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Existence Suzuki Type Fixed Point Results in A_b-Metric Spaces With Application

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Abstract. In this paper, we give some applications to integral equations as well as homotopy theory via Suzuki contractive type common coupled fixed point results in complete A_b-metric space. We also furnish an example which supports our main result.

1. Introduction

The study of fixed points is aexquisite synthesis of analysis, topology, and geometry. Its numerous applications in areas such as homotopy theory, integral, integro-differential, and impulsive differential equations, finding solutions to optimization problems, approximation theory, non-linear analysis, biomechanics, and algorithms have made it an essential tool.

Suzuki [1] recently established expanded versions of Edelstein's and Banach's fundamental conclusions, sparking a great deal of research in this area (See [2–6]). The b-metric space was first introduced by I.A. Bakhtin [7] in 1989. Numerous generalizations of metric spaces were created as a result of the development of b-metric space.The n-tuple metric space was first introduced and its topological features were examined by M.Abbas et al. in 2015 [8]. A_b-metric spaces were first described by M. Ughade et al. [9] as a generalized version of n-tuple metric space. Then, in partially ordered A_b-metric

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spaces, N.Mlaiki et al. [19] and K.Ravibabu et al. [11, 12] discovered original coupled common fixed point theorems.

The idea of coupled fixed point was first suggested in 1987 by Guo and Lakshmikantham [13]. Later, employing a weak contractivity type assumption, Bhaskar and Lakshmikantham [14] derived a novel fixed point theorem for a mixed monotone mapping on a metric space powered with partial ordering. See study findings in [15–20] and related sources for additional results on coupled fixed point outcomes.

In the context of A_b-metric spaces, the purpose of the current research is to establish original common coupled fixed point theorems for Suzuki contractive type mapping. We can also provide examples that are appropriate and relevant applications to homotopy and integral equations.

Before we can demonstrate the primary findings, we need certain fundamental definitions, results and examples from the literature.

2. Preliminaries

Definition 2.1. *[9] Let $\mathcal{P}$ be a non-empty set and $b \geq 1$ be given real number. A mapping $A_b : \mathcal{P}^n \to [0, \infty)$ is called an A_b-metric on $\mathcal{P}$ if and only if for all $\lambda_i, \nu \in \mathcal{P}$ $i = 1, 2, 3, ..n$; the following conditions hold :*

(A_b1) $A_b(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) \geq 0$,

(A_b2) $A_b(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) = 0 \Leftrightarrow \lambda_1 = \lambda_2 = \cdots\cdots = \lambda_{n-1} = \lambda_n$,

$$(A_b3)\ A_b(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) \leq b \begin{pmatrix} A_b(\lambda_1, \lambda_1, ..., (\lambda_1)_{n-1}, \nu) \\ +A_b(\lambda_2, \lambda_2, ..., (\lambda_2)_{n-1}, \nu) \\ +\cdots\cdots + A_b(\lambda_{n-1}, \lambda_{n-1}, ..., (\lambda_{n-1})_{n-1}, \nu) \\ +A_b(\lambda_n, \lambda_n, ..., (\lambda_n)_{n-1}, \nu) \end{pmatrix}$$

Then the pair $(\mathcal{P}, A_b)$ is called an A_b-metric space.

Remark 2.1. *[9] It should be noted that, the class of A_b-metric spaces is effectively larger than that of A-metric spaces. Indeed each A-metric space is a A_b-metric space with $b = 1$.However, the converse is not always true. Also A_b-metric space is an "n-dimensional S_b-metric space. Therefore the S_b-metric are special cases of an A_b-metric with $n = 3$.*

Following example shows that a A_b-metric on $\mathcal{P}$ need not be a A-metric on $\mathcal{P}$.

Example 2.1. *[9] Let $\mathcal{P} = [0, +\infty)$, define $A_b : \mathcal{P}^n \to [0, +\infty)$ as $A_b(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) = \sum_{i=1}^{n} \sum_{i<j} |\lambda_i - \lambda_j|^2$ for all $\lambda_i \in \mathcal{P}, i = 1, 2, \cdots$. Then $(\mathcal{P}, A_b)$ is an A_b-metric space with $b = 2 > 1$.*

Definition 2.2. *[9] A A_b- metric space $(\mathcal{P}, A_b)$ is said to be symmetric if $A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \varrho) = A_b(\varrho, \varrho, \cdots, (\varrho)_{n-1}, \lambda)$ for all $\lambda, \varrho \in \mathcal{P}$.*

Definition 2.3. *[9] Let $(\mathcal{P}, A_b)$ be a A_b-metric space. Then, for $\lambda \in \mathcal{P}$, $r > 0$ we defined the open ball $B_{A_b}(\lambda, r)$ and closed ball $B_{A_b}[\lambda, r]$ with center λ and radius r as follows respectively:*

$$B_{A_b}(\lambda, r) = \{\varrho \in \mathcal{P} : A_b(\varrho, \varrho, \cdots, (\varrho)_{n-1}, \lambda) < r\},$$

and

$$B_{A_b}[\lambda, r] = \{\varrho \in \mathcal{P} : A_b(\varrho, \varrho, \cdots, (\varrho)_{n-1}, \lambda) \leq r\}.$$

Lemma 2.1. *[9] In a A_b-metric space, we have*

(1) $A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \varrho) \leq bA_b(\varrho, \varrho, \cdots, (\varrho)_{n-1}, \lambda)$;
(2) $A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \zeta) \leq b(n-1)A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \varrho) + b^2 A_b(\varrho, \varrho, \cdots, (\varrho)_{n-1}, \zeta)$.

Definition 2.4. *[9] If $(\mathcal{P}, A_b)$ be a A_b-metric space. A sequence $\{\lambda_k\}$ in $\mathcal{P}$ is said to be:*

(1) *A_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $A_b(\lambda_k, \lambda_k, \cdots\cdots(\lambda_k)_{n-1}, \lambda_m) < \epsilon$ for each $m, k \geq n_0$.*
(2) *A_b-convergent to a point $\lambda \in \mathcal{P}$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $A_b(\lambda_k, \lambda_k, \cdots\cdots(\lambda_k)_{n-1}, \lambda) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim_{k\to\infty} \lambda_k = \lambda$.*
(3) *A A_b-metric space $(\mathcal{P}, A_b)$ is called complete if every A_b-Cauchy sequence is A_b-convergent in $\mathcal{P}$.*

Lemma 2.2. *[9] If $(\mathcal{P}, A_b)$ be a A_b-metric space with $b \geq 1$ and suppose that $\{\lambda_k\}$ is a A_b-convergent to λ and $\{\zeta_k\}$ is a A_b-convergent to ζ, then we have*

(i)

$$\begin{aligned}\frac{1}{b^2}A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \zeta) &\leq \lim_{k\to\infty}\inf\ A_b(\lambda_k, \lambda_k, \cdots, (\lambda_k)_{n-1}, \zeta_k)\\ &\leq \lim_{k\to\infty}\sup A_b(\lambda_k, \lambda_k, \cdots, (\lambda_k)_{n-1}, \zeta_k)\\ &\leq b^2 A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \zeta).\end{aligned}$$

In particular, if $\zeta_k = \zeta$ is constant, then

(ii)

$$\begin{aligned}\frac{1}{b^2}A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \zeta) &\leq \lim_{k\to\infty}\inf\ A_b(\lambda_k, \lambda_k, \cdots, (\lambda_k)_{n-1}, \zeta)\\ &\leq \lim_{k\to\infty}\sup A_b(\lambda_k, \lambda_k, \cdots, (\lambda_k)_{n-1}, \zeta)\\ &\leq b^2 A_b(\lambda, \lambda, \cdots, (\lambda)_{n-1}, \zeta).\end{aligned}$$

Theorem 2.1. *[1] Let $(\mathcal{P}; d)$ be a complete metric space, let $T : \mathcal{P} \to \mathcal{P}$ be a mapping and define a non increasing function*

$$\theta : [0; 1) \to (0; 1] \text{ by } \theta(t) = \begin{cases} 1, & 0 \leq t \leq \frac{\sqrt{5}-1}{2} \\ (1-t)t^{-2}, & \frac{\sqrt{5}-1}{2} \leq t \leq \frac{1}{\sqrt{2}} \\ (1+t)^{-1}, & \frac{1}{\sqrt{2}} < t \leq 1 \end{cases}.$$

Assume that there exists $t \in [0;1)$ *such that*

$$\theta(t)d(\rho, T\rho) \leq d(\rho; \varrho) \text{ implies } d(T\rho; T\varrho) \leq td(\rho; \varrho)$$

for all $\rho, \varrho \in \mathcal{P}$. *Then there exists a unique fixed point* a *of* T.
Moreover, $\lim_{n\to\infty} T^n\rho = a$ *for all* $\rho \in \mathcal{P}$.

In order to obtain our results we need to consider the followings.

3. Main Results

Definition 3.1. *Let* $(\mathcal{P}, A_b)$ *be a* A_b*-metric spaces and suppose* $F : \mathcal{P}^2 \to \mathcal{P}$ *be a mapping. If* $F(\rho, \varrho) = \rho$, $F(\varrho, \rho) = \varrho$ *for* $\rho, \varrho \in \mathcal{P}$ *then* (ρ, ϱ) *is called a coupled fixed point of* F.

