.*

Proof. (1) ⇒ (2) Let B be a bi-ideal of T and $b \in B$. By (1), there exists $x \in T$ such that $b \leq bxb$. Since $bxb \in BTB$, it follows that $b \in (BTB]$. On the other hand, let $a \in (BTB]$. Then $a \leq b_1 t_1 b_2$

for some $b_1, b_2 \in B$ and $t_1 \in T$. Since T is regular, there exists $t_2 \in T$ such that $b_1 \leq b_1 t_2 b_1$. Thus $a \leq b_1 t_1 b_2 \leq (b_1 t_2 b_1) t_1 b_2 \in BTBTB \subseteq B$. Hence, $B = (BTB]$.

(2) $\Rightarrow$ (3) Since every quasi-ideal of T is a bi-ideal, by (2), we have $Q = (QTQ]$.

(3) $\Rightarrow$ (1) Let $a \in T$. Since $(a)_l$ is a left ideal, $(a)_m$ a lateral ideal and $(a)_r$ a right ideal of T. By Lemma 2.2, $(a)_r \cap (a)_m \cap (a)_l$ is a quasi-ideal of T. By (3),

$$\begin{aligned} a \in (a)_r \cap (a)_m \cap (a)_l &= (((a)_r \cap (a)_m \cap (a)_l) T ((a)_r \cap (a)_m \cap (a)_l)] \\ &\subseteq ((a)_r T (a)_l] \\ &= ((a \cup aTT] T (a \cup TTa]] \\ &\subseteq ((aTa \cup aTTTa \cup aTTTa \cup aTTTTTa]] \\ &= (aTa]. \end{aligned}$$

Thus T is regular. □

Theorem 3.4. *Let T be an ordered ternary semigroup. Then T is regular if every bi-ideal of T is idempotent.*

Proof. The proof is similar to that of Theorem 3.1. □

Theorem 3.5. *Let T be an ordered ternary semigroup. The following assertions are equivalent:*

(1) *T is regular;*
(2) *$M \cap B =(BMB]$ every lateral ideal M and for every bi-ideal B;*
(3) *$M \cap Q = (QMQ]$ for every lateral ideal M and every quasi-ideal Q.*

Proof. (1) $\Rightarrow$ (2) Let M be a lateral ideal and B a bi-ideal of T. Then, we have $(BMB] \subseteq (TMT] \subseteq (M] = M$. By Theorem 3.3, we have $B = (BTB]$. Hence

$$(BMB] = ((BTB]MB] \subseteq ((BTBTB]] \subseteq ((B]] = B.$$

Therefore, $(BMB] \subseteq M \cap B$. For the reverse, we let $a \in M \cap B$. By (1), there exists $x \in T$ such that $a \leq axa$. As $a \leq axa \leq (axa)xa = a(xax)a \in BMB \subseteq (BMB]$, we have $M \cap B \subseteq (BMB]$. Consequently, $M \cap B = (BMB]$.

(2) $\Rightarrow$ (3) Since every quasi-ideal of an ordered ternary semigroup T is also a bi-ideal, by assumption, $M \cap Q = (QMQ]$.

(3) $\Rightarrow$ (1) Assume Q is a quasi-ideal of T. We obtain that $Q = T \cap Q = (QTQ]$, by (3). Hence by Theorem 3.3, T is regular. □

Theorem 3.6. *Let T be an ordered ternary semigroup. The following assertions are equivalent:*

(1) *T is regular;*
(2) *$B \cap L \subseteq (BTL]$ for every bi-ideal B and every left ideal L of T;*
(3) *$Q \cap L \subseteq (QTL]$ for every quasi-ideal Q and every left ideal L of T;*
(4) *$B \cap R \subseteq (RTB]$ for every bi-ideal B and every right ideal R of T;*
(5) *$Q \cap R \subseteq (RTQ]$ for every quasi-ideal Q and every right ideal R of T.*

Proof. (1) ⇒ (2) Let B be a bi-ideal and L a left ideal of T. Let $a \in B \cap L$. Because T is regular, there exists $x \in T$ such that $a \leq axa$. But $a \leq axa \in BTL \subseteq (BTL]$. Hence (2) follows.

(2) ⇒ (3) Since every quasi-ideal of T is a bi-ideal, by (2), $Q \cap L \subseteq (QTL]$.

