Int. J. Anal. Appl. (2022), 20:63

International Journal of Analysis and Applications

Numerical Simulation of Singularly Perturbed Delay Differential Equations With Large Delay Using an Exponential Spline

Ramavath Omkar, K. Phaneendra*

Department of Mathematics, University College of Science, Osmania University, India

**Corresponding author:* kollojuphaneendra@yahoo.co.in

Abstract. In this study, numerical solution of a differential-difference equation with a boundary layer at one end of the domain is suggested using an exponential spline. The numerical scheme is developed using an exponential spline with a special type of mesh. A fitting parameter is inserted in the scheme to improve the accuracy and to control the oscillations in the solution due to large delay. Convergence of the method is examined. The error profiles are represented by tabulating the maximum absolute errors in the solution. Graphs are being used to show that how the fitting parameter influence the layer structure.

1. Introduction

Differential-difference equations (DDEs) are ones in which a state variable's time evolution is inconsistently dependent on a particular history, i.e., a physical system's rate of change is dependent not only on its current condition but also on its prior history. These equations have been widely utilized in population dynamics [7], nonlinear delay differential equations relating to physiological control systems [14], red blood cell system [13], predator-prey models [15], neuronal variability problems connected to patterns of nerve action potentials formed by unit quantal inputs occurring at random are studied [22].

Bellman and Cooke [1], Doolan et al. [2] and Driver [3] just are a few of the authors who have produced papers and books in recent years explaining various methods for solving differential-difference equations with perturbation. In [4] authors solved singularly perturbed delay differential equation (SPDE) using a fitted scheme on an uniform mesh. In [5], Glizer solved linear quadratic optimal control problem with

Received: Oct. 15, 2022.

2010 *Mathematics Subject Classification.* 65L11, 65L12.

Key words and phrases. exponential spline; differential-difference equation; large delay; truncation error.

https://doi.org/10.28924/2291-8639-20-2022-63

ISSN: 2291-8639

delay by Hamiltonian boundary-value problem, with boundary function method. In [6], Kadalbajoo et. al. proposed numerical research utilizing finite difference techniques. The authors in [8] developed a technique for solving SPDEs with twin layers or oscillatory behaviour using a computational scheme. The authors in [9] used a nonpolynomial spline to develop a numerical solution for a DDEs having layer with a small and large delay in the differentiated terms. Using domain decomposition, the authors of [10] suggested a mixed difference technique to solve DDEs with mixed shifts. For a class of linear second-order DDE type, Lange and Miura [11, 12] developed an asymptotic method. They developed a mathematical model by random synaptic inputs in dendrites for estimating the approximate time for the activation of action potentials in nerve cells and also discussed issues with solutions that had rapid oscillations in their study. The authors of [16] utilized a quadrature method for solving the SPDE, as well as a two-point quadrature rule to obtain a tridiagonal system. The authors in [17] developed a finite difference approach for solving SPDEs with turning points and mixed shifts. A finite difference scheme on Shishkin mesh is proposed to solve singularly perturbed delay differential equations with or without a turning point in [18]. The authors of [19, 20] used tension splines and an exponentially fitted spline for the problem with convection delay-dominated diffusion equation. The authors of [23] solved a two parameter semi linear differential equation by using an exponential spline. The authors of [21] solved SPDE with a numerical integration technique.

2. Statement of the problem

Consider the continuous problem of a second order SPDE

$$\varepsilon z'' + p(u)z'(u-\delta) + q(u)z(u) = r(u) \tag{2.1}$$

with boundary conditions

$$z(u) = \varphi(u) \quad -\delta \leq u \leq 0 \ \ , \ z(1) = \tau \tag{2.2}$$

where $0 < \varepsilon << 1$ is a perturbation, $p(u), q(u), r(u)$ and $\varphi(u)$ are bounded continuous functions on $(0,1)$, and τ is finite constant and δ is delay parameter. It is well known that when $\delta = 0$ the above delay differential equation is reducing to a SPDE. The solution $z(u)$ exhibit boundary layer on left side when $p(u)$ is positive or on right side when $p(u)$ is negative throughout the interval $[0,1]$. If $\delta(\varepsilon)$ is of order $O(\varepsilon)$, layer profile of the solution is no longer retained and the solution oscillates.

3. An exponential spline

The region $[\upsilon, \Upsilon]$ is subdivided into l equal subregions of mesh size $h = \frac{(\upsilon-\Upsilon)}{l}$ so that $u_i = \upsilon + ih, i = 0, 1, \ldots, l$ are the mesh points. To manage the delay term, the mesh size is chosen as $h = \frac{\delta}{\nu}$, where ν= mantissa of δ which is a positive integer.

Let the exact solution be $z(u)$ to Eq. (2.1) and u_i be an approximation solution to the $z(u_i)$ attained by the segment $Q_i(u)$ passing (u_i, Q_i) and (u_{i+1}, Q_{i+1}). Each exponential spline segment $Q_i(u)$ has the form.

$$Q_i(u) = a_i e^k(u-u_i) + b_i e^{-k}(u-u_i) + c_i(u-u_i) + d_i \quad for \quad i = 0, 1, 2, \ldots, l. \tag{3.1}$$

where a_i, b_i, c_i and d_i are the constants and k is a free parameter.

To find the values of the coefficients in Eq. (3.1), the interpolatory conditions $Q_i(u_i) = S_i, Q_i(u_{i+1}) = S_{i+1}, Q_i''(u_i) = M_i, Q_i''(u_{i+1}) = M_{i+1}$ are used.

Using these conditions, we get the following expressions

$a_i = \frac{h^2\left(M_{i+1}-e^{-\theta}M_i\right)}{2\theta^2 sinh(\theta)}$, $b_i = \frac{h^2\left(M_i-e^{-\theta}M_{i+1}\right)}{2\theta^2 sinh(\theta)}$, $c_i = \frac{(S_{i+1}-S_i)}{h} - \frac{h(M_{i+1}-M_i)}{\theta^2}$ and $d_i = S_i - \frac{h^2 M_i}{\theta^2}$

where $\theta = kh$ and $i = 0, 1, 2, \ldots, l$.

Using the first derivative continuity i.e., $Q_{i-1}'(u) = Q_i{}'(u)$ at the point (u_i, Q_i), we get the relation

$$(z_{i+1} - 2z_i + z_{i-1}) = h^2\left(\alpha M_{i+1} + \beta M_i + \alpha M_{i-1}\right) \quad for \ i = 0, 1, \ldots, l-1 \tag{3.2}$$

where $\alpha = \left(\frac{sinh(\theta)-\theta}{\theta^2 sinh(\theta)}\right)$ and $\beta = \frac{2\theta cosh(\theta) - 2sinh(\theta)}{\theta^2 sinh(\theta)}$.