Definition 3.2. *Let* $(\mathcal{P}, A_b)$ *be a* A_b*-metric spaces and suppose* $F : \mathcal{P}^2 \to \mathcal{P}$ *and* $f : \mathcal{P} \to \mathcal{P}$ *be two mappings. An element* (ρ, ϱ) *is said to be a coupled coincident point of* F *and* f *if* $F(\rho, \varrho) = f\rho$, $F(\varrho, \rho) = f\varrho$.

Definition 3.3. *Let* $(\mathcal{P}, A_b)$ *be a* A_b*-metric spaces and suppose* $F : \mathcal{P}^2 \to \mathcal{P}$, $f : \mathcal{P} \to \mathcal{P}$ *be two mappings. An element* (ρ, ϱ) *is said to be a coupled common point of* F *and* f *if* $F(\rho, \varrho) = f\rho = \rho$, $F(\varrho, \rho) = f\varrho = \varrho$,

Definition 3.4. *Let* $(\mathcal{P}, A_b)$ *be a* A_b*-metric space. A pair* (F, f) *is called weakly compatible if* $f(F(\rho, \varrho)) = F(f\rho, f\varrho)$ *whenever for all* $\rho, \varrho \in \mathcal{P}$ *such that*
$F(\rho, \varrho) = f\rho$, $F(\varrho, \rho) = f\varrho$.

Theorem 3.1. *Let* $(\mathcal{P}, A_b)$ *be a* A_b*-metric space. Suppose that* $T : \mathcal{P}^2 \to \mathcal{P}$ *and* $f : \mathcal{P} \to \mathcal{P}$ *be a two mappings satisfying the following:*

$$\eta(\theta)A_b(f\lambda, f\lambda, \cdots, (f\lambda)_{n-1}, T(\lambda, \zeta)) \leq \max\left\{\begin{array}{c} A_b(f\lambda, f\lambda, \cdots, (f\lambda)_{n-1}, f\rho), \\ A_b(f\zeta, f\zeta, \cdots, (f\zeta)_{n-1}, f\varrho) \\ A_b(f\lambda, f\lambda, \cdots, (f\lambda)_{n-1}, T(\lambda, \zeta)), \\ A_b(f\zeta, f\zeta, \cdots, (f\zeta)_{n-1}, T(\zeta, \lambda)), \end{array}\right\} \text{ implies}$$

$$A_b(T(\lambda, \zeta), T(\lambda, \zeta), \cdots, (T(\lambda, \zeta))_{n-1}, T(\rho, \varrho)) \leq \theta \max\left\{\begin{array}{c} A_b(f\lambda, f\lambda, \cdots, (f\lambda)_{n-1}, f\rho), \\ A_b(f\zeta, f\zeta, \cdots, (f\zeta)_{n-1} f\varrho), \\ A_b(f\lambda, f\lambda, \cdots, (f\lambda)_{n-1}, T(\lambda, \zeta)), \\ A_b(f\zeta, f\zeta, \cdots, (f\zeta)_{n-1}, T(\zeta, \lambda)), \\ A_b(f\rho, f\rho, \cdots, (f\rho)_{n-1}, T(\rho, \varrho)), \\ A_b(f\varrho, f\varrho, \cdots, (f\varrho)_{n-1}, T(\varrho, \rho)), \\ A_b(f\rho, f\rho, \cdots, (f\rho)_{n-1}, T(\lambda, \zeta)), \\ A_b(f\varrho, f\varrho, \cdots, (f\varrho)_{n-1} T(\zeta, \lambda)) \end{array}\right\}. \tag{3.1}$$

For all $\lambda, \zeta, \rho, \varrho \in \mathcal{P}$, where $\theta \in [0,1)$ and $\eta : [0,1) \to (0,1]$ defined as $\eta(\theta) = \frac{1}{b^2((n-1)+\theta)}$ is a strictly decreasing function,

a) $T(\mathcal{P}^2) \subseteq f(\mathcal{P})$ and $f(\mathcal{P})$ is complete,

b) pair (T, f) is ω-compatible.

Then there is a unique common coupled fixed point of T and f in $\mathcal{P}$.

Proof. Let $\lambda_0, \zeta_0 \in \mathcal{P}$ be arbitrary, and from (a), we construct the sequences $\{\lambda_p\}, \{\zeta_p\}$, in $\mathcal{P}$ as

$$T(\lambda_p, \zeta_p) = f\lambda_{p+1}, \quad T(\zeta_p, \lambda_p) = f\zeta_{p+1}, \text{ where } p = 0, 1, 2, \ldots.$$

Case (i): Assume that

$$f\lambda_p \neq f\lambda_{p+1} \text{ or } f\zeta_p \neq f\zeta_{p+1} \forall\, p. \tag{3.2}$$

Since

$$\begin{aligned} \eta(\theta)A_b(f\lambda_0, f\lambda_0, \cdots, T(\lambda_0,\zeta_0)) &= \eta(\theta)A_b(f\lambda_0, f\lambda_0, \cdots, f\lambda_1) \\ &\leq A_b(f\lambda_0, f\lambda_0, \cdots, f\lambda_1) \\ &\leq \max\left\{\begin{array}{c} A_b(f\lambda_0, f\lambda_0, \cdots, f\lambda_1), \\ A_b(f\zeta_0, f\zeta_0, \cdots, f\zeta_1), \\ A_b(f\lambda_0, f\lambda_0, \cdots, T(\lambda_0,\zeta_0)), \\ A_b(f\zeta_0, f\zeta_0, \cdots, T(\zeta_0,\lambda_0)) \end{array}\right\}. \end{aligned}$$

Then from (3.1), we can get

$$\begin{aligned} A_b(f\lambda_1, f\lambda_1, \cdots, f\lambda_2) &= A_b(T(\lambda_0,\zeta_0), T(\lambda_0,\zeta_0), \cdots, T(\lambda_1,\zeta_1)) \\ &\leq \theta\max\left\{\begin{array}{c} A_b(f\lambda_0, f\lambda_0, \cdots, f\lambda_1), A_b(f\zeta_0, f\zeta_0, \cdots, f\zeta_1), \\ A_b(f\lambda_0, f\lambda_0, \cdots, T(\lambda_0,\zeta_0)), A_b(f\zeta_0, f\zeta_0, \cdots, T(\zeta_0,\lambda_0)), \\ A_b(f\lambda_1, f\lambda_1, \cdots, T(\lambda_1,\zeta_1)), A_b(f\zeta_1, f\zeta_1, \cdots, T(\zeta_1,\lambda_1)), \\ A_b(f\lambda_1, f\lambda_1, \cdots, T(\lambda_0,\zeta_0)), A_b(f\zeta_1, f\zeta_1, \cdots, T(\zeta_0,\lambda_0)) \end{array}\right\} \\ &\leq \theta\max\left\{\begin{array}{c} A_b(f\lambda_0, f\lambda_0, \cdots, f\lambda_1), A_b(f\zeta_0, f\zeta_0, \cdots, f\zeta_1), \\ A_b(f\lambda_1, f\lambda_1, \cdots, f\lambda_2), A_b(f\zeta_1, f\zeta_1, \cdots, f\zeta_2) \end{array}\right\} \end{aligned} \tag{3.3}$$

Similarly, we can prove that

$$A_b(f\zeta_1, f\zeta_1, \cdots, f\zeta_2) \leq \theta\max\left\{\begin{array}{c} A_b(f\lambda_0, f\lambda_0, \cdots, f\lambda_1), \\ A_b(f\zeta_0, f\zeta_0, \cdots, f\zeta_1), \\ A_b(f\lambda_1, f\lambda_1, \cdots, f\lambda_2), \\ A_b(f\zeta_1, f\zeta_1, \cdots, f\zeta_2) \end{array}\right\}. \tag{3.4}$$

Due to (3.3) − (3.4), we conclude that

$$\max\left\{\begin{array}{c} A_b(f\lambda_1, f\lambda_1, \cdots, f\lambda_2), \\ A_b(f\zeta_1, f\zeta_1, \cdots, f\zeta_2) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right), \\ A_b\left(f\lambda_1, f\lambda_1, \cdots, f\lambda_2\right), \\ A_b\left(f\zeta_1, f\zeta_1, \cdots, f\zeta_2\right) \end{array}\right\}. \tag{3.5}$$

If $\max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right) \end{array}\right\} \leq \max\left\{\begin{array}{c} A_b\left(f\lambda_1, f\lambda_1, \cdots, f\lambda_2\right), \\ A_b\left(f\zeta_1, f\zeta_1, \cdots, f\zeta_2\right) \end{array}\right\}$.
Then from (3.5), we have $f\lambda_1 = f\lambda_2$ or $f\zeta_1 = f\zeta_2$. It is contradiction to (3.2).
Hence from (3.5), we have

$$\max\left\{\begin{array}{c} A_b\left(f\lambda_1, f\lambda_1, \cdots, f\lambda_2\right), \\ A_b\left(f\zeta_1, f\zeta_1, \cdots, f\zeta_2\right) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right) \end{array}\right\}.$$