(3) ⇒ (1) Let R be a right ideal of T. From every right ideal of T is a quasi-ideal of T, so take $Q = R$, by (3), $R \cap L \subseteq (RTL]$. Because $(RTL] \subseteq R \cap L$, it follows that $R \cap L = (RTL]$. Then, by Theorem 3.2, T is regular.

Similarly we can show that (1) ⇒ (4) ⇒ (5) ⇒ (1). □

Theorem 3.7. *Let T be an ordered ternary semigroup. The following assertions are equivalent:*

(1) *T is regular;*
(2) *$B_1 \cap B_2 \subseteq (B_1TB_2] \cap (B_2TB_1]$ for each bi-ideals B_1 and B_2 of T;*
(3) *$B \cap Q \subseteq (BTQ] \cap (QTB]$ for each bi-ideal B and each quasi-ideal Q of T;*
(4) *$B \cap L \subseteq (BTL] \cap (LTB]$ for each bi-ideal B and each left ideal L of T;*
(5) *$Q \cap L \subseteq (QTL] \cap (LTQ]$ for each quasi-ideal Q and each left ideal L of T;*
(6) *$R \cap L \subseteq (RTL] \cap (LTR]$ for each left ideal L and each right ideal R of T;*
(7) *$B \cap R \subseteq (BTR] \cap (RTQ]$ for each bi-ideal B and each right ideal R of T;*
(8) *$Q \cap R \subseteq (QTR] \cap (RTQ]$ for each quasi-ideal Q and each right ideal R of T.*

Proof. (1) ⇒ (2) Assume that B_1 and B_2 are bi-ideals of T, and let $a \in B_1 \cap B_2$. By (1), there exists $x \in T$ such that $a \leq axa \in B_1TB_2 \subseteq (B_1TB_2]$ and $a \leq axa \in B_2TB_1 \subseteq (B_1TB_2]$; thus, $B_1 \cap B_2 \subseteq (B_1TB_2] \cap (B_2TB_1]$.

(2) ⇒ (3) Since every quasi-ideal of T is a bi-ideal, by (2), we have $B \cap Q \subseteq (BTQ] \cap (QTB]$.

(3) ⇒ (4) Since every one-sided ideal of T is a quasi-ideal, by (3), we have $B \cap L \subseteq (BTL] \cap (LTB]$ for every bi-ideal B and for every left ideal L of T.

(4) ⇒ (5) From for every quasi-ideal of T is a bi-ideal, by (4), it follows that $Q \cap L \subseteq (QTL] \cap (LTQ]$ for every quasi-ideal Q and for every left ideal L of T.

(5) ⇒ (6) This follows similarly to the proof of (3) ⇒ (4).

(6) ⇒ (1) Let L be a left and R a right ideal of T. Then by (6), $R \cap L \subseteq (RTL] \cap (LTR] \subseteq (RTL]$. From $(RTL] \subseteq (RTT] \subseteq (R] = R$ and $(RTL] \subseteq (TTL] \subseteq (L] = L$, whence $(RTL] \subseteq R \cap L$. We deduce that $(RTL] = R \cap L$. By Theorem 3.2; hence T is regular.

Similarly we can prove that (1) ⇒ (7) ⇒ (8) ⇒ (1). □

4. Prime, strongly prime, semiprime bi-ideals and irreducible bi-ideals of ordered ternary semigroups.

In this section, we introduce prime, strongly prime, semiprime bi-ideals and irreducible bi-ideals of an ordered ternary semigroup. Throughout this section, T will be considered as an ordered ternary semigroup with zero.

Definition 4.1. A bi-ideal B of an ordered ternary semigroup T is called *prime bi-ideal*, if $(B_1B_2B_3] \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for every bi-ideal B_1, B_2, B_3 of T.

Definition 4.2. A bi-ideal B of an ordered ternary semigroup T is called *strongly prime bi-ideal*, if $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideal B_1, B_2, B_3 of T.

Definition 4.3. Let T be an ordered ternary semigroup. A bi-ideal B of T is called a *semiprime bi-ideal* of T if, for each bi-ideal B_1 of T, $(B_1B_1B_1] \subseteq B$ implies $B_1 \subseteq B$.