4. Numerical Approach

Eq. (2.1) can be discretized at the mesh point u_j by

$$\varepsilon M_j = r\left(u_j\right) - p\left(u_j\right) z'_{j-\nu} - q\left(u_j\right) z_j \quad for \quad j = i-1, \ i, \ i+1 \tag{4.1}$$

Using the following difference approximations of z':

$$z_{i-\nu}' \approx \left[\frac{1 + 2\omega h^2 q_{i+1} + \omega h\left[3\ p_{i+1} + \ p_{i-1}\right]}{2h}\right] z_{i-\nu+1} - 2\omega\left[p_{i+1} + p_{i-1}\right] z_{i-\nu}$$
$$-\left[\frac{1 + 2\ \omega h^2\ q_{i-1} - \omega h\left[p_{i+1} + 3\ p_{i-1}\right]}{2h}\right] z_{i-\nu-1} + \omega h\left[r_{i+1} - r_{i-1}\right]$$
$$z_{i-\nu+1}' \approx \frac{3z_{i-\nu+1} - 4z_{i-\nu} + z_{i-\nu-1}}{2h},$$

$$z_{i-\nu-1}' \approx \frac{-z_{i-\nu+1} + 4z_{i-\nu} - 3z_{i-\nu-1}}{2h}$$

Now substituting the above approximations in Eq. (4.1) and in Eq. (3.2) we obtained.

$$\left(\frac{\varepsilon}{h^2} + \alpha q_{i+1}\right) z_{i+1} + \left(\frac{-2\varepsilon}{h^2} + \beta q_i\right) z_i + \left(\frac{\varepsilon}{h^2} + \alpha q_{i-1}\right) z_{i-1} +$$
$$\left(\frac{-\alpha p_{i-1}}{2h} + \frac{\beta p_i}{2h}\left(1 + 2\omega h^2 q_{i+1} + \omega h\left(3p_{i+1} + p_{i-1}\right)\right) + \frac{3\alpha}{2h} p_{i+1}\right) z_{i-\nu+1} +$$
$$\left(\frac{2\alpha p_{i-1}}{h} - 2\beta\omega p_i\left(p_{i+1} + p_{i-1}\right) - \frac{2\alpha}{h} p_{i+1}\right) z_{i-\nu} +$$
$$\left(\frac{-3\alpha p_{i-1}}{2h} - \frac{\beta p_i}{2h}\left(1 + 2\omega h^2 q_{i-1} - \omega h\left(p_{i+1} + 3p_{i-1}\right)\right) + \frac{\alpha}{2h} p_{i+1}\right) z_{i-\nu-1}$$

$$= \left(\alpha r_{i+1} + \beta r_i + \alpha r_{i-1}\right) - \beta\omega p_i h\left(r_{i+1} - r_{i-1}\right) \tag{4.2}$$

When the shift parameter is small in comparison to the perturbation parameter, the layer behavior of the solution is maintained and reliable results are achieved. The layer behavior, however, no longer holds good if $\delta(\varepsilon)$ is of order $O(\varepsilon)$ and oscillations manifest. We are therefore constructing a numerical approach with fitting parameter based on an exponential spline method and a particular mesh type. In order to demonstrate how significant the fitting parameter is in our suggested approach, we also examine how the solution layer behaves when there are large delays. Now inserting the fitting parameter σ in the above scheme, we have

$$\begin{gathered}(\sigma\epsilon + h^2\alpha q_{i+1})z_{i+1} + (-2\sigma\epsilon + h^2\beta q_i)z_i + (\sigma\epsilon + h^2\sigma q_{i-1})z_{i-1} + \\ (\frac{-\alpha h p_{i-1}}{2} + \frac{\beta h p_i}{2}(1 + 2\omega h^2 q_{i+1} + \omega h(3p_{i+1} + p_{i-1})) + \frac{3\alpha h}{2}p_{i+1})z_{i-\nu+1} + \\ (2\alpha h p_{i-1} - 2\beta\omega p_i h^2(p_{i+1} + p_{i-1}) - 2\alpha h p_{i+1})z_{i-\nu} + \\ (\frac{-3\alpha h p_{i-1}}{2} - \frac{\beta h p_i}{2}(1 + 2\omega h^2 q_{i-1} - \omega h(p_{i+1} + 3p_{i-1})) + \frac{\alpha h}{2}p_{i+1})z_{i-\nu-1} \\ \\ = h^2(\alpha r_{i+1} + \beta r_i + \alpha r_{i-1}) - \beta\omega p_i h^3(r_{i+1} - r_{i-1}) \end{gathered} \tag{4.3}$$

Re write the above equation

$$A_{i+1}z_{i+1} + B_i z_i + C_{i-1}z_{i-1} + D_{i-\nu+1}z_{i-\nu+1} + E_{i-\nu}z_{i-\nu} + F_{i-\nu-1}z_{i-\nu-1} = G_i \tag{4.4}$$

where

$$\begin{gathered} A_{i+1} = \sigma\epsilon + \alpha h^2 q_{i+1},\ B_i = -2\sigma\epsilon + \beta h^2 q_i,\ C_{i-1} = \sigma\epsilon + \alpha h^2 q_{i-1} \\ D_{i-\nu+1} = \frac{-\alpha h p_{i-1}}{2} + \frac{\beta h p_i}{2}(1 + 2\omega h^2 q_{i+1} + \omega h(3p_{i+1} + p_{i-1})) + \frac{3\alpha h}{2}p_{i+1} \\ E_{i-\nu} = 2\alpha h p_{i-1} - 2\beta\omega p_i h^2(p_{i+1} + p_{i-1}) - 2\alpha h p_{i+1} \\ F_{i-\nu-1} = \frac{-3\alpha h p_{i-1}}{2} - \frac{\beta p_i h}{2}(1 + 2\omega h^2 q_{i-1} - \omega h(p_{i+1} + 3p_{i-1})) + \frac{\alpha h}{2}p_{i+1} \\ G_i = h^2(\alpha r_{i+1} + \beta r_i + \alpha r_{i-1}) - \beta\omega p_i h^3(r_{i+1} - r_{i-1}) \\ p(u_i) = p_i, q(u_i) = q_i, r(u_i) = r_i \end{gathered}$$

The above scheme can be written by using the boundary conditions

$$\begin{gathered} A_{i+1}z_{i+1} + B_i z_i + C_{i-1}z_{i-1} = G_i - D_{i-\nu+1}z_{i-\nu+1} - E_{i-\nu}z_{i-\nu} - F_{i-\nu+1}z_{i-\nu+1}, \forall\ 1 \le i \le \nu - 1 \\ A_{i+1}z_{i+1} + B_i z_i + C_{i-1}z_{i-1} + D_{i-\nu+1}z_{i-\nu+1} = G_i - E_{i-\nu}z_{i-\nu} - F_{i-\nu-1}z_{i-\nu-1}, \quad \forall\ i = \nu \\ A_{i+1}z_{i+1} + B_i z_i + C_{i-1}z_{i-1} + D_{i-\nu+1}z_{i-\nu+1} + E_{i-\nu}z_{i-\nu} = G_i - F_{i-\nu-1}z_{i-\nu-1}, \quad \forall\ i = \nu + 1 \end{gathered}$$

$$A_{i+1}z_{i+1} + B_i z_i + C_{i-1}z_{i-1} + D_{i-\nu+1}z_{i-\nu+1} + E_{i-\nu}z_{i-\nu} + F_{i-\nu-1}z_{i-\nu-1} = G_i\ \forall\ \nu + 2 \le i \le I - 1. \tag{4.5}$$

By using Gauss elimination method with partial pivoting or any other method resolves the above system of I equations.