Continuing in this way, we get

$$\begin{aligned}
\max\left\{\begin{array}{c} A_b\left(f\lambda_p, f\lambda_p, \cdots, f\lambda_{p+1}\right), \\ A_b\left(f\zeta_p, f\zeta_p, \cdots, f\zeta_{p+1}\right) \end{array}\right\} &\leq \theta \max\left\{\begin{array}{c} A_b\left(f\lambda_{p-1}, f\lambda_{p-1}, \cdots, f\lambda_p\right), \\ A_b\left(f\zeta_{p-1}, f\zeta_{p-1}, \cdots, f\zeta_p\right) \end{array}\right\} \\
&\leq \theta^2 \max\left\{\begin{array}{c} A_b\left(f\lambda_{p-2}, f\lambda_{p-2}, \cdots, f\lambda_{p-1}\right), \\ A_b\left(f\zeta_{p-2}, f\zeta_{p-2}, \cdots, f\zeta_{p-1}\right) \end{array}\right\} \\
&\vdots \\
&\leq \theta^p \max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right) \end{array}\right\}.
\end{aligned}$$

Thus $A_b\left(f\lambda_p, f\lambda_p, \cdots, f\lambda_{p+1}\right) \leq \theta^p \max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right) \end{array}\right\}$,
and
$A_b\left(f\zeta_p, f\zeta_p, \cdots, f\zeta_{p+1}\right) \leq \theta^p \max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right) \end{array}\right\}$
Now for $q > p$, by use of (A_b3), we have

$$\begin{aligned}
A_b\left(f\lambda_p, f\lambda_p, \cdots, (f\lambda_p)_{n-1}, f\lambda_q\right) &\leq b\left(\begin{array}{c} A_b\left(f\lambda_p, f\lambda_p, \dots\dots\dots, (f\lambda_p)_{n-1}, f\lambda_{p+1}\right) \\ +A_b\left(f\lambda_p, f\lambda_p, \dots\dots\dots, (f\lambda_p)_{n-1}, f\lambda_{p+1}\right) \\ +\cdots\cdots+A_b\left(f\lambda_p, \lambda_p, \dots\dots\dots, (f\lambda_p)_{n-1}, f\lambda_{p+1}\right) \\ +A_b\left(f\lambda_q, f\lambda_q, \dots\dots\dots, (f\lambda_q)_{n-1}, f\lambda_{p+1}\right) \end{array}\right) \\
&\leq b(n-1)A_b\left(f\lambda_p, f\lambda_p, \dots\dots\dots, (f\lambda_p)_{n-1}, f\lambda_{p+1}\right) \\
&\quad +bA_b\left(f\lambda_q, f\lambda_q, \dots\dots\dots, (f\lambda_q)_{n-1}, f\lambda_{p+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq b(n-1)A_b\left(f\lambda_p, f\lambda_p, ..., (f\lambda_p)_{n-1}, f\lambda_{p+1}\right)\\
&\qquad +b^2A_b\left(f\lambda_{p+1}, f\lambda_{p+1}, ..., (f\lambda_{p+1})_{n-1}, f\lambda_q\right)\\
&\leq b(n-1)A_b\left(f\lambda_p, f\lambda_p, ..., (f\lambda_p)_{n-1}, f\lambda_{p+1}\right)\\
&\qquad +b^3(n-1)A_b\left(f\lambda_{p+1}, f\lambda_{p+1}, ..., (f\lambda_{p+1})_{n-1}, f\lambda_{p+2}\right)\\
&\qquad +b^4A_b\left(f\lambda_{p+2}, f\lambda_{p+2}, ..., (f\lambda_{p+2})_{n-1}, f\lambda_q\right)\\
&\leq b(n-1)A_b\left(f\lambda_p, f\lambda_p, ..., (f\lambda_p)_{n-1}, f\lambda_{p+1}\right)\\
&\qquad +b^3(n-1)A_b\left(f\lambda_{p+1}, f\lambda_{p+1}, ..., (f\lambda_{p+1})_{n-1}, f\lambda_{p+2}\right)\\
&\qquad +b^5(n-1)A_b\left(f\lambda_{p+2}, f\lambda_{p+2}, ..., (f\lambda_{p+2})_{n-1}, f\lambda_{p+3}\right)\\
&\qquad +b^7(n-1)A_b\left(f\lambda_{p+3}, f\lambda_{p+3}, ..., (f\lambda_{p+3})_{n-1}, f\lambda_{p+4}\right)\\
&\qquad +\ldots+b^{2q-2p-2}(n-1)A_b\left(f\zeta_{q-2}, f\zeta_{q-2}, ..., (f\zeta_{q-2})_{n-1}, f\lambda_{q-1}\right)\\
&\qquad +b^{2q-2p-3}A_b\left(f\lambda_{q-1}, f\lambda_{q-1}, ..., (f\lambda_{q-1})_{n-1}, f\lambda_q\right)\\
&\leq (n-1)\left(b\theta^p+b^3\theta^{p+1}+b^5\theta^{p+2}+\ldots+b^{2q-2p-2}\theta^{q-2}\right)\max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right)\end{array}\right\}\\
&\qquad +b^{2q-2p-3}\theta^{q-1}\max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right)\end{array}\right\}\\
&\leq (n-1)b\theta^p\left(1+b^2\theta+b^4\theta^2+\ldots+b^{2q-2p-4}\theta^{q-p-2}\right)\max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right)\end{array}\right\}\\
&\qquad +b^{2q-2p-3}\theta^{q-p-1}\max\left\{ A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right)\right\}\\
&\leq (n-1)b\theta^p\left(1+b^2\theta+b^4\theta^2+b^6\theta^3+\ldots\right)\max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right)\end{array}\right\}\\
&\leq \frac{(n-1)b\theta^p}{1-b^2\theta}\max\left\{\begin{array}{c} A_b\left(f\lambda_0, f\lambda_0, \cdots, f\lambda_1\right), \\ A_b\left(f\zeta_0, f\zeta_0, \cdots, f\zeta_1\right)\end{array}\right\}\to 0 \text{ as } p, q\to\infty.
\end{aligned}
$$

Hence $\{f\lambda_p\}$ is a Cauchy sequence in $f(\mathcal{P})$. Similarly we can show that $\{f\zeta_p\}$, is Cauchy sequence in $f(\mathcal{P})$. Since $f(\mathcal{P})$ is complete, there exist α, β and a, b in $\mathcal{P}$ such that

$$\lim_{p\to\infty} f\lambda_p = \alpha = fa \quad \lim_{p\to\infty} f\zeta_p = \beta = fb$$

Since $f\lambda_p \to \alpha$, $f\zeta_p \to \beta$, we may assume that $f\lambda_p \neq \alpha$, $f\zeta_p \neq \beta$ for infinitely many p. We claim that

$$\max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots, T(\lambda,\zeta)\right), \\ A_b\left(fb, fb, \cdots, T(\zeta,\lambda)\right), \end{array}\right\} \leq \theta\max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots, f\lambda\right), A_b\left(fb, fb, \cdots, f\zeta\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), \\ A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right) \end{array}\right\}$$

for all $\lambda, \zeta \in \mathcal{P}$ with $fa \neq f\lambda, fb \neq f\zeta$.
Let $\lambda, \zeta \in \mathcal{P}$ with $fa \neq f\lambda, fb \neq f\zeta$. Then there exists a positive integer p_0 such that for $p \geq p_0$, we have $A_b(fa, fa, \cdots, f\lambda_p) \leq \frac{1}{2b^2(n-1)} A_b(fa, fa, \cdots, f\lambda)$,
$A_b(fb, fb, \cdots, f\zeta_p) \leq \frac{1}{2b^2(n-1)} A_b(fb, fb, \cdots, f\zeta)$.
Now for $p \geq p_0$,

$$\begin{aligned}
& \eta(\theta) A_b(f\lambda_p, f\lambda_p, \cdots, (f\lambda_p)_{n-1}, T(\lambda_p, \zeta_p)) \\
\leq\ & A_b(f\lambda_p, f\lambda_p, \cdots, (f\lambda_p)_{n-1}, T(\lambda_p, \zeta_p)) \\
=\ & A_b(f\lambda_p, f\lambda_p, \cdots, (f\lambda_p)_{n-1}, f\lambda_{p+1}) \\
\leq\ & b(n-1) A_b(f\lambda_p, f\lambda_p, \cdots (f\lambda_p)_{n-1}, fa) \\
& \quad + b^2 A_b(fa, fa, \cdots, (fa)_{n-1}, f\lambda_{p+1}) \\
\leq\ & b^2(n-1) A_b(fa, fa \cdots, (fa)_{n-1}, f\lambda_p) \\
& \quad + b^2 A_b(fa, fa, \cdots, (fa)_{n-1}, f\lambda_{p+1}) \\
\leq\ & b^2(n-1) A_b(fa, fa \cdots, (fa)_{n-1}, f\lambda_p) \\
& \quad + b^2(n-1) A_b(fa, fa, \cdots, (fa)_{n-1}, f\lambda_{p+1}) \\
\leq\ & \frac{1}{2} A_b(fa, fa \cdots, (fa)_{n-1}, f\lambda) \\
& \quad + \frac{1}{2} A_b(fa, fa, \cdots, (fa)_{n-1}, f\lambda) \\
\leq\ & A_b(fa, fa \cdots, (fa)_{n-1}, f\lambda) \\
\leq\ & A_b(f\lambda, f\lambda, \cdots, f\lambda_p) \\
\leq\ & \max \left\{ \begin{array}{c} A_b(f\lambda, f\lambda, \cdots, f\lambda_p), A_b(f\zeta, f\zeta, \cdots, f\zeta_p), \\ A_b(f\lambda_p, f\lambda_p, \cdots, T(\lambda_p, \zeta_p)), A_b(f\zeta_p, f\zeta_p, \cdots, T(\zeta_p, \lambda_p)) \end{array} \right\}.
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
& A_b(T(\lambda_p, \zeta_p), T(\lambda_p, \zeta_p), \cdots, T(\lambda, \zeta)) \\
\leq\ & \theta \max \left\{ \begin{array}{c} A_b(f\lambda_p, f\lambda_p, f\lambda), A_b(f\zeta_p, f\zeta_p, f\zeta), \\ A_b(f\lambda_p, f\lambda_p, f\lambda_{p+1}), A_b(f\zeta_p, f\zeta_p, f\zeta_{p+1}), \\ A_b(f\lambda, f\lambda, \cdots, T(\lambda, \zeta)), A_b(f\zeta, f\zeta, \cdots, T(\zeta, \lambda)), \\ A_b(f\lambda, f\lambda, \cdots, f\lambda_{p+1}), A_b(f\zeta, f\zeta, \cdots, f\zeta_{p+1}) \end{array} \right\}.
\end{aligned}$$