Every strongly prime bi-ideal of an ordered ternary semigroup T is a prime bi-ideal of T and every prime bi-ideal of T is a semiprime bi-ideal of T. But a prime bi-ideal of T is need not be a strongly prime bi-ideal of T and a semiprime bi-ideal of T need not be a prime bi-ideal of T. The following example shows that a semiprime bi-ideal of T need not be a prime bi-ideal of T.

Example 4.1. *Let $\mathbb{Z}_0^-$ be the set of all non-positive integers and $T = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}_0^-\}$ Then $\mathbb{Z}_0^-$ is an ordered ternary semigroup under usual multiplication of numbers with partial ordered relation $\leq_{\mathbb{Z}_0^-}$(less than or equal to). Now, we define partial order relation $\leq_T$ on T by, for each $A, B \in T$*

$$A \leq_T B \Longleftrightarrow a_{ij} \leq_{\mathbb{Z}_0^-} b_{ij} \text{ for all } a_{ij} \in A \text{ and } b_{ij} \in B.$$

Then T is an ordered ternary semigroup under usual multiplication of matrices with partial order relation $\leq_T$. Let $A = \{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_0^-\}$. It is very easy to verify that A is a semiprime bi-ideal of T. But A is not a prime bi-ideal of T. Since $B_1 = \{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{Z}_0^-\}$, $B_2 = \{\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \mid c \in \mathbb{Z}_0^-\}$ and $B_3 = \{\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \mid d \in \mathbb{Z}_0^-\}$ are bi-ideals of T and $(B_1B_2B_3] \subseteq A$. But $B_1 \nsubseteq A$, $B_2 \nsubseteq A$ and $B_3 \nsubseteq A$.

Definition 4.4. Let T be an ordered ternary semigroup. A bi-ideal B of T is called *irreducible bi-ideal* of T if for any bi-ideals B_1, B_2, B_3 of T such that $B_1 \cap B_2 \cap B_3 = B$ implies $B_1 = B$ or $B_2 = B$ or $B_3 = B$.

Definition 4.5. Let T be an ordered ternary semigroup. A bi-ideal B of T is called *strongly irreducible bi-ideal* of T if for any bi-ideals B_1, B_2, B_3 of T such that $B_1 \cap B_2 \cap B_3 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$.

Every strongly irreducible bi-ideal of an ordered ternary semigroup T is an irreducible bi-ideal of T but the converse is not true in general.

Theorem 4.1. *Let T be an ordered ternary semigroup. Then every strongly prime bi-ideal of T is a prime bi-ideal.*

Proof. Suppose that B is a strongly prime bi-ideal of T. Let B_1, B_2, B_3 be three bi-ideals of T such that $(B_1B_2B_3] \subseteq B$. Then $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq B$. By assumption, we get $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus, B is a prime bi-ideal of T. □

Theorem 4.2. *Let $A_i : i \in I$ be a family of prime bi-ideals of an ordered ternary semigroup T. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is a semiprime bi-ideal of T.*

Proof. Let $\{A_i : i \in I\}$ be a family of prime bi-ideals of an ordered ternary semigroup T. Assume that $\bigcap_{i\in I} A_i \neq \emptyset$. Then $\bigcap_{i\in I} A_i$ is a bi-ideal of T. Let B be a bi-ideal of T such that $(BBB] \subseteq \bigcap_{i\in I} A_i$. Then $(BBB] \subseteq A_i$ for all $i \in I$. Consequently, $B \subseteq A_i$ for all $i \in I$. Hence, $B \subseteq \bigcap_{i\in I} A_i$. It follows that $\bigcap_{i\in I} A_i$ is a semiprime bi-ideal of T. □

Theorem 4.3. *Let T be an ordered ternary semigroup. Every strongly irreducible semiprime bi-ideal of T is a strong prime bi-ideal of T.*

Proof. Suppose that B is a strongly irreducible semiprime bi-ideal of T. Let B_1, B_2, B_3 be three bi-ideals of T such that $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq B$. From $B_1 \cap B_2 \cap B_3 \subseteq B_1$, $B_1 \cap B_2 \cap B_3 \subseteq B_2$ and $B_1 \cap B_2 \cap B_3 \subseteq B_3$, it follows that $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_1B_2B_3]$, $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_2B_3B_1]$ and $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_3B_1B_2]$. Hence, $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq B$. Since B is a semiprime bi-ideal, so $B_1 \cap B_2 \cap B_3 \subseteq B$. By assumption, $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$, that is, B is a strongly prime bi-ideal of T. □