4.1. **Left-end layer.** Assume $p(u) \geq \mathcal{M} > 0$ and $q(u) \leq -\subseteq < 0$ where θ and $\mathcal{M}$ are positive constants. Then the problem (2.1) - (2.2) shows layer structure at $u = 0$ for small values of perturbation. Now, to enhance the scheme's effectiveness, the fitting parameter inserted in Eq. (4.4) is determined by using the following approximate solution given in [2].

$$lim_{h\to 0} z(ih) \approx z_0(ih) + (\phi(0) - z_0(0))exp(-p(0)i\rho) + O(\varepsilon),$$

where $\rho = \frac{h}{\epsilon}$. Using this expression in Eq. (4.4) and following the procedure given in [2], we get

$$\sigma = \rho(\alpha + \beta/2)p(0)e^{p(0)\nu\rho} coth(\frac{p(0)\rho}{2})$$

Lemma 4.1. *Suppose* $\Psi_0 \geq 0$ *and* $\Psi_l \geq 0$*, then* $L^l\Psi_i \leq 0$ $\forall i = 1, 2, 3, \ldots.., l-1$ *implies that* $\Psi_i \geq 0$ $\forall i = 0, 1, 2, 3. \ldots .l$.

Proof. Let $j \in (0, 1, \ldots .l)$ such that $\Psi_j = max_{0\leq i\leq l}\Psi_i$. Let if possible $\Psi_j < 0$. We will show that it leads to contradiction. Clearly $j \notin (0, l)$. Now we have

$$L^l\Psi_j = \begin{cases} \varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i = G_i - D_{i-\nu+1}\Psi_{i-\nu+1} - E_{i-\nu}\Psi_{i-\nu} - F_{i-\nu-1}\Psi_{i-\nu-1}, \\ \forall i = 1, 2, \ldots, \nu. \\ \varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i + D_{i-\nu+1}\Psi_{i-\nu+1} = G_i - E_{i-\nu}\Psi_{i-\nu} - F_{i-\nu-1}\Psi_{i-\nu-1}, \\ \forall\ i = \nu+1, \ldots., l_1 - 1. \\ \varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i + D_{i-\nu+1}\Psi_{i-\nu+1} + E_{i-\nu}\Psi_{i-\nu} = G_i - F_{i-\nu-1}\Psi_{i-\nu-1}, \\ \forall i = l_1, \ldots.., l-1. \end{cases} \tag{4.6}$$

Case 1: For $i = 1, 2, \ldots, \nu$

$$L^l\Psi_j = (\varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i = G_i - D_{i-\nu+1}\Psi_{i-\nu+1} - E_{i-\nu}\Psi_{i-\nu} - F_{i-\nu-1}\Psi_{i-\nu-1},$$

$$\forall i = 1, 2, \ldots, \nu.) \geq 0(\because q_i) < 0, i = 1, 2, \ldots .\nu$$

Case 2: For $i = \nu + 1, \ldots., l_1 - 1$

$$L^l\Psi_j = (\varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i + D_{i-\nu+1}\Psi_{i-\nu+1} = G_i - E_{i-\nu}\Psi_{i-\nu} - F_{i-\nu-1}\Psi_{i-\nu-1},$$

$$\forall\ i = \nu+1, \ldots., l_1 - 1.) \geq 0$$

Case 3: For $i = l_1, \ldots.., l-1$

$$L^l\Psi_j = (\varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i + D_{i-\nu+1}\Psi_{i-\nu+1} + E_{i-\nu}\Psi_{i-\nu} = G_i - F_{i-\nu-1}\Psi_{i-\nu-1},$$

$$\forall i = l_1, \ldots.., l-1.) \quad \geq 0$$

As a result, we have $L^l\Psi_j > 0$, which is contradicts the hypothesis that $L^l\Psi_j \leq 0,\ 1 \leq j \leq l-1$. As a consequence, we assume that $\Psi_j < 0$ is incorrect and that $\Psi_j > 0$ is correct. Since j was chosen an arbitrary, we have

$$\Psi_j \geq 0, \forall j = 0, 1, 2, \ldots, l \tag{4.7}$$

□

Theorem 4.1. *Let $p(u) \geq \mathcal{M} > 0$ and $q(u) \leq -\theta < 0$ where $\mathcal{M}$ and θ are both positive constants. The problem (2.1) - (2.2) with boundary conditions has a uniqueness, existence and satisfying solution.*

$$||Z||_{h,\infty} \leq ||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + \tau) \tag{4.8}$$

where $C_2 \geq 1$ is a positive constant.

Proof. To demonstrate the uniqueness and existence, assume $<v>_{i=0}^{l}$ and $<w>_{i=0}^{l}$ be the two solutions to the discrete problem (2.1) - (2.2). Then $t_i = v_i - w_i$ is a mesh function that meets the conditions $t_0 = 0 = t_l$ and for $1 \leq i \leq l-1$ we have $L^l t_i = L^l v_i - L^l w_i$. Since v_i and w_i satisfy Eq. (4.5), therefore $L^l t_i = 0, 1 \leq i \leq l-1$. As a result, the mesh function t_i meets the discrete minimum principle hypothesis, and when we apply it to the mesh function t_i, we get

$$t_i = (v_i - w_i) \geq 0 \quad for \quad 0 \leq i \leq l \tag{4.9}$$

Again, if we set $t_i = -(v_i - w_i)$, then t_i is a mesh function that satisfies $t_0 = 0 = t_l$ and we have $L^l t_i = 0, \quad 1 \leq i \leq l-1$ along the same lines as before. As a result, the discrete minimum principle is applied to the mesh function t_i gives $t_i = -(v_i - w_i) \geq 0$

$$i.e., (v_i - w_i) \leq 0 \quad for \quad 0 \leq i \leq l \tag{4.10}$$

From Eq. (4.9) and Eq. (4.10), we get $v_i - w_i = 0$. This suggests that the discrete problem (2.1)-(2.2) solution is unique. The existence of linear equation is assumed by their uniqueness. Now we'll explain how to prove the bound on $< z_i >_i^l = 0$. We do this by introducing two barrier functions $\Psi_i^{\pm}$ Defined by

$$\Psi_i^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm Z_i, 0 \leq i \leq l$$

where $C_2 \geq 1$ is a positive constant that can be chosen at random. Then we have

$$\Psi_i^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm Z_i, 0 \leq i \leq l$$

Now we have

$$\Psi_0^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm Z_0,$$
$$\Psi_0^{\pm} = (||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm C_2\varphi_0) \geq 0$$

since $||\varphi||_{h,\infty} + |\tau| \geq \varphi_0$ and $C_2 \geq 1$,

$$\Psi_n^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm Z_n$$
$$\Psi_n^{\pm} = (||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm \tau) \geq 0$$

Case 1. For $i = 1, 2, 3, \ldots., \nu$

$$L^l\Psi_i^{\pm} = \varepsilon\rho_i(\Psi_i^{\pm})'' + (2\alpha + \beta)q_i\Psi_i^{\pm} = G_i - D_{i-\nu+1}\Psi_{i-\nu+1}^{\pm} - E_{i-\nu}\Psi_{i-\nu}^{\pm} - F_{i-\nu-1}\Psi_{i-\nu-1}^{\pm}$$
$$= -q_i(||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm L^l Z_i)$$

$$= (\frac{-q_i}{\theta}||r||_{h,\infty} \pm r_i) - q_i C_2(||\varphi||_{h,\infty} + |\tau|) \leq 0$$

Case 2. For $i = \nu + 1, \ldots, l_1 - 1$

$$L^l \Psi_i^{\pm} = \varepsilon \rho_i (\Psi_i^{\pm})'' + (2\alpha + \beta) q_i \Psi_i^{\pm} + D_{i-\nu+1} \Psi_{i-\nu+1}^{\pm} = G_i - E_{i-\nu} \Psi_{i-\nu}^{\pm} - F_{i-\nu-1} \Psi_{i-\nu-1}^{\pm}$$

$$= -q_i(||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm L^l Z_i)$$

$$= (\frac{-q_i}{\theta}||r||_{h,\infty} \pm r_i) - q_i C_2(||\varphi||_{h,\infty} + |\tau|) \leq 0$$

Case 3. For $i = l_1, \ldots.., l - 1$

$$L^l \Psi_i^{\pm} = \varepsilon \rho_i (\Psi_i^{\pm})'' + 2\alpha + \beta q_i \Psi_i^{\pm} + D_{i-\nu+1} \Psi_{i-\nu+1}^{\pm} + E_{i-\nu} \Psi_{i-\nu}^{\pm} = G_i - F_{i-\nu-1} \Psi_{i-\nu-1}^{\pm}$$

$$= -q_i(||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm L^l Z_i)$$

$$= (\frac{-q_i}{\theta}||r||_{h,\infty} \pm r_i) - q_i C_2(||\varphi||_{h,\infty} + |\tau|) \leq 0$$

Combining above cases, we have $L^l \Psi_i^{\pm} \leq 0, 1 \leq i \leq l$.