Letting $p \to \infty$, we get

$$A_b(fa, fa, \cdots, T(\lambda, \zeta)) \leq \theta \max \left\{ \begin{array}{c} A_b(fa, fa, \cdots, f\lambda), A_b(fb, fb, \cdots, f\zeta), \\ A_b(f\lambda, f\lambda, \cdots, T(\lambda, \zeta)), A_b(f\zeta, f\zeta, \cdots, T(\zeta, \lambda)) \end{array} \right\}.$$

Similarly we can show that

$$A_b\left(fb, fb, \cdots, T(\zeta,\lambda)\right) \leq \theta \max\left\{\begin{array}{c} A_b\left(fb, fb, \cdots, f\zeta\right), A_b\left(fa, fa, \cdots, f\lambda\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right) \end{array}\right\}.$$

Thus

$$\max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots, T(\lambda,\zeta)\right), \\ A_b\left(fb, fb, \cdots, T(\zeta,\lambda)\right), \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots . f\lambda\right), A_b\left(fb, fb, \cdots, f\zeta\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), \\ A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right) \end{array}\right\}.$$

Hence the claim. Now consider

$$\begin{aligned} A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right) &\leq (n-1)bA_b\left(f\lambda, f\lambda, \cdots, fa\right) + b^2 A_b\left(fa, fa, \cdots, T(\lambda,\zeta)\right) \\ &\leq (n-1)b^2 A_b\left(fa, fa, \cdots, f\lambda\right) \\ &\quad + b^2\theta \max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots, f\lambda\right), A_b\left(fb, fb, \cdots, f\zeta\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right), \end{array}\right\} \\ &\leq b^2\left((n-1)+\theta\right) \max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots, f\lambda\right), A_b\left(fb, fb, \cdots, f\zeta\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), \\ A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right) \end{array}\right\}. \end{aligned}$$

Thus

$$\eta(\theta) A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right) \leq \max\left\{\begin{array}{c} A_b\left(fa, fa, \cdots, f\lambda\right), A_b\left(fb, fb, \cdots, f\zeta\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right) \end{array}\right\}.$$

Hence from (3.1), we have

$$\begin{aligned} & A_b\left(T(\lambda,\zeta), T(\lambda,\zeta), \cdots, T(a,b)\right) \\ \leq\ & \theta \max\left\{\begin{array}{c} A_b\left(f\lambda, f\lambda, \cdots, fa\right), A_b\left(f\zeta, f\zeta, \cdots, fb\right), \\ A_b\left(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)\right), A_b\left(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)\right), \\ A_b\left(fa, fa, \cdots, T(a,b)\right), A_b\left(fb, fb, \cdots, T(b,a)\right), \\ A_b\left(fa, fa, \cdots, T(\lambda,\zeta)\right), A_b\left(fb, fb, \cdots, T(\zeta,\lambda)\right) \end{array}\right\}. \end{aligned}$$

Now

$$\begin{aligned} & A_b\left(fa, fa, \cdots, T(a,b)\right) = \lim_{p\to\infty} A_b\left(f\lambda_{p+1}, f\lambda_{p+1}, \cdots, T(a,b)\right) \\ =\ & \lim_{p\to\infty} A_b\left(T(\lambda_p, y_p), T(\lambda_p,\zeta_p), \cdots, T(a,b)\right) \\ \leq\ & \lim_{p\to\infty} \theta \max\left\{\begin{array}{c} A_b\left(f\lambda_p, f\lambda_p, \cdots, fa\right), A_b\left(f\zeta_p, f\zeta_p, \cdots fb\right), \\ A_b\left(fa, fa, \cdots, T(a,b)\right), A_b\left(fb, fb, \cdots, T(b,a)\right), \\ A_b\left(f\lambda_p, f\lambda_p, \cdots, T(\lambda_p,\zeta_p)\right), A_b\left(f\zeta_p, f\zeta_p, \cdots, T(\zeta_p,\lambda_p)\right), \\ A_b\left(fa, fa, \cdots, T(\lambda_p,\zeta_p)\right), A_b\left(fb, fb, \cdots, T(\zeta_p,\lambda_p)\right), \end{array}\right\} \\ \leq\ & \theta \max\left\{ A_b\left(fa, fa, \cdots, T(a,b)\right), A_b\left(fb, fb, \cdots, T(b,a)\right) \right\}. \end{aligned}$$

Similarly, we can have

$$A_b(fb, fb, \cdots, T(b,a)) \leq \theta \max\left\{ A_b(fa, fa, \cdots, T(a,b)), A_b(fb, fb, \cdots, T(b,a)) \right\}$$

Thus

$$\max\left\{ \begin{array}{c} A_b(fa, fa, \cdots, T(a,b)), \\ A_b(fb, fb, \cdots, T(b,a)), \end{array} \right\} \leq \theta \max\left\{ \begin{array}{c} A_b(fa, fa, \cdots, T(a,b)), \\ A_b(fb, fb, \cdots, T(b,a)) \end{array} \right\}.$$

So that $T(a,b) = fa$ and $T(b,a) = fb$. Thus (a,b) is a coupled coincidence point of T and f . Since the pair (T,f) is ω-compatible, we have

$$\begin{aligned} f\alpha &= f^2a = f(T(a,b)) = T(fa, fb) = T(\alpha, \beta) \\ f\beta &= f^2b = f(T(b,a)) = T(fb, fa) = T(\beta, \alpha) \end{aligned} \tag{3.6}$$

Now

$$\eta(\theta) A_b(f\alpha, f\alpha, \cdots, T(\alpha,\beta)) = 0 \leq \max\left\{ \begin{array}{c} A_b(fa, fa, \cdots, f\alpha), A_b(fb, fb, \cdots, f\beta), \\ A_b(f\alpha, f\alpha, \cdots, T(\alpha,\beta)), A_b(f\beta, f\beta, \cdots, T(\beta,\alpha)) \end{array} \right\}.$$

Hence from (3.1), we have

$$\begin{aligned} A_b(f\alpha, f\alpha, \cdots, fa) &= A_b(T(\alpha,\beta), T(\alpha,\beta), \cdots, T(a,b)) \\ &\leq \theta \max\left\{ \begin{array}{c} A_b(f\alpha, f\alpha, \cdots, fa), A_b(f\beta, f\beta, \cdots, fb), \\ A_b(f\alpha, f\alpha, \cdots, T(\alpha,\beta)), A_b(f\beta, f\beta, \cdots, T(\beta,\alpha)), \\ A_b(fa, fa, \cdots, T(a,b)), A_b(fb, fb, \cdots, T(b,a)), \\ A_b(fa, fa, \cdots, T(\alpha,\beta)), A_b(fb, fb, \cdots, T(\beta,\alpha)) \end{array} \right\}. \\ &\leq \theta \max\left\{ A_b(f\alpha, f\alpha, \cdots, fa), A_b(f\beta, f\beta, \cdots, fb) \right\}. \end{aligned}$$

Similarly, we have

$$A_b(f\beta, f\beta, \cdots, fb) \leq \theta \max\left\{ A_b(f\alpha, f\alpha, \cdots, fa), A_b(f\beta, f\beta, \cdots, fb) \right\}.$$

Thus

$$\max\left\{ A_b(f\alpha, f\alpha, \cdots, fa), A_b(f\beta, f\beta, \cdots, fb), \right\} \leq \theta \max\left\{ \begin{array}{c} A_b(f\alpha, f\alpha, \cdots, fa), \\ A_b(f\beta, f\beta, \cdots, fb) \end{array} \right\}.$$