Theorem 4.4. *Let A be a bi-ideal of an ordered ternary semigroup T, and let $a \in T \setminus A$. Then there exists an irreducible bi-ideal M such that $A \subseteq M$ and $a \in T \setminus M$.*

Proof. Suppose that $a \in T \setminus A$. Let

$$P = \{X_i : X_i \text{ is a bi-ideal of } T \text{ such that } A \subseteq X_i \text{ and } a \in T \setminus X_i\}.$$

It is clear that $P \neq \emptyset$ because $A \in P$. Since P is a partially ordered set by inclusion. Now, let $\{X_i : i \in I\}$ be any totally ordered subset of P. It is observed that $\bigcup_{i\in I} X_i$ is a bi-ideal of T containing A and $a \notin \bigcup_{i\in I} X_i$. By Zorn's Lemma, we get P has a maximal element, say M, that is, M is bi-ideal where $A \subseteq M$ and $a \notin M$. Next, we want to show that M is an irreducible bi-ideal of T. Suppose that B_1, B_2 and B_3 are three bi-ideals of T such that $B_1 \cap B_2 \cap B_3 = M$. If $M \subsetneq B_j$ for $j = 1, 2, 3$, then the maximality of M in P implies that $a \in B_j$ for each j. Hence, $a \in B_1 \cap B_2 \cap B_3 = M$, a contradiction. We conclude that $M = B_1$ or $M = B_2$ or $M = B_3$. □

Theorem 4.5. *Let T be a regular ordered ternary semigroup. The following are equivalent:*

(1) *every bi-ideal of T is idempotent;*
(2) $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] = B_1 \cap B_2 \cap B_3$ *for any bi-ideals B_1, B_2, B_3 of T;*
(3) *every bi-ideal of T is semiprime;*
(4) *every proper bi-ideal of T is the intersection of irreducible semiprime bi-ideals of T such that contain it.*

Proof. (1) ⇒ (2) Let B_1, B_2, B_3 be bi-ideals of T. Then $B_1 \cap B_2 \cap B_3$ is a bi-ideal of T. According to hypothesis, we have $(B_1 \cap B_2 \cap B_3)^3 = B_1 \cap B_2 \cap B_3$. As $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_1B_2B_3]$, $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_2B_3B_1]$ and $(B_1 \cap B_2 \cap B_3)^3 \subseteq (B_3B_1B_2]$. Hence,

$$B_1 \cap B_2 \cap B_3 \subseteq (B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2].$$

By Lemma 2.1 and the intersection property of bi-ideals, $(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1]$ is a bi-ideal of T. By assumption, we obtain

$$\begin{aligned}(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] &= ((B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1])^3 \\ &\subseteq (B_1B_2B_3](B_3B_1B_2](B_2B_3B_1] \\ &\subseteq (B_1TT](TB_1T](TTB_1] \\ &\subseteq (B_1(TTT)B_1(TTT)B_1] \\ &\subseteq (B_1TB_1TB_1] \\ &\subseteq B_1.\end{aligned}$$

Similarly, we have $(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] \subseteq B_2$ and $(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] \subseteq B_3$. Hence, $(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] \subseteq B_1 \cap B_2 \cap B_3$.

(2) $\Rightarrow$ (3) Let B be a bi-ideal of T, and let A be a bi-ideal of T such that $(A^3] \subseteq B$. Then, by (2),

$$B = B \cap B \cap B = (B^3] \cap (B^3] \cap (B^3] \subseteq A.$$

Thus, A is a semiprime of T.

(3) $\Rightarrow$ (4) Assume that A is a proper bi-ideal of T. Let $\bigcap_{i\in I} I_i$ be the intersection of all irreducible semiprime bi-ideals of T containing A. We need to show that $A = \bigcap_{i\in I} J_i$. By assumption, it is clear that $A \subseteq \bigcap_{i\in I} J_i$. Now, let $a \in \bigcap_{i\in I} J_i$. If $a \notin A$ then by Theorem 4.4, there exists an irreducible bi-ideal J of T such that $A \subseteq J$ and $a \notin J$. This means that $\bigcap_{i\in I} J_i \subseteq J$; hence, $a \in J$ which is a contradiction, so $a \in A$. Thus, each bi-ideal of T is the intersection of irreducible semiprime bi-ideals of T which contain it.