Using the discrete maximum principle

$$\Psi_i^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm Z_n \geq 0, \quad 0 \leq i \leq l.$$

which proves the desired results. □

4.2. **Right-end layer.** Assume $p(u) \leq -\mathcal{M} < 0$ and $q(u) \leq -\subseteq < 0$ where θ and $\mathcal{M}$ are positive constants. Then the problem (2.1) - (2.2) shows layer structure at $u = 1$ for small values of ε. In this case, the fitting parameter inserted in Eq. (4.4) is determined by using the following approximate solution given in [2].

$$lim_{h \to 0} z(ih) \approx z_0(0) + (\phi(0) - z_0(1)) e^{-p(1)(\frac{1}{\varepsilon} - i\rho)} + O(\varepsilon),$$

where $\rho = \frac{h}{\epsilon}$. Using this expression in Eq. (4.4) and following the process given in [2], we get

$$\sigma = \rho(\alpha + \beta/2) p(0) e^{-p(0)\nu\rho} coth(\frac{p(1)\rho}{2}) \tag{4.11}$$

Lemma 4.2. . *Suppose $\Psi_0 \geq 0$ and $\Psi_l \geq 0$, then $L^l \Psi_i \leq 0$ $\forall$ $i = 1, 2, 3, \ldots..l - 1$ implies that $\Psi_i \geq 0$ $\forall$ $i = 0, 1, 2, 3. \ldots.l$.*

Proof. Let $j \in (0, 1, \ldots.l)$ such that $\Psi_j = min_{0 \leq i \leq l} \Psi_i$. Let if possible $\Psi_j < 0$. We will show that it leads to contradiction. Clearly $j \notin (0, l)$. Now we have

$$L^l \Psi_j = \begin{cases} \varepsilon \rho_i \Psi_i'' + (2\alpha + \beta) q_i \Psi_i = G_i - D_{i-\nu+1} \Psi_{i-\nu+1} - E_{i-\nu} \Psi_{i-\nu} - F_{i-\nu-1} \Psi_{i-\nu-1}, \\ \forall i = 1, 2, \ldots, \nu. \\ \varepsilon \rho_i \Psi_i'' + (2\alpha + \beta) q_i \Psi_i + D_{i-\nu+1} \Psi_{i-\nu+1} = G_i - E_{i-\nu} \Psi_{i-\nu} - F_{i-\nu-1} \Psi_{i-\nu-1}, \\ \forall\ i = \nu + 1, \ldots., l_1 - 1. \\ \varepsilon \rho_i \Psi_i'' + (2\alpha + \beta) q_i \Psi_i + D_{i-\nu+1} \Psi_{i-\nu+1} + E_{i-\nu} \Psi_{i-\nu} = G_i - F_{i-\nu-1} \Psi_{i-\nu-1}, \\ \forall i = l_1, \ldots.., l - 1. \end{cases} \tag{4.12}$$

Case 1: For $i = 1, 2, \ldots, \nu$

$$L^l\Psi_j = (\varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i = G_i - D_{i-\nu+1}\Psi_{i-\nu+1} - E_{i-\nu}\Psi_{i-\nu} - F_{i-\nu-1}\Psi_{i-\nu-1},$$

$$\forall i = 1, 2, \ldots, \nu.) \geq 0 \ (\because q_i < 0, i = 1, 2, \ldots.\nu)$$

Case 2: For $i = \nu+1, \ldots., l_1 - 1$

$$L^l\Psi_j = (\varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i + D_{i-\nu+1}\Psi_{i-\nu+1} = G_i - E_{i-\nu}\Psi_{i-\nu} - F_{i-\nu-1}\Psi_{i-\nu-1},$$

$$\forall \ i = \nu+1, \ldots, l_1 - 1.) \geq 0$$

Case 3: For $i = l_1, \ldots.., l-1$

$$L^l\Psi_j = (\varepsilon\rho_i\Psi_i'' + (2\alpha+\beta)q_i\Psi_i + D_{i-\nu+1}\Psi_{i-\nu+1} + E_{i-\nu}\Psi_{i-\nu} = G_i - F_{i-\nu-1}\Psi_{i-\nu-1},$$

$$\forall i = l_1, \ldots.., l-1.) \quad \geq 0$$

which is contradiction to

$$L^l\Psi_j \leq 0, \ 1 \leq j \leq l-1$$

Therefore, the assumption $\Psi_j < 0$ is false and $\Psi_j > 0$. Since j was choosing arbitrary, we have $\Psi_j \geq 0, \forall j = 0, 1, 2, \ldots l$

Theorem 4.2. *Let $p(u) \geq \mathcal{M} > 0$ and $q(u) \geq \theta > 0$ where $\mathcal{M}$ and θ are both positive constants. The solutions to the problem (2.1) - (2.2) with boundary conditions exist, are unique and satisfy.*

$$||Z||_{h,\infty} \leq ||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + \tau) \tag{4.13}$$

where $C_1 \geq 1$ is a positive constant

Proof. To prove the existence and uniqueness, assume $<v>_{i=0}^{l}$ and $<w>_{i=0}^{l}$ be the two solutions to the problem (2.1) - (2.2). Then $t_i = v_i - w_i$ is a mesh function that meets the conditions $t_0 = 0 = t_l$ and for $1 \leq i \leq l-1$ we have $L^l t_i = L^l v_i - L^l w_i$. Since v_i and w_i satisfy Eq. (4.5), therefore $L^l t_i = 0, 1 \leq i \leq l-1$. As a result, the mesh function t_i meets the discrete minimum principle hypothesis, and when we apply it to the mesh function t_i, we get

$$t_i = (v_i - w_i) \geq 0 \ \ for \ \ 0 \leq i \leq l \tag{4.14}$$

Again, if we set $t_i = -(v_i - w_i)$, then t_i is a mesh that satisfies $t_0 = 0 = t_l$ and we have $L^l t_i = 0, \ 1 \leq i \leq l-1$ along the same lines as before. As a result, the discrete minimum principle is applied to the mesh function t_i gives $t_i = -(v_i - w_i) \geq 0$

$$i.e., (v_i - w_i) \leq 0 \ \ for \ \ 0 \leq i \leq l \tag{4.15}$$

From Eq. (4.14) and Eq. (4.15), we get $v_i - w_i = 0$. This suggests that the discrete problem (2.1)-(2.2) solution is unique. The existence of linear equation is assumed by their uniqueness. Now we'll explain how to prove the bound on $< z_i >_i^l = 0$. We do this by introducing two barrier functions $\Psi_i^{\pm}$ Defined by

$$\Psi_i^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm Z_i, 0 \leq i \leq l$$

where $C_1 \geq 1$ is a positive constant that can be chosen at random. Then we have

$$\Psi_i^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm Z_i, 0 \leq i \leq l$$