Hence $\alpha = fa = f\alpha$ and $\beta = fb = f\beta$,. Hence from (3.6), we have (α,β) is a common coupled fixed point of T and f. In the following we will show the uniqueness of common coupled fixed point in $\mathcal{P}$. For this purpose, assume that there is another coupled fixed point (α',β') of T, f. Now consider,

$$\eta(\theta) A_b(f\alpha, f\alpha, \cdots, T(\alpha,\beta)) = 0 \leq \max\left\{ \begin{array}{c} A_b(f\alpha, f\alpha, \cdots, f\alpha'), A_b(f\beta, f\beta, \cdots, f\beta'), \\ A_b(f\alpha, f\alpha, \cdots, T(\alpha,\beta)), A_b(f\beta, f\beta, \cdots, T(\beta,\alpha)) \end{array} \right\}$$

by (3.1), we have

$$\begin{aligned} A_b(\alpha,\alpha,\cdots,\alpha') &= A_b\left(T(\alpha,\beta),T(\alpha,\beta),\cdots,T(\alpha',\beta')\right) \\ &\leq \theta \max\left\{ A_b\left(\alpha,\alpha,\cdots,\alpha'\right), A_b\left(\beta,\beta,\cdots,\beta'\right) \right\}. \end{aligned}$$

Similarly, we can show that

$$A_b(\beta,\beta,\cdots,\beta') \leq \theta \max\left\{ A_b\left(\alpha,\alpha,\cdots,\alpha'\right), A_b\left(\beta,\beta,\cdots,\beta'\right) \right\}.$$

Thus

$$\max\left\{ A_b\left(\alpha,\alpha,\cdots,\alpha'\right), A_b\left(\beta,\beta,\cdots,\beta'\right) \right\} \leq \theta \max\left\{ A_b\left(\alpha,\alpha,\cdots,\alpha'\right), A_b\left(\beta,\beta,\cdots,\beta'\right) \right\}.$$

Hence $\alpha = \alpha', \beta = \beta'$. Thus (α,β) is the unique common coupled fixed point of T and f.
case(ii): If $f\lambda_p = f\lambda_{p+1}, f\zeta_p = f\zeta_{p+1}$ for some p then
$f\lambda_p = T(\lambda_p,\zeta_p)$, $f\zeta_p = T(\zeta_p,\lambda_p)$ so that (λ_p,ζ_p) is a coupled coincidence point of T and f . Now proceeding as in case (i) with $f\lambda_p = \alpha$, $f\zeta_p = \beta$, we can show that (α,β) is the unique common coupled fixed point of T and f. □

Example 3.1. *Let $\mathcal{P} = [0,+\infty)$, define $A_b : \mathcal{P}^n \to [0,+\infty)$ as $A_b(\lambda_1,\lambda_2,........,\lambda_{n-1},\lambda_n) = \sum_{i=1}^n \sum_{i<j} |\lambda_i - \lambda_j|^2$ for all $\lambda_i \in \mathcal{P}, i = 1,2,\cdots$. Then $(\mathcal{P}, A_b)$ is an complete A_b-metric space with $b = 2$.*

Let $T : \mathcal{P}^2 \to \mathcal{P}$ and $f : \mathcal{P} \to \mathcal{P}$ be given by $T(\lambda,\zeta) = \frac{4\lambda-2\zeta+16n-2}{16n}$ and $f(\lambda) = \frac{6\lambda+32n-6}{32n}$. Then obviously, $T(\mathcal{P}^2) \subseteq f(\mathcal{P})$ and the pair (T,f) are ω-compatible and clearly for all $a,b \in \mathcal{P}$

$$\begin{aligned} \frac{4}{5}A_b(fa,fa,\cdots,T(a,b)) &\leq A_b(fa,fa,\cdots,T(a,b)) \\ &\leq \max\left\{ \begin{array}{c} A_b(fa,fa,\cdots,f\lambda), A_b(fb,fb,\cdots,f\zeta), \\ A_b(fa,fa,\cdots,T(a,b)), A_b(fb,fb,\cdots,T(b,a)), \end{array} \right\} \end{aligned}$$

Now we have

$$\begin{aligned} & A_b(T(a,b),T(a,b),\cdots,T(\lambda,\zeta)) \\ = \ & (n-1)|T(a,b) - T(\lambda,\zeta)|^2 \\ = \ & (n-1)|\frac{4a-2b+16n-2}{16n} - \frac{4\lambda-2\zeta+16n-2}{16n})|^2 \\ = \ & (n-1)|\frac{4a-2b-4\lambda+2\zeta}{16n}|^2 \\ \leq \ & \frac{(n-1)}{2}|\frac{6a+32n-6}{32n} - \frac{4\lambda-2\zeta+16n-2}{16n}|^2 \\ \leq \ & \frac{(n-1)}{2}|fa - T(\lambda,\zeta)|^2 = \frac{1}{2}A_b(fa,fa,\cdots,T(\lambda,\zeta)) \end{aligned}$$

$$\leq \frac{1}{2}\max\left\{\begin{array}{c} A_b(fa, fa, \cdots, f\lambda), A_b(fb, fb, \cdots, f\zeta), \\ A_b(fa, fa, \cdots, T(a,b)), A_b(fb, fb, \cdots, T(b,a)), \\ A_b(f\lambda, f\lambda, \cdots, T(\lambda,\zeta)), A_b(f\zeta, f\zeta, \cdots, T(\zeta,\lambda)), \\ A_b(f\lambda, f\lambda, \cdots, T(a,b)), A_b(f\zeta, f\zeta, \cdots, T(b,a)) \end{array}\right\}.$$

Put $n = 2$ and $b = 2$, since $\theta = \frac{1}{2} < 1$.Thus all the conditions of the Theorem 3.1 are satisfied and $(1, 1)$ is unique common coupled fixed point of T and f.

Corollary 3.1. *Let $(\mathcal{P}, A_b)$ be a complete A_b-metric space. Suppose that $T : \mathcal{P}^2 \to \mathcal{P}$ be a mapping satisfying:*

$$\eta(\theta)A_b(\lambda, \lambda, \cdots, T(\lambda,\zeta)) \leq \max\left\{\begin{array}{c} A_b(\lambda, \lambda, \cdots, \rho), A_b(\zeta, \zeta, \cdots, \varrho), \\ A_b(\lambda, \lambda, \cdots, T(\lambda,\zeta)), A_b(\zeta, \zeta, \cdots, T(\zeta,\lambda)) \end{array}\right\} \text{ implies}$$

$$A_b(T(\lambda,\zeta), T(\lambda,\zeta), \cdots, T(\rho,\varrho)) \leq \theta\max\left\{\begin{array}{c} A_b(\lambda, \lambda, \cdots, \rho), A_b(\zeta, \zeta, \cdots, \varrho), \\ A_b(\lambda, \lambda, \cdots, T(\lambda,\zeta)), A_b(\zeta, \zeta, \cdots, T(\zeta,\lambda)), \\ A_b(\rho, \rho, \cdots, T(\rho,\varrho)), A_b(\varrho, \varrho, \cdots, T(\varrho,\rho)), \\ A_b(\rho, \rho, \cdots, T(\lambda,\zeta)), A_b(\varrho, \varrho, \cdots, T(\zeta,\lambda)) \end{array}\right\}.$$

for all $\lambda, \zeta, \rho, \varrho \in \mathcal{P}$, where $\theta \in [0, 1)$ and $\eta : [0, 1) \to (0, 1]$ defined as $\eta(\theta) = \frac{1}{b^2((n-1)+\theta)}$ is a strictly decreasing function. Then there is a unique coupled fixed point of T in $\mathcal{P}$.

4. Application to Integral Equations

In this section, we study the existence of an unique solution to an initial value problem, as an application to Corollary 3.1.

Theorem 4.1. *Consider the initial value problem*

$$\lambda'(t) = T(t, (\lambda,\zeta)(t)),\ t \in I = [0, 1],\ (\lambda,\zeta)(0) = (\lambda_0, \zeta_0) \tag{4.1}$$

where $T : I \times \mathbb{R} \to \mathbb{R}$ with $\int_0^t T(s, (\lambda,\zeta)(s))ds = \max\left\{ \int_0^t T(s, \lambda(s))ds, \int_0^t T(s, \zeta(s))ds \right\}$ and $\lambda_0, \zeta_0 \in \mathbb{R}$. Then there exists unique solution in $C(I, \mathbb{R})$ for the initial value problem (4.1).