(4) $\Rightarrow$ (1) Let A be a proper bi-ideal of T. By Lemma 2.1, we get $(A^3]$ is a bi-ideal of T. If $(A^3] = T$ then $A \subseteq T = (A^3]$. But $(A^3] \subseteq A$, so $(A^3] = A$. If $(A^3] \subset T$ then $(A^3] = \bigcap_{i\in I} J_i$ such that J_i is an irreducible semiprime bi-ideal of T where $(A^3] \subseteq J_i$ for all $i \in I$. Then $A \subseteq J_i$ for all $i \in I$. We have that $A \subseteq \bigcap_{i\in I} J_i = (A^3]$. But $(A^3] \subseteq A$, it follows that $(A^3] = A$; therefore, A is idempotent of T. □

Theorem 4.6. *Let T be a regular ordered ternary semigroup. If every bi-ideal of T is idempotent, then a bi-ideal A of T is strongly irreducible if and only if A is strongly prime.*

Proof. Assume that A is a strongly irreducible bi-ideal of T. Let B_1, B_2, B_3 be bi-ideals of T such that $(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] \subseteq A$. Using Theorem 4.5,

$$(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] = B_1 \cap B_2 \cap B_3.$$

Therefore, $B_1 \cap B_2 \cap B_3 \subseteq A$. Because A is strongly irreducible, it follows that $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$, that is, A is strongly prime.

Conversely, suppose that A is a strongly prime bi-ideal of T. To show that A is a strongly irreducible. Now, let B_1, B_2, B_3 be three bi-ideals of T such that $B_1 \cap B_2 \cap B_3 \subseteq A$. From Theorem 4.5, we have

$$(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] = B_1 \cap B_2 \cap B_3.$$

Thus, $(B_1B_2B_3] \cap (B_3B_1B_2] \cap (B_2B_3B_1] \subseteq A$. By assumption, $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$. □

Next, we characterize an ordered ternary semigroup T in which every bi-ideal of T is strongly irreducible and strongly prime.

Theorem 4.7. *Let T be a regular ordered ternary semigroup. Then every bi-ideal of T is strongly prime if and only if every bi-ideal of T is idempotent and the set of bi-ideals of T is totally ordered under inclusion.*

Proof. Assume that every bi-ideal of T is strongly prime. Then every bi-ideal of T is a semiprime. By Theorem 4.5, we obtain that every bi-ideal of T is idempotent. Now, let B_1, B_2 be bi-ideals of T. According to Theorem 4.5,

$$B_1 \cap B_2 = B_1 \cap B_2 \cap T = (B_1B_2T] \cap (B_2TB_1] \cap (TB_1B_2].$$

Hence, $(B_1B_2T] \cap (B_2TB_1] \cap (TB_1B_2] \subseteq B_1 \cap B_2$. Since $B_1 \cap B_2$ is a bi-ideal of T, by assumption, $B_1 \cap B_2$ is strongly prime. Therefore,

$$B_1 \subseteq B_1 \cap B_2, \quad B_2 \subseteq B_1 \cap B_2, \quad \text{or} \quad T \subseteq B_1 \cap B_2.$$

If $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$. If $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$. In the remaining case, $T \subseteq B_1 \cap B_2$, and hence $B_1 = B_2 = T$. Thus, the set of all bi-ideals of T is totally ordered under inclusion.

Conversely, suppose that every bi-ideal of T is idempotent and the set of bi-ideals of T is totally ordered under inclusion. Let A, B_1, B_2, B_3 be bi-ideals of T such that $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq A$. Since $B_1 \cap B_2 \cap B_3$ is a bi-ideal, according to Theorem 4.5, we have

$$B_1 \cap B_2 \cap B_3 = (B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2].$$

By assumption, we have the following six possibilities:

(1) if $B_1 \subseteq B_2 \subseteq B_3$, then $B_1 \cap B_2 \cap B_3 = B_1$;
(2) if $B_1 \subseteq B_3 \subseteq B_2$, then $B_1 \cap B_2 \cap B_3 = B_1$;
(3) if $B_2 \subseteq B_1 \subseteq B_3$, then $B_1 \cap B_2 \cap B_3 = B_2$;
(4) if $B_2 \subseteq B_3 \subseteq B_1$, then $B_1 \cap B_2 \cap B_3 = B_2$;
(5) if $B_3 \subseteq B_1 \subseteq B_2$, then $B_1 \cap B_2 \cap B_3 = B_3$; and
(6) if $B_3 \subseteq B_2 \subseteq B_1$, then $B_1 \cap B_2 \cap B_3 = B_3$.