Now we have

$$\Psi_0^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm Z_0,$$

$$\Psi_0^{\pm} = (||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm C_1\varphi_0) \geq 0$$

since $||\varphi||_{h,\infty} + |\tau| \geq \varphi_0$ and $C_1 \geq 1$,

$$\Psi_l^{\pm} = ||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm Z_l$$

$$\Psi_l^{\pm} = (||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm \tau) \geq 0$$

Case 1. For $i = 1, 2, 3, \ldots., \nu$

$$L^l\Psi_i^{\pm} = \varepsilon\rho_i(\Psi_i^{\pm})'' + (2\alpha+\beta)q_i\Psi_i^{\pm} = G_i - D_{i-\nu+1}\Psi_{i-\nu+1}^{\pm} - E_{i-\nu}\Psi_{i-\nu}^{\pm} - F_{i-\nu-1}\Psi_{i-\nu-1}^{\pm}$$

$$= -q_i(||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm L^l Z_i)$$

$$= (\frac{-q_i}{\theta}||r||_{h,\infty} \pm r_i) - q_iC_2(||\varphi||_{h,\infty} + |\tau|) \leq 0$$

Case 2. For $i = \nu+1, \ldots., l_1 - 1$

$$L^l\Psi_i^{\pm} = \varepsilon\rho_i(\Psi_i^{\pm})'' + 2\alpha + \beta q_i\Psi_i^{\pm} + D_{i-\nu+1}\Psi_{i-\nu+1}^{\pm} = G_i - E_{i-\nu}\Psi_{i-\nu}^{\pm} - F_{i-\nu-1}\Psi_{i-\nu-1}^{\pm}$$

$$= -q_i(||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm L^l Z_i)$$

$$= (\frac{-q_i}{\theta}||r||_{h,\infty} \pm r_i) - q_iC_2(||\varphi||_{h,\infty} + |\tau|) \leq 0$$

Case 3. For $i = l_1, \ldots.., l - 1$

$$L^l\Psi_i^{\pm} = \varepsilon\rho_i(\Psi_i^{\pm})'' + (2\alpha+\beta)q_i\Psi_i^{\pm} + D_{i-\nu+1}\Psi_{i-\nu+1}^{\pm} + E_{i-\nu}\Psi_{i-\nu}^{\pm} = G_i - F_{i-\nu-1}\Psi_{i-\nu-1}^{\pm}$$

$$= -q_i(||r||_{h,\infty}\theta^{-1} + C_2(||\varphi||_{h,\infty} + |\tau|) \pm L^l Z_i)$$

$$= (\frac{-q_i}{\theta}||r||_{h,\infty} \pm r_i) - q_iC_2(||\varphi||_{h,\infty} + |\tau|) \leq 0$$

Using the above cases, we have $L^l\Psi_i \leq 0, 1 \leq i \leq l$. Using the discrete minimum principle

$$\Psi_i^{\pm} = (||r||_{h,\infty}\theta^{-1} + C_1(||\varphi||_{h,\infty} + |\tau|) \pm Z_l) \geq 0, \ 0 \leq i \leq l$$

which proves the desired results.

As a result of the previous theorems, the problem (2.1)-(2.2) solution is uniformly bounded, regardless of the mesh size h and the perturbation ε, demonstrating that the proposed scheme is stable for all h. $\square$

5. Truncation error

Using Taylor series, the truncation error in the method Eq. (4.2) is

$$T\,(h) = \{(2\alpha+\beta)-1\}\,\sigma\varepsilon z_i'' h^2 + \left\{\left[\left(\frac{-2\alpha}{3}\right)+\left(2\omega\sigma\varepsilon+\frac{1}{6}\right)\beta\right] p_i z_i^3 + \left(\frac{2\alpha-1}{12}\right)\sigma\varepsilon z_i^4\right\} h^4 + O(h^6) \tag{5.1}$$

The truncation error is fourth-order for any arbitrary α and β with $2\alpha+\beta=1$ and any value of ω.

6. Convergence analysis

The matrix form of the Eq. (4.5) is

$$\mathcal{A}z + \mathcal{B} + T(h) = 0 \tag{6.1}$$

$$\mathcal{A} = \begin{bmatrix} B_1 & A_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ C_2 & B_2 & A_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & C_3 & B_3 & A_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_\nu & 0 & \cdots & C_\nu & B_\nu & A_\nu & 0 & \cdots & \cdots & 0 \\ E_{\nu+1} & D_{\nu+1} & 0 & \cdots & C_{\nu+1} & B_{\nu+1} & A_{\nu+1} & 0 & 0 & \\ F_{\nu+2} & E_{\nu+2} & D_{\nu+2} & 0 & \cdots & C_{\nu+2} & B_{\nu+2} & A_{\nu+2} & 0 & 0 \\ 0 & F_{\nu+3} & E_{\nu+3} & D_{\nu+3} & 0 & C_{\nu+3} & B_{\nu+3} & A_{\nu+3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \cdots & 0 & F_{l-2} & E_{l-2} & D_{l-2} & 0 & \cdots & C_{l-2} & B_{l-2} & A_{l-2} \\ \cdots & \cdots & 0 & F_{l-1} & E_{l-1} & D_{l-1} & 0 & \cdots & C_{l-1} & B_{l-1} \end{bmatrix}$$

and $\mathcal{B} = [\kappa_1, \kappa_2, ..\kappa_\nu, \kappa_{\nu+1},\ \kappa_{\nu+2}, ..., \kappa_{l-2}, \kappa_{l-1}]$
where

$$\kappa_i = \begin{cases} G_i - D_{i-\nu+1}z_{i-\nu+1} - E_{i-\nu}z_{i-\nu} - F_{i-\nu-1}z_{i-\nu-1} - C_1 z_0, & \forall\ 1 \le i \le \nu-1 \\ G_i - E_{i-\nu}z_{i-\nu} - F_{i-\nu-1}z_{i-\nu-1}, & \forall\ i=\nu \\ G_i - F_{i-\nu-1}z_{i-\nu-1}, & \forall\ i=\nu+1 \\ G_i - A_{l-1}z_l, & \forall\ \nu+2 \le i \le l-1 \end{cases}$$

where

$$A_{i+1} = \sigma\epsilon + \alpha h^2 q_{i+1}, B_i = -2\sigma\epsilon + \beta h^2 q_i, C_{i-1} = \sigma\epsilon + \alpha h^2 q_{i-1}$$

$$D_{i-\nu+1} = \frac{-\alpha h p_{i-1}}{2} + \frac{\beta h p_i}{2}(1 + 2\omega h^2 q_{i+1} + \omega h(3p_{i+1} + p_{i-1})) + \frac{3\alpha h}{2} p_{i+1}$$

$$E_{i-\nu} = 2\alpha h p_{i-1} - 2\beta\omega p_i h^2 (p_{i+1} + p_{i-1}) - 2\alpha h p_{i+1}$$

$$F_{i-\nu-1} = \frac{-3\alpha h p_{i-1}}{2} - \frac{\beta p_i h}{2}(1 + 2\omega h^2 q_{i-1} - \omega h(p_{i+1} + 3p_{i-1})) + \frac{\alpha h}{2} p_{i+1}$$

$$G_i = h^2(\alpha r_{i+1} + \beta r_i + \alpha r_{i-1}) - \beta\omega p_i h^3 (r_{i+1} - r_{i-1})$$

for $i = 0, 1, \ldots, l-1$ and

$$T(h) = O\left(h^4\right), z = [z_1, z_2, \ldots, z_{l-1}]^T, T(h) = [T_1, T_2, \ldots, T_{l-1}]^T, O = [0, 0, \ldots, 0]^T$$

are related vectors of Eq. (6.1).