Proof. The integral equation corresponding to initial value problem (4.1) is

$$\lambda(t) = \lambda_0 + \frac{\sqrt{n-1}}{2}b^2\int_0^t T(s, (\lambda,\zeta)(s))ds.$$

Let $\mathcal{P} = C(I, \mathbb{R})$ and $A_b(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) = \sum_{i=1}^n \sum_{i<j} |\lambda_i - \lambda_j|^2$ for all $\lambda_i \in \mathcal{P}, i = 1, 2, \cdots,$ define $R : \mathcal{P}^2 \to \mathcal{P}$ by

$$R(\lambda,\zeta)(t) = \frac{2\lambda_0}{\sqrt{(n-1)b^2}} + \int_0^t T(s, (\lambda,\zeta)(s))ds.$$

Clearly, for all $\lambda, \zeta \in \mathcal{P}$, we have

$$\frac{3}{5}A_b(\lambda, \lambda, \cdots, R(\lambda, \zeta)) \le \max\left\{\begin{array}{c} A_b(\lambda, \lambda, \cdots, a), A_b(\zeta, \zeta, \cdots, b), \\ A_b(\lambda, \lambda, \cdots, R(\lambda, \zeta)), A_b(\zeta, \zeta, \cdots, R(\zeta, \lambda)) \end{array}\right\}.$$

Now

$$\begin{aligned} & A_b(R(\lambda, \zeta)(t), R(\lambda, \zeta)(t), \cdots, R(a, b)(t)) \\ = \ & (n-1)|R(\lambda, \zeta)(t) - R(a, b)(t)|^2 \\ = \ & |\frac{2\lambda_0}{\sqrt{(n-1)b^2}} + \int_0^t T(s, (\lambda, \zeta)(s))ds - \frac{2a_0}{\sqrt{(n-1)b^2}} - \int_0^t T(s, (a, b)(s))ds|^2 \\ = \ & \frac{4}{b^4}|\lambda(t) - a(t)|^2 \le \frac{1}{2}(n-1)|\lambda(t) - a(t)|^2 \le \frac{1}{2}A_b(\lambda, \lambda, \cdots, a) \\ \le \ & \frac{1}{2}\max\left\{\begin{array}{c} A_b(\lambda, \lambda, \cdots, a), A_b(\zeta, \zeta, \cdots, b), \\ A_b(\lambda, \lambda, \cdots, R(\lambda, \zeta)), A_b(\zeta, \zeta, \cdots, R(\zeta, \lambda)), \\ A_b(a, a, \cdots, R(a, b)), A_b(b, b, \cdots, R(b, a)), \\ A_b(a, a, \cdots, R(\lambda, \zeta)), A_b(b, b, \cdots, R(\zeta, \lambda)) \end{array}\right\}. \end{aligned}$$

It follows from Corollary 3.1, we conclude that R has a unique fixed point in $\mathcal{P}$. □

5. Application to Homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 5.1. *Let $(\mathcal{P}, A_b)$ be complete A_b-metric space, U and $\overline{U}$ be an open and closed subset of $\mathcal{P}$ such that $U \subseteq \overline{U}$. Suppose $H : \overline{U}^2 \times [0, 1] \to \mathcal{P}$ be an operator with following conditions are satisfying,*
τ_0) $\lambda \neq H(\lambda, \zeta, \kappa)$, $\zeta \neq H(\zeta, \lambda, \kappa)$, for each $\lambda, \zeta \in \partial U$ and $\kappa \in [0, 1]$ (Here ∂U is boundary of U in $\mathcal{P}$);
τ_1) for all $\lambda, \zeta, \rho, \varrho \in \overline{U}$ and $\kappa \in [0, 1]$ such that

$$\eta(\theta)A_b(\lambda, \lambda, \cdots, H(\lambda, \zeta, \kappa)) \le \max\left\{\begin{array}{c} A_b(\lambda, \lambda, \cdots, \rho), A_b(\zeta, \zeta, \cdots, \varrho), \\ A_b(\lambda, \lambda, \cdots, H(\lambda, \zeta, \kappa)), A_b(\zeta, \zeta, \cdots, H(\zeta, \lambda, \kappa)) \end{array}\right\} \text{ implies}$$

$$A_b\Big(H(\lambda, \zeta, \kappa), H(\lambda, \zeta, \kappa), \cdots, H(\rho, \varrho, \kappa)\Big) \le \theta \max\left\{\begin{array}{c} A_b(\lambda, \lambda, \cdots, \rho), A_b(\zeta, \zeta, \cdots, \varrho), \\ A_b(\lambda, \lambda, \cdots, H(\lambda, \zeta, \kappa)), \\ A_b(\zeta, \zeta, \cdots, H(\zeta, \lambda, \kappa)), \\ A_b(\rho, \rho, \cdots, H(\rho, \varrho, \kappa)), \\ A_b(\varrho, \varrho, \cdots, H(\varrho, \rho, \kappa)), \\ A_b(\rho, \rho, \cdots, H(\lambda, \zeta, \kappa)), \\ A_b(\varrho, \varrho, \cdots, H(\zeta, \lambda, \kappa)) \end{array}\right\}. \quad (5.1)$$

where $\theta \in [0, 1)$ and $\eta : [0, 1) \to (0, 1]$ defined as $\eta(\theta) = \frac{1}{b^2((n-1)+\theta)}$ is a strictly decreasing function.
τ_2) $\exists\ M \ge 0 \ni A_b(H(\lambda, \zeta, \kappa), H(\lambda, \zeta, \kappa), \cdots, H(\lambda, \zeta, \zeta)) \le M|\kappa - \zeta|$

for every $\lambda,\zeta \in \overline{U}$and $\kappa,\zeta \in [0,1]$.
Then $H(.,0)$ has a coupled fixed point $\iff$ $H(.,1)$ has a coupled fixed point.

Proof. Let the set

$$\mathcal{P}=\left\{ \ \kappa\in[0,1]:H(\lambda,\zeta,\kappa)=\lambda, H(\zeta,\lambda,\kappa)=\zeta \text{ for some } \lambda,\zeta\in U \ \right\}.$$

Suppose that $H(.,0)$ has a coupled fixed point in U^2, we have that $(0,0)\in\mathcal{P}^2$. So that $\mathcal{P}$ is non-empty set. Now we show that $\mathcal{P}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathcal{P}=[0,1]$. As a result, $H(.,1)$ has a coupled fixed point in U^2. First we show that $\mathcal{P}$ closed in $[0,1]$. To see this, Let $\{\kappa_p\}_{p=1}^{\infty}\subseteq\mathcal{P}$ with $\kappa_p\to\kappa\in[0,1]$ as $p\to\infty$. We must show that $\kappa\in\mathcal{P}$. Since $\kappa_p\in\mathcal{P}$ for $p=0,1,2,3,\cdots$, there exists sequences $\{\lambda_p\},\{\zeta_p\}$ with $\lambda_{p+1}=H(\lambda_p,\zeta_p,\kappa_p)$, $\zeta_{p+1}=H(\zeta_p,\lambda_p,\kappa_p)$.
Consider

$$\begin{aligned}
& A_b(\lambda_p,\lambda_p,\cdots,(\lambda_p)_{n-1},H(\lambda_p,\zeta_p,\kappa_p)) \\
= \ & A_b\left(H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}),H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}),\cdots,(H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}))_{n-1},H(\lambda_p,\zeta_p,\kappa_p)\right) \\
\leq \ & \begin{array}{c} b(n-1)A_b\left(\ H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}),H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}),\cdots,H(\lambda_{p-1},\zeta_{p-1},\kappa_p) \ \right) \\ +b^2A_b\left(\ H(\lambda_{p-1},\zeta_{p-1},\kappa_p),H(\lambda_{p-1},\zeta_{p-1},\kappa_p),\cdots,H(\lambda_p,\zeta_p,\kappa_p) \ \right) \end{array} \\
\leq \ & \begin{array}{c} b(n-1)A_b\left(\ H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}),H(\lambda_{p-1},\zeta_{p-1},\kappa_{p-1}),\cdots,H(\lambda_{p-1},\zeta_{p-1},\kappa_p) \ \right) \\ +b^2\theta\max\left\{ \begin{array}{c} A_b(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}),A_b(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}), \\ A_b(\lambda_p,\lambda_p,\cdots,H(\lambda_{p-1},\zeta_{p-1},\kappa_p)), \\ A_b(\zeta_p,\zeta_p,\cdots,H(\zeta_{p-1},\lambda_{p-1},\kappa_p)) \end{array} \right\} \end{array} \\
\leq \ & b^2((n-1)+\theta)\max\left\{ \begin{array}{c} A_b(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}),A_b(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}), \\ A_b(\lambda_p,\lambda_p,\cdots,H(\lambda_{p-1},\zeta_{p-1},\kappa_p)), \\ A_b(\zeta_p,\zeta_p,\cdots,H(\zeta_{p-1},\lambda_{p-1},\kappa_p)) \end{array} \right\} \\
\leq \ & b^2((n-1)+\theta)\max\left\{ \begin{array}{c} A_b(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}),A_b(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}), \\ A_b(H(\lambda_p,\zeta_p,\kappa_p),H(\lambda_p,\zeta_p,\kappa_p),\cdots,H(\lambda_p,\zeta_p,\kappa_{p+1})), \\ A_b(H(\zeta_p,\lambda_p.\kappa_p),H(\zeta_p,\lambda_p.\kappa_p),\cdots,H(\zeta_p,\lambda_p,\kappa_{p+1})) \end{array} \right\} \\
\leq \ & b^2((n-1)+\theta)\max\left\{ \begin{array}{c} A_b(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}),A_b(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}), \\ M|\kappa_p-\kappa_{p+1}| \end{array} \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \eta(\theta)A_b(\lambda_p,\lambda_p,\cdots,(\lambda_p)_{n-1},H(\lambda_p,\zeta_p,\kappa_p)) \\
\leq \ & \max\left\{ \begin{array}{c} A_b(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}),A_b(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}), \\ A_b(H(\lambda_p,\zeta_p,\kappa_p),H(\lambda_p,\zeta_p,\kappa_p),\cdots,H(\lambda_p,\zeta_p,\kappa_{p+1})), \\ A_b(H(\zeta_p,\lambda_p.\kappa_p),H(\zeta_p,\lambda_p.\kappa_p),\cdots,H(\zeta_p,\lambda_p,\kappa_{p+1})) \end{array} \right\}.
\end{aligned}$$