Since $B_1 \cap B_2 \cap B_3 \subseteq A$, we have $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$. This proves that A is a strongly prime; hence, the proof is completed. □

Theorem 4.8. *Let T be a regular ordered ternary semigroup. If the set of bi-ideals of T is totally ordered, then every bi-ideal of T is idempotent if and only if every bi-ideal of T is prime.*

Proof. Assume first that every bi-ideal of T is idempotent. Let A, B_1, B_2, B_3 be four bi-ideals of T such that $(B_1B_2B_3] \subseteq A$. From the set of bi-ideals of T is totally ordered, we have the following six possibilities: (1) $B_1 \subseteq B_2 \subseteq B_3$ (2) $B_1 \subseteq B_3 \subseteq B_2$ (3) $B_2 \subseteq B_1 \subseteq B_3$ (4) $B_2 \subseteq B_3 \subseteq B_1$ (5) $B_3 \subseteq B_1 \subseteq B_2$ (6) $B_3 \subseteq B_2 \subseteq B_1$. In cases (1) and (2), we obtain that

$$B_1 = B_1B_1B_1 \subseteq (B_1B_1B_1] \subseteq (B_1B_2B_3] \subseteq A.$$

Consequently, $B_1 \subseteq A$. Similarly, cases (3) and (4) yield $B_2 \subseteq A$, while cases (5) and (6) yield $B_3 \subseteq A$. Thus, A is prime.

Conversely, suppose that every bi-ideal of T is prime. Then every bi-ideal is semiprime. Hence, we have every bi-ideal is idempotent by Theorem 4.5. □

Theorem 4.9. *Let T be a regular ordered ternary semigroup. If the set of bi-ideals of T is totally ordered, then each prime bi-ideal of T is strongly prime.*

Proof. Let A, B_1, B_2, B_3 be bi-ideals of T such that $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq A$. Since the set of bi-ideals of T is totally ordered, the following six possibilities arise: (1) $B_1 \subseteq B_2 \subseteq B_3$ (2) $B_1 \subseteq B_3 \subseteq B_2$ (3) $B_2 \subseteq B_1 \subseteq B_3$ (4) $B_2 \subseteq B_3 \subseteq B_1$ (5) $B_3 \subseteq B_1 \subseteq B_2$ (6) $B_3 \subseteq B_2 \subseteq B_1$. In cases (1) and (2), we have

$$B_1^3 = B_1^3 \cap B_1^3 \cap B_1^3 \subseteq (B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq A.$$

Since A is prime and $(B_1^3] \subseteq A$, it follows that $B_1 \subseteq A$. Similarly, cases (3) and (4) yield $B_2 \subseteq A$, while cases (5) and (6) yield $B_3 \subseteq A$. Accordingly, A is strongly prime. □

Theorem 4.10. *Let T be a regular ordered ternary semigroup and the set of bi-ideals of T be totally ordered. Then each strongly prime bi-ideal of T is prime.*

Proof. Let A, B_1, B_2, B_3 be bi-ideals of T such that $(B_1B_2B_3] \subseteq A$. Then, we have $(B_1B_2B_3] \cap (B_2B_3B_1] \cap (B_3B_1B_2] \subseteq A$. According to Theorem 4.9, A is strongly prime. Whence $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$. Hence, every strongly prime bi-ideal of T is prime bi-ideal of T. □

Theorem 4.11. *Let T be an ordered ternary semigroup. The following are equivalent:*

(1) *the set of bi-ideals of T is totally ordered by set inclusion;*
(2) *every bi-ideal of T is strongly irreducible;*
(3) *every bi-ideal of T is irreducible.*

Proof. (1) $\Rightarrow$ (2) Assume that the set of bi-ideals of T is totally ordered by set inclusion. Let A, B_1, B_2, B_3 be bi-ideals of T such that $B_1 \cap B_2 \cap B_3 \subseteq A$. By assumption, we have $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$. Therefore, A is strongly irreducible.