Let

$$\widetilde{z} = [\widetilde{z}_1, \widetilde{z}_2, \ldots, \widetilde{z}_{l-1}]^T \cong Z$$

satisfies the equation

$$\mathcal{A}\widetilde{z} + \mathcal{B} = 0 \tag{6.2}$$

Let $e_i = z_i - Z_i, i = 1\,(1)\,l-1$ be the error so that $E = [e_1, e_2, ..., e_{l-1}]^T = u - U$.

Subtracting Eq. (6.1) from Eq. (6.2), we obtain the error equation

$$\mathcal{A}E = T(h) \tag{6.3}$$

Let the sum of the i^{th} row elements of the matrix $\mathcal{A}$ be s_i. Then

$s_i = h^2\left(\alpha q_{i-1} + \beta q_i + \alpha q_{i+1}\right)$ for $1 \leq i \leq \nu - 1$

$$s_i = \frac{h}{2}\left(-\alpha p_{i-1} + \beta p_i\left(1 + 2\omega h^2 q_{i+1} + \omega h\left(3p_{i+1} + p_{i-1}\right)\right) + 3\alpha p_{i+1}\right) + h^2\left(\alpha q_{i-1} + \beta q_i + \alpha q_{i+1}\right)$$

for $i = \nu$

$$s_i = \frac{h}{2}\left(-3\alpha p_{i-1} - \beta p_i\left(1 + 2\omega h^2 q_{i-1} - \omega h\left(p_{i+1} + 3p_{i-1}\right)\right) + \alpha p_{i+1}\right) + h^2\left(\alpha q_{i-1} + \beta q_i + \alpha q_{i+1}\right)$$

for $i = \nu + 1$

$s_i = h^2\left(\alpha q_{i-1} + \beta q_i + \alpha q_{i+1}\right)$ for $\nu + 2 \leq i \leq l - 1$.

Let $\zeta_{1^*} = min_{1\leq i\leq l}\left|p\left(u_i\right)\right|, \zeta_1^* = max_{1\leq i\leq l}\left|p\left(u_i\right)\right|, \zeta_{2^*} = min_{1\leq i\leq l}\left|q\left(u_i\right)\right|$

and $\zeta_2^* = max_{1\leq i\leq l}\left|q\left(u_i\right)\right|$.

It is verified that $\mathcal{A}$ is irreducible and monotone since $0 < \varepsilon \ll 1$ and $\varepsilon \propto O(h)$ with sufficiently small h. Hence $\mathcal{A}^{-1}$ exists and $\mathcal{A}^{-1} \geq 0$. Therefore, using Eq. (6.3) we have

$$||E|| \leq ||\mathcal{A}^{-1}||\ ||T|| \tag{6.4}$$

For small h, we have

$s_i \geq h^2\left[2\left(\alpha + \beta/2\right)\zeta_{2^*}\right]$ for $1 \leq i \leq \nu - 1$

$s_i \geq h^2\left[2\left(\alpha + \beta/2\right)\zeta_{2^*}\right]$ for $i = \nu$

$s_i \geq h^2\left[2\left(\alpha + \beta/2\right)\zeta_{2^*}\right]$ for $i = \nu + 1$

$s_i \geq h^2\left[2\left(\alpha + \beta/2\right)\zeta_{2^*}\right]$ for $\nu + 2 \leq i \leq l - 1$

Let $\mathcal{A}_{i,k}^{-1}$ be the $(i,k)^{th}$ element of $\mathcal{A}^{-1}$ and we define

$$||\ \mathcal{A}|| = max_{1\leq i\leq l-1}\sum_{k=1}^{l-1}\mathcal{A}_{i,k}^{-1}$$

and

$$||T(h)|| = max_{1\leq i\leq l-1}|T_i(h)|.$$

Since $\mathcal{A}_{i,k}^{-1} \geq 0$ and $\sum_{k=1}^{l-1} \mathcal{A}_{i,k}^{-1}.\quad s_k = 1$ for $i = 12, 3, ..., l-1$. Hence

$$\sum_{k=1}^{\nu-1} \mathcal{A}_{i,k}^{-1} \leq \frac{1}{min_{1 \leq k \leq \nu-1} s\ _k} < \frac{1}{h^2 \left[2\left(\alpha+\beta/2\right)\zeta_{2^*}\right]}\ ,\quad i = 1, 2, 3, \ldots, \nu-1 \tag{6.5}$$

$$\mathcal{A}_{i,k}^{-1} \leq \frac{1}{s_\nu} < \frac{1}{h^2 \left[2\left(\alpha+\beta/2\right)\zeta_{2^*}\right]}\ ,\ i = \nu, \nu+1 \tag{6.6}$$

Furthermore

$$\sum_{k=\nu+2}^{l-1} \mathcal{A}_{i,k}^{-1} \leq \frac{1}{min_{1 \leq k \leq \nu-1} s_k} < \frac{1}{h^2 \left[2\left(\alpha+\beta/2\right)\zeta_{2^*}\right]},\quad i = \nu+2,\ \nu+3, ..., l-1. \tag{6.7}$$

Using Eqs. (6.5) – (6.6) and Eq. (6.7), we get

$$||E|| \leq O(h^2). \tag{6.8}$$

As a result, the proposed exponential spline strategy converges to second order. The convergence analysis for the right end boundary layer can be examined using the same procedure.

7. Numerical Examples

The effectiveness of the numerical approach can be demonstrated by the following instances. MAE (maximum absolute error) in the solution is computed using the double mesh principle $E^l = max_{0 \leq i \leq l} \left| z_i^l - z_{2i}^{2l} \right|$. The order of convergence is also investigated by $r^l = \log_2 \left({E_i}^l / {E_i}^{2l} \right)$. We exhibit graphs showing the computed solution of the issue for various δ values.

Example 1: $\varepsilon z'' + z'(u-\delta) - z(u) = 0$ with $z(u) = 1,\ -\delta \leq u \leq 0$ and $z(1) = 0$.

Example 2: $\varepsilon z'' + exp(-0.25u) - z(u) = 0$ with $z(u) = 1$ for $-\delta \leq u \leq 0, z\left(1\right) = 1$.

Example 3: $\varepsilon z'' - z'(u-\delta) - z(u) = 0$ with $z(u) = 1,\ -\delta \leq u \leq 0$ and $z(1) = -1$.

Example 4: $\varepsilon z'' - z'(u-\delta) + z(u) = 0$ with $z(u) = 1$ for $-\delta \leq u \leq 0, z\left(1\right) = 1$.

8. Conclusion

A computational approach for solving the singularly perturbed differential equation with the large delay is derived using a special type of mesh. A numerical scheme consisting of a fitting parameter is developed to minimize the error and to control the layer structure in the solution. Four examples are solved and computational results with large delay are shown in Table 1-4. In the proposed method, we also analyzed the effect of the large delay on the layer structure or oscillatory behaviour of the solutions with and without the fitting parameter in the Figures 1-8. The graphs depict the layer behaviour in the solution of the examples with and without the fitting parameter. The impact of the fitting factor in controlling the oscillations in the solution is also depicted in the graphs. We have clearly noticed that the fitting parameter controls the oscillations in the layer for the large delay values. The proposed method is simple and it works very well with small as well as large delay.