from (5.1), we have that

$$
\begin{aligned}
& A_b(H(\lambda_p,\zeta_p,\kappa_p)H(\lambda_p,\zeta_p,\kappa_p),\cdots,H(\lambda_{p+1},\zeta_{p+1},\kappa_{p+1})) \\
\leq \quad & \theta\max\left\{\begin{array}{c} A_b\left(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}\right), A_b\left(\zeta_p,\zeta_p,\cdots,\zeta_{q+1}\right), \\ A_b\left(\lambda_p,\lambda_p,\cdots,H(\lambda_p,\zeta_p,\kappa_p)\right), \\ A_b\left(\zeta_p,\zeta_p,\cdots,H(\zeta_p,\lambda_p,\kappa_p)\right), \\ A_b\left(\lambda_{p+1},\lambda_{p+1},\cdots,H(\lambda_{p+1},\zeta_{p+1},\kappa_{p+1})\right), \\ A_b\left(\zeta_{p+1},\zeta_{p+1},\cdots,H(\zeta_{p+1},\lambda_{p+1},\kappa_{p+1})\right), \\ A_b\left(\lambda_{p+1},\lambda_{p+1},\cdots,H(\lambda_p,\zeta_p,\kappa_p)\right), \\ A_b\left(\zeta_{p+1},\zeta_{p+1},\cdots,H(\zeta_p,\lambda_p,\kappa_p)\right) \end{array}\right\} \\
\leq \quad & \theta\max\left\{\begin{array}{c} A_b\left(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}\right), A_b\left(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}\right), \\ A_b\left(\lambda_{p+1},\lambda_{p+1},\cdots,\lambda_{p+2}\right), \\ A_b\left(\zeta_{p+1},\zeta_{p+1},\cdots,y_{p+2}\right) \end{array}\right\}.
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
A_b(H(\lambda_p,\zeta_p,\kappa_p),H(\lambda_p,\zeta_p,\kappa_p),\cdots,H(\lambda_{p+1},\zeta_{p+1},\kappa_{p+1})) \quad & \leq \quad \theta\max\left\{\begin{array}{c} A_b\left(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}\right), \\ A_b\left(\zeta_p,\zeta_p,\cdots,\zeta_{p+1}\right) \end{array}\right\} \\
& \vdots \\
& \leq \quad \theta^p\max\left\{\begin{array}{c} A_b\left(\lambda_0,\lambda_0,\cdots,\lambda_1\right), \\ A_b\left(\zeta_0,\zeta_0,\cdots,\zeta_1\right) \end{array}\right\}.
\end{aligned}
$$

Similarly, we can show that

$$
A_b(H(\zeta_p,\lambda_p,\kappa_p),H(\zeta_p,\lambda_p,\kappa_p),\cdots,H(\zeta_{p+1},\lambda_{p+1},\kappa_{p+1})) \quad \leq \quad \theta^p\max\left\{\begin{array}{c} A_b\left(\lambda_0,\lambda_0,\cdots,\lambda_1\right), \\ A_b\left(\zeta_0,\zeta_0,\cdots,\zeta_1\right) \end{array}\right\}.
$$

Thus

$$
\begin{aligned}
& \max\left\{\begin{array}{c} A_b(H(\lambda_p,\zeta_p,\kappa_p),H(\lambda_p,\zeta_p,\kappa_p),\cdots,H(\lambda_{p+1},\zeta_{p+1},\kappa_{p+1})), \\ A_b(H(\zeta_p,\lambda_p,\kappa_p),H(\zeta_p,\lambda_p,\kappa_p),\cdots,H(\zeta_{p+1},\lambda_{p+1},\kappa_{p+1})) \end{array}\right\} \\
\leq \quad & \theta^p\max\left\{\begin{array}{c} A_b\left(\lambda_0,\lambda_0,\cdots,\lambda_1\right), \\ A_b\left(\zeta_0,\zeta_0,\cdots,\zeta_1\right) \end{array}\right\}. \to o \text{ as } p\to\infty.
\end{aligned}
$$

Now for $q>p$, by use of (A_b3), we have

$$
A_b\left(\lambda_p,\lambda_p,\cdots,\lambda_q\right) \leq b(n-1)A_b\left(\lambda_p,\lambda_p,\cdots,\lambda_{p+1}\right) + b^2A_b\left(\lambda_{p+1},\lambda_{p+1},\cdots,\lambda_q\right)
$$

Letting $p\to\infty$, we get

$$
\begin{aligned}
& \lim_{p\to\infty} A_b\left(\lambda_p,\lambda_p,\cdots,\lambda_q\right) \\
\leq \quad & \lim_{p\to\infty} b^2A_b\left(H(\lambda_p,\zeta_p,\kappa_p),H(\lambda_p,\zeta_p,\kappa_p),\cdots,H(\lambda_{q-1},\zeta_{q-1},\kappa_{q-1})\right) \\
\leq \quad & \lim_{p\to\infty} b^{2q-2p-3}A_b\left(H(\zeta_{q-2},\zeta_{q-2},\kappa_{q-2}),H(\zeta_{q-2},\zeta_{q-2},\kappa_{q-2}),\cdots,H(\lambda_{q-1},\zeta_{q-1},\kappa_{q-1})\right)
\end{aligned}
$$

$$\leq \lim_{p\to\infty} b^{2q-2p-3}\theta \max\left\{\begin{array}{c} A_b\left(\zeta_{q-2},\zeta_{q-2},\cdots,\lambda_{q-1}\right), A_b\left(\zeta_{q-2},\zeta_{q-2},\cdots,\zeta_{q-1}\right),\\ A_b\left(\zeta_{q-2},\zeta_{q-2},\cdots,H(\zeta_{q-2},\zeta_{q-2},\kappa_{q-2})\right),\\ A_b\left(\zeta_{q-2},\zeta_{q-2},\cdots,H(\zeta_{q-2},\zeta_{q-2},\kappa_{q-2})\right),\\ A_b\left(\lambda_{q-1},\lambda_{q-1},\cdots,H(\lambda_{q-1},y_{q-1},\kappa_{q-1})\right),\\ A_b\left(\zeta_{q-1},\zeta_{q-1},\cdots,H(\zeta_{q-1},\lambda_{q-1},\kappa_{q-1})\right),\\ A_b\left(\lambda_{q-1},\lambda_{q-1},\cdots,H(\zeta_{q-2},\zeta_{q-2},\kappa_{q-2})\right),\\ A_b\left(\zeta_{q-1},\zeta_{q-1},\cdots,H(\zeta_{q-2},\zeta_{q-2},\kappa_{q-2})\right) \end{array}\right\}.$$

$$\leq \lim_{p\to\infty} b^{2q-2p-3}\theta^{q-p-1}\max\left\{ A_b\left(\lambda_0,\lambda_0,\cdots,\lambda_1\right), A_b\left(\zeta_0,\zeta_0,\cdots,\zeta_1\right)\right\}\to 0 \text{ as } q\to\infty.$$

Hence $\{\lambda_p\}$ is a Cauchy sequence in A_b metric spaces $(\mathcal{P}, A_b)$. Similarly we can show that $\{\zeta_p\}$, is Cauchy sequence in $(\mathcal{P}, A_b)$ and by the completeness of $(\mathcal{P}, A_b)$, there exist $a, b \in \mathcal{P}$ with

$$\lim_{p\to\infty}\lambda_{p+1} = a \quad \lim_{p\to\infty}\zeta_{p+1} = b \tag{5.2}$$

Since

$$\eta(\theta)A_b\left(a,a,\cdots,H(a,b,\kappa)\right) \leq \max\left\{\begin{array}{c} A_b\left(a,a,\cdots,\lambda_p\right), A_b\left(b,b,\cdots,\zeta_p\right),\\ A_b\left(a,a,\cdots,H(a,b,\kappa)\right), A_b\left(b,b,\cdots,H(b,a,\kappa)\right)\end{array}\right\}$$

we have

$$\begin{aligned} & A_b\left(a,a,\cdots,H(a,b,\kappa)\right)\\ \leq\ & \lim_{p\to\infty} A_b\left(H(\lambda_p,\zeta_p,\kappa),H(\lambda_p,\zeta_p,\kappa),\cdots,H(a,b,\kappa)\right)\\ \leq\ & \lim_{n\to\infty}\theta\max\left\{\begin{array}{c} A_b\left(\lambda_p,\lambda_p,\cdots,a\right), A_b\left(\zeta_p,\zeta_p,\cdots,b\right),\\ A_b\left(a,a,\cdots,H(a,b,\kappa)\right), A_b\left(b,b,\cdots,H(b,a,\kappa)\right),\\ A_b\left(\lambda_p,\lambda_p,\cdots,H(\lambda_p,\zeta_p,\kappa)\right), A_b\left(\zeta_p,\zeta_p,\cdots,H(\zeta_p,\lambda_p,\kappa)\right),\\ A_b\left(a,a,\cdots,H(\lambda_p,\zeta_p,\kappa)\right), A_b\left(b,b,\cdots,H(\zeta_p,\lambda_p,\kappa)\right)\end{array}\right\}\\ \leq\ & \theta\max\left\{ A_b\left(a,a,\cdots,H(a,b,\kappa)\right), A_b\left(b,b,\cdots,H(b,a,\kappa)\right)\right\}. \end{aligned}$$