(2) $\Rightarrow$ (3) Assume that every bi-ideal of T is strongly irreducible. Let A, B_1, B_2, B_3 be bi-ideals of T such that $B_1 \cap B_2 \cap B_3 = A$. Since A is strongly irreducible, we have $B_1 \subseteq A$ or $B_2 \subseteq A$ or $B_3 \subseteq A$. On the other hand, since $A = B_1 \cap B_2 \cap B_3$, we have $A \subseteq B_i$ for each $i = 1, 2, 3$. Therefore, $A = B_1$ or $A = B_2$ or $A = B_3$. Hence, A is irreducible.

(3) $\Rightarrow$ (1) Assume that every bi-ideal of T is irreducible. To show that the set of bi-ideals of T is totally ordered by set inclusion. Now, let B_1 and B_2 be any two bi-ideals of T. Then $B_1 \cap B_2$ is a bi-ideal of T and so is irreducible. Because $B_1 \cap B_2 \cap T = B_1 \cap B_2$, by assumption, $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$ or $T = B_1 \cap B_2$. Whence $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2 = T$.

□

Let T be an ordered ternary semigroup in which every bi-ideal is idempotent. The set of all bi-ideals of T and the set of all strongly prime proper bi-ideals of T will be denoted by $\mathcal{B}(T)$ and $\mathcal{S}(T)$, respectively. For each $A \in \mathcal{B}(T)$, let

$$\mathcal{J}_A = \{J \in \mathcal{S}(T) \mid A \nsubseteq J\} \text{ and } \tau(T) = \{\mathcal{J}_A \mid A \in \mathcal{B}(T)\}.$$

Theorem 4.12. *$\tau(T)$ forms a topology on $\mathcal{S}(T)$.*

Proof. Since $\{0\}$ is a bi-ideal of T and $\mathcal{J}_{\{0\}} = \emptyset$, $\emptyset \in \tau(T)$. Because T is a bi-ideal of T, we have $\mathcal{J}_T = \mathcal{S}(T)$, and so $\mathcal{S}(T) \in \tau(T)$. Now, let $\{\mathcal{J}_{A_i} : i \in I\} \subseteq \tau(T)$. Then

$$\bigcup_{i\in I} \mathcal{J}_{A_i} = \bigcup_{i\in I}\{J \in \mathcal{S}(T) \mid A_i \nsubseteq J\} = \{J \in \mathcal{S}(T) \mid \bigcup_{i\in I} A_i \nsubseteq J\} = \mathcal{J}_{\bigcup_{i\in I} A_i}$$

Since $\bigcup_{i\in I} A_i \in \mathcal{B}(T)$, it follows that $\mathcal{J}_{(\bigcup_{i\in I} A_i)} \in \tau(T)$. Finally, we show that $\tau(T)$ is closed under finite intersections. Let $\mathcal{J}_{A_1}, \mathcal{J}_{A_2} \in \tau(T)$. We claim that $\mathcal{J}_{A_1} \cap \mathcal{J}_{A_2} = \mathcal{J}_{A_1\cap A_2}$. Let $J \in \mathcal{J}_{A_1\cap A_2}$. Then $J \in \mathcal{S}(T)$ and $A_1 \cap A_2 \nsubseteq J$, that is, $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$. Assume that $A_1 \cap A_2 \subseteq J$. According to Theorem 4.5, we have $[A_1A_2T) \cap [A_2TA_1) \cap [TA_1A_2) = A_1 \cap A_2 \cap T = A_1 \cap A_2 \subseteq J$. Since J is a strongly prime bi-ideal of T, it follows that $A_1 \subseteq J$ or $A_2 \subseteq J$. This is a contradiction; as a result, $A_1 \cap A_2 \nsubseteq J$. That is $J \in \mathcal{J}_{A_1\cap A_2}$. For the reverse inclusion, let $J \in \mathcal{J}_{A_1\cap A_2}$. Then $A_1 \cap A_2 \nsubseteq J$. Hence, there exists $x \in A_1 \cap A_2$ such that $x \notin J$. Thus, $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$, and consequently $J \in \mathcal{J}_{A_1} \cap \mathcal{J}_{A_2}$. So $J \in \mathcal{J}_{A_1}$ and $J \in \mathcal{J}_{A_2}$. We obtain that $J \in \mathcal{J}_{A_1} \cap \mathcal{J}_{A_2}$. Hence, $\tau(T)$ is a topology on $\mathcal{S}(T)$. □

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