Table 1. The MAEs and the order of convergence in Example 1 for $\delta = 0.03$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
2^{-1}	1.4419e-04	3.6260e-05	9.0922e-06	2.2765e-06	5.6956e-07
	1.9915	1.9957	1.9978	1.9989	
2^{-2}	2.7820e-04	7.0357e-05	1.7692e-05	4.4358e-06	1.1106e-06
	1.9834	1.9916	1.9958	1.9979	
2^{-3}	5.3252e-04	1.3580e-04	3.4283e-05	8.6123e-06	2.1583e-06
	1.9992	1.9859	1.9930	1.9965	
2^{-4}	1.0850e-03	2.7618e-04	6.9566e-05	1.7450e-05	4.3695e-06
	1.9740	1.9892	1.9952	1.9977	
2^{-5}	2.6527e-03	6.6814e-04	1.6755e-04	4.1911e-05	1.0479e-05
	1.9892	1.9956	1.9992	1.9998	
2^{-6}	7.4932e-03	1.9264e-03	4.8498e-04	1.2146e-04	3.0379e-05
	1.9597	1.9899	1.9974	1.9993	
2^{-7}	1.7702e-02	4.9420e-03	1.2718e-03	3.2029e-04	8.0219e-05
	1.8407	1.9582	1.9894	1.9974	
2^{-8}	3.1170e-03	1.2306e-03	3.5392e-04	9.1838e-05	2.3178e-05
	1.3408	1.7979	1.9463	1.9863	

Table 2. The MAEs and the order of convergence in Example 2 for $\delta = 0.03$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
2^{-1}	2.2573e-04	7.3086e-05	2.6515e-05	1.0740e-05	4.7393e-06
	1.6269	1.4628	1.3038	1.1802	
2^{-2}	4.2612e-04	1.3699e-04	4.9174e-05	1.9718e-05	8.6367e-06
	1.6372	1.4781	1.3184	1.1910	
2^{-3}	7.7550e-04	2.4419e-04	8.5262e-05	3.3306e-05	1.4307e-05
	1.6671	1.5180	1.3561	1.2191	
2^{-4}	1.4151e-03	4.1821e-04	1.3513e-04	4.8916e-05	1.9780e-05
	1.7586	1.6299	1.4660	1.3063	
2^{-5}	2.9428e-03	7.6106e-04	2.0667e-04	6.0787e-05	1.9899e-05
	1.9511	1.8807	1.7655	1.6111	
2^{-6}	7.6116e-03	1.8207e-03	3.9430e-04	7.5748e-05	1.5215e-05
	2.0637	2.2071	2.3800	2.3157	
2^{-7}	1.7691e-02	4.7307e-03	1.0985e-03	1.1542e-04	2.3304e-05
	1.9029	2.1065	2.3503	2.3082	
2^{-8}	3.0452e-03	1.1738e-03	3.2141e-04	7.4958e-05	1.4643e-05
	1.3754	1.8687	2.1003	2.3559	

Table 3. The MAEs and the order of convergence in Example 3 for $\delta = 0.03$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
2^{-1}	1.3799e-04	3.4901e-05	8.7768e-06	2.2007e-06	5.5099e-07
	1.9832	1.9915	1.9957	1.9979	
2^{-2}	2.5428e-04	6.5014e-05	1.6440e-05	4.1336e-06	1.0364e-06
	1.9676	1.9835	1.9917	1.9958	
2^{-3}	4.4129e-04	1.1407e-04	2.9002e-05	7.3122e-06	1.8358e-06
	1.9518	1.9757	1.9878	1.9939	
2^{-4}	6.8107e-04	1.7779e-04	4.5418e-05	1.1478e-05	2.8849e-06
	1.9376	1.9688	1.9844	1.9923	
2^{-5}	1.2357e-03	3.1508e-04	7.9465e-05	1.9949e-05	4.9974e-06
	1.9715	1.9873	1.9940	1.9971	
2^{-6}	4.3006e-03	1.1004e-03	2.7687e-04	6.9351e-05	1.7349e-05
	1.9665	1.9907	1.9972	1.9991	
2^{-7}	1.4413e-02	3.9925e-03	1.0253e-03	2.5808e-04	6.4629e-05
	1.8520	1.9612	1.9902	1.9976	
2^{-8}	3.0843e-03	1.2152e-03	3.4925e-04	9.0609e-05	2.2867e-05
	1.3437	1.7989	1.9465	1.9864	

Table 4. The MAEs and the order of convergence in Example 4 for $\delta = 0.03$

$\varepsilon \downarrow$	$N = 100$	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
2^{-1}	1.6785e-04	4.1860e-05	1.0451e-05	2.6111e-06	6.5256e-07
	2.0035	2.0019	2.0009	2.0005	
2^{-2}	3.4720e-04	8.6528e-05	2.1595e-05	5.3939e-06	1.3479e-06
	2.0045	2.0025	2.0013	2.0006	
2^{-3}	6.8995e-04	1.7230e-04	4.3045e-05	1.0757e-05	2.6886e-06
	2.0016	2.0010	2.0006	2.0003	
2^{-4}	1.3788e-03	3.4472e-04	8.6138e-05	2.1527e-05	5.3806e-06
	1.9999	2.0007	2.0005	2.0003	
2^{-5}	3.5095e-03	8.8050e-04	2.2015e-04	5.5021e-05	1.3752e-05
	1.9949	1.9998	2.0004	2.0003	
2^{-6}	1.3569e-02	3.4562e-03	8.6809e-04	2.1726e-04	5.4328e-05
	1.9731	1.9933	1.9984	1.9997	
2^{-7}	3.3612e-02	9.6117e-03	2.4894e-03	6.2795e-04	1.5734e-04
	1.8061	1.9490	1.9871	1.9968	
2^{-8}	3.2040e-03	1.2716e-03	3.6643e-04	9.5138e-05	2.4014e-05
	1.3332	1.7950	1.9454	1.9861	

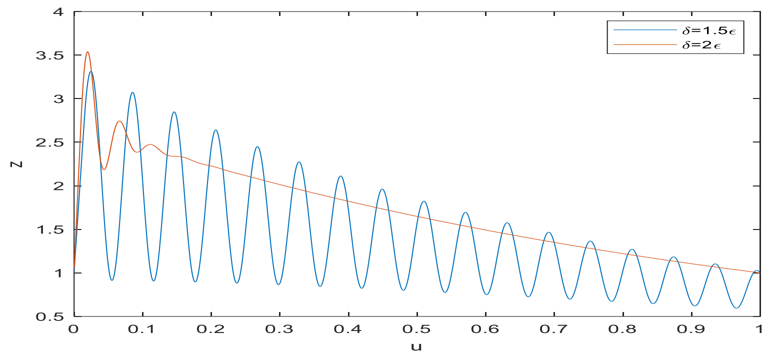

Figure 1. Layer profile of Example 1 for $\varepsilon = 0.01$ without fitting factor.

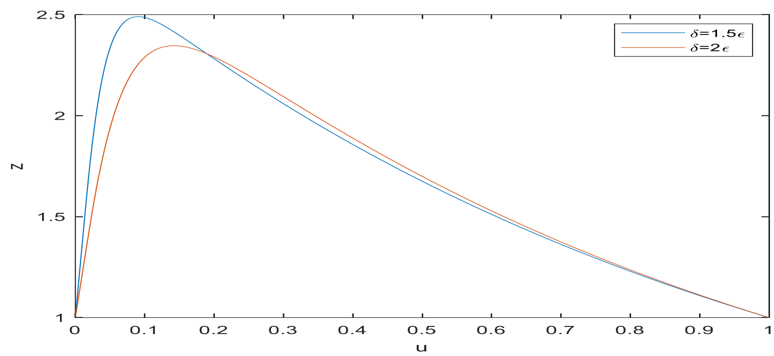

Figure 2. Layer profile of Example 1 for $\varepsilon = 0.01$ with fitting factor.

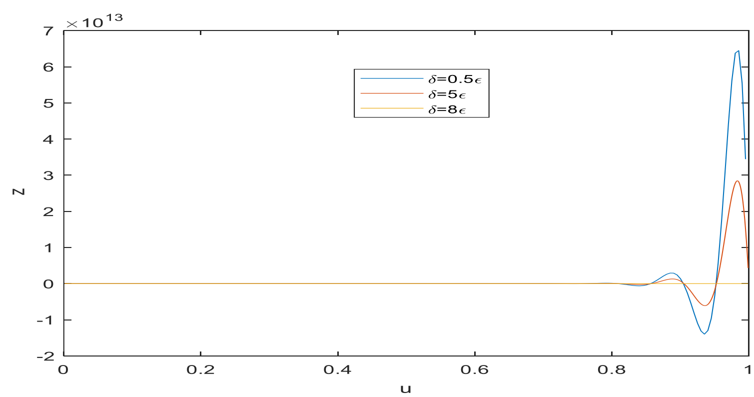

Figure 3. Layer profile of Example 2 for $\varepsilon = 0.01$ without fitting factor.

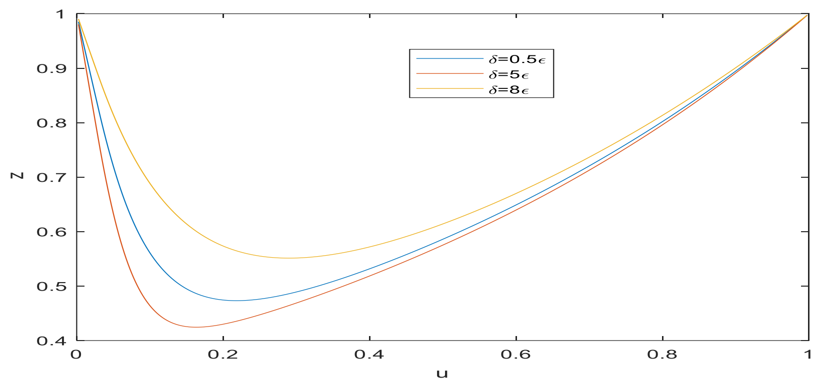

Figure 4. Layer profile of Example 2 for $\varepsilon = 0.01$ with fitting factor.

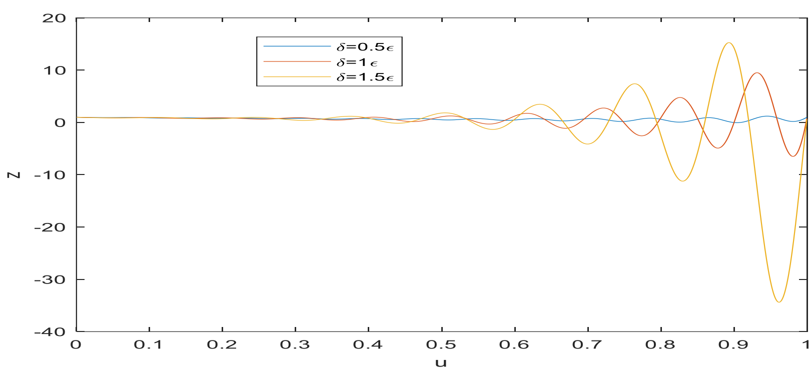

Figure 5. Layer profile of Example 3 for $\varepsilon = 0.01$ without fitting factor.

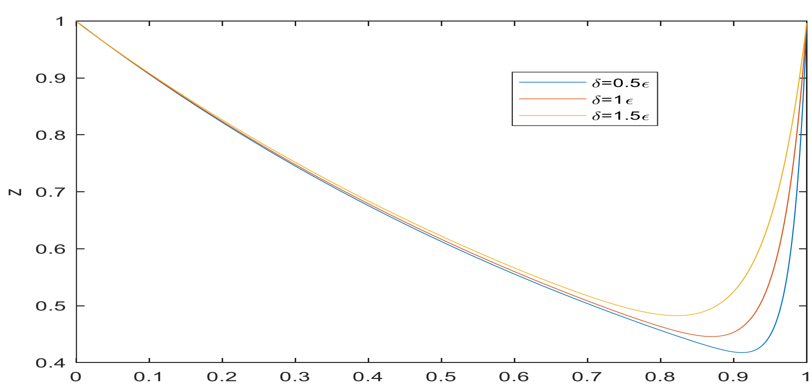

Figure 6. Layer profile of Example 3 for $\varepsilon = 0.01$ with fitting factor.

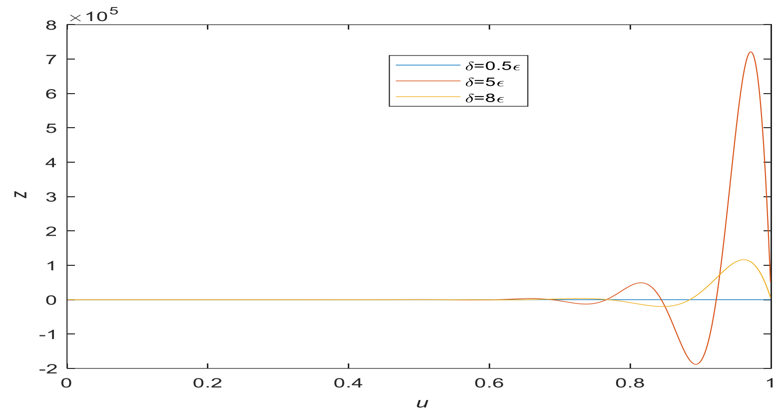

Figure 7. Layer profile of Example 4 for $\varepsilon = 0.01$ without fitting factor.

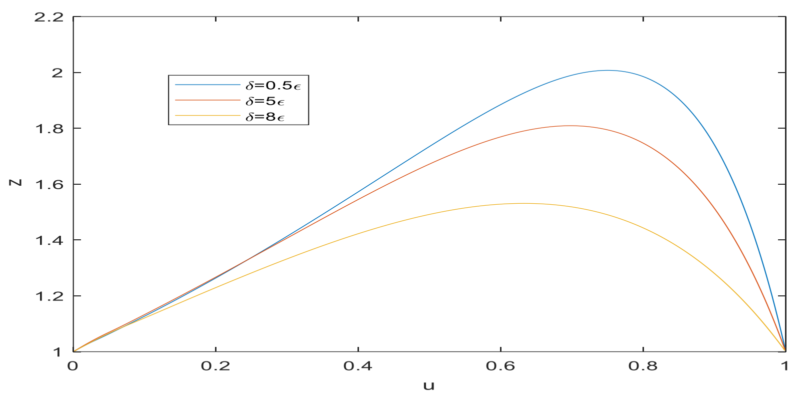

Figure 8. Layer profile of Example 4 for $\varepsilon = 0.01$ with fitting factor.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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