Therefore,

$$\max\left\{\begin{array}{c} A_b\left(a,a,\cdots,H(a,b,\kappa)\right),\\ A_b\left(b,b,\cdots,H(b,a,\kappa)\right)\end{array}\right\} \leq \theta\max\left\{\begin{array}{c} A_b\left(a,a,\cdots,H(a,b,\kappa)\right),\\ A_b\left(b,b,\cdots,H(b,a,\kappa)\right)\end{array}\right\}.$$

It follows that $H(a,b,\kappa) = a, H(b,a,\kappa) = b$. Thus $\kappa \in \mathcal{P}$. Hence $\mathcal{P}$ is closed in $[0,1]$. Let $\kappa_0 \in \mathcal{P}$, then there exist $\lambda_0, \zeta_0 \in U$ with $\lambda_0 = H(\lambda_0,\zeta_0,\kappa_0)$,
$\zeta_0 = H(\zeta_0,\lambda_0,\kappa_0)$. Since U is open, then there exist $r > 0$ such that
$B_{A_b}(\lambda_0, r) \subseteq U$. Choose $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \kappa_0| \leq \frac{1}{M^p} < \frac{\epsilon}{2}$, then for
$\lambda \in \overline{B_{A_b}(\lambda_0, r)} = \{\lambda \in \mathcal{P}/A_b(\lambda,\lambda,\cdots,\lambda_0) \leq r + A_b(\lambda_0,\lambda_0,\cdots,\lambda_0)\}$. Also

$$\eta(\theta)A_b\left(\lambda,\lambda,\cdots,H(\lambda_0,\zeta_0,\kappa)\right) \leq \max\left\{\begin{array}{c} A_b\left(\lambda,\lambda,\cdots,\lambda_0\right), A_b\left(\zeta,\zeta,\cdots,\zeta_0\right),\\ A_b\left(\lambda,\lambda,\cdots,H(\lambda_0,\zeta_0,\kappa)\right), A_b\left(\zeta,\zeta,\cdots,H(\zeta_0,\lambda_0,\kappa)\right)\end{array}\right\}$$

Now we have

$$\begin{aligned}
& A_b\left(H(\lambda,\zeta,\kappa),H(\lambda,\zeta,\kappa),\cdots,\lambda_0\right)\\
= \ & A_b\left(H(\lambda,\zeta,\kappa),H(\lambda,\zeta,\kappa),\cdots,H(\lambda_0,\zeta_0,\kappa_0)\right)\\
\leq \ & (n-1)bA_b\left(H(\lambda,\zeta,\kappa),H(\lambda,\zeta,\kappa),\cdots,H(\lambda,\zeta,\kappa_0)\right)\\
& +b^2A_b\left(H(\lambda,\zeta,\kappa_0),H(\lambda,\zeta,\kappa_0),\cdots,H(\lambda_0,\zeta_0,\kappa_0)\right)\\
\leq \ & b(n-1)M|\kappa-\kappa_0|+b^2A_b\left(H(\lambda,\zeta,\kappa_0),H(\lambda,\zeta,\kappa_0),\cdots,H(\lambda_0,\zeta_0,\kappa_0)\right)\\
\leq \ & b(n-1)\frac{1}{M^{p-1}}+b^2A_b\left(H(\lambda,\zeta,\kappa_0),H(\lambda,\zeta,\kappa_0),\cdots,H(\lambda_0,\zeta_0,\kappa_0)\right).
\end{aligned}$$

Letting $p\to\infty$, we obtain

$$\begin{aligned}
& A_b\left(H(\lambda,\zeta,\kappa),H(\lambda,\zeta,\kappa),\cdots,\lambda_0\right)\\
& \leq b^2A_b\left(H(\lambda,\zeta,\kappa_0),H(\lambda,\zeta,\kappa_0),\cdots,H(\lambda_0,\zeta_0,\kappa_0)\right).\\
\leq \ & b^2\theta\max\left\{\begin{array}{c}
A_b\left(\lambda,\lambda,\cdots,\lambda_0\right),A_b\left(\zeta,\zeta,\cdots,\zeta_0\right),\\
A_b\left(\lambda,\lambda,\cdots,H(\lambda,\zeta,\kappa)\right),A_b\left(\zeta,\zeta,\cdots,H(\zeta,\lambda,\kappa)\right),\\
A_b\left(\lambda_0,\lambda_0,\cdots,H(\lambda_0,\zeta_0,\kappa)\right),A_b\left(\zeta_0,\zeta_0,\cdots,H(\zeta_0,\lambda_0,\kappa)\right),\\
A_b\left(\lambda_0,\lambda_0,\cdots,H(\lambda,\zeta,\kappa)\right),A_b\left(\zeta_0,\zeta_0,\cdots,H(\zeta,\lambda,\kappa)\right)
\end{array}\right\}.\\
\leq \ & b^2\theta\max\left\{\ A_b\left(\lambda,\lambda,\cdots,\lambda_0\right),A_b\left(\zeta,\zeta,\cdots,\zeta_0\right)\ \right\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\max\left\{\begin{array}{c}
A_b\left(H(\lambda,\zeta,\kappa),H(\lambda,\zeta,\kappa),\cdots,\lambda_0\right)\\
A_b\left(H(\zeta,\lambda,\kappa),H(\zeta,\lambda,\kappa),\cdots,\zeta_0\right)
\end{array}\right\} & \leq b^2\theta\max\left\{\ A_b\left(\lambda,\lambda,\cdots,\lambda_0\right),A_b\left(\zeta,\zeta,\cdots,\zeta_0\right)\ \right\}\\
& \leq b^2\theta\max\left\{\begin{array}{c}
r+A_b\left(\lambda_0,\lambda_0,\cdots,\lambda_0\right),\\
r+A_b\left(\zeta_0,\zeta_0,\cdots,\zeta_0\right)
\end{array}\right\}.
\end{aligned}$$

Thus for each fixed $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$, $H(.,\kappa):\overline{B_{A_b}(\lambda_0,r)}\to\overline{B_{A_b}(\lambda_0,r)}$, $H(.,\kappa):\overline{B_{A_b}(\zeta_0,r)}\to\overline{B_{A_b}(\zeta_0,r)}$. Then all conditions of Theorem 5.1 are satisfied. Thus we conclude that $H(.,\kappa)$ has a coupled fixed point in $\overline{U}^2$. But this must be in U^2. Since (τ_0) holds. Thus, $\kappa\in\mathcal{P}$ for any $\kappa\in(\kappa_0-\epsilon,\kappa_0+\epsilon)$. Hence $(\kappa_0-\epsilon,\kappa_0+\epsilon)\subseteq\mathcal{P}$. Clearly $\mathcal{P}$ is open in [0, 1]. For the reverse implication, we use the same strategy. □

6. Conclusion

In this paper we conclude some applications to homotopy theory and integral equations by using Suzuki contractive type fixed point theorems in the set up of A_b-metric spaces.

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References

[1] T. Suzuki, A Generalized Banach Contraction Principle That Characterizes Metric Completeness, Proc. Amer. Math. Soc. 136 (2008), 1861-1869. https://www.jstor.org/stable/20535364.

[2] I. Altun, A. Erduran, A Suzuki Type Fixed-Point Theorem, Int. J. Math. Math. Sci. 2011 (2011), 736063. https://doi.org/10.1155/2011/736063.

[3] M. Aggarwal, R. Chugh, R. Kamal, Suzuki-Type Fixed Point Results in G-Metric Spaces and Applications, Int. J. Computer Appl. 47 (2012), 14–17. https://doi.org/10.5120/7239-0073.

[4] N. Hussain, D. Đorić, Z. Kadelburg, S. Radenović, Suzuki-Type Fixed Point Results in Metric Type Spaces, Fixed Point Theory Appl. 2012 (2012), 126. https://doi.org/10.1186/1687-1812-2012-126.

[5] K.P.R. Rao, K.R.K. Rao, V.C.C. Raju, A Suzuki Type Unique Common Coupled Fixed Point Theorem in Metric Spaces, Int. J. Innov. Res. Sci. Eng. Technol. 2 (2013), 5187-5192.

[6] T. Suzuki, A New Type of Fixed Point Theorem in Metric Spaces, Nonlinear Anal.: Theory Methods Appl. 71 (2009), 5313–5317. https://doi.org/10.1016/j.na.2009.04.017.

[7] I.A. Bakhtin, The Contraction Mapping Principle in Quasimetric Spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30 (1989), 26–37.

[8] M. Abbas, B. Ali, Y.I. Suleiman, Generalized Coupled Common Fixed Point Results in Partially Ordered A-Metric Spaces, Fixed Point Theory Appl. 2015 (2015), 64. https://doi.org/10.1186/s13663-015-0309-2.

[9] M. Ughade, D. Turkoglu, S.R. Singh, R.D. Daheriya, Some Fixed Point Theorems in A_b-Metric Space, Br. J. Math. Computer Sci. 19 (2016), 1-24.

[10] N. Mlaiki, Y. Rohen, Some Coupled Fixed Point Theorems in Partially Ordered A_b-Metric Space, J. Nonlinear Sci. Appl. 10 (2017), 1731–1743. https://doi.org/10.22436/jnsa.010.04.35.

[11] R. Konchada, S.R. Chindirala, R.N. Chappa, A Novel Coupled Fixed Point Results Pertinent to A_b-Metric Spaces With Application to Integral Equations, Math. Anal. Contemp. Appl. 4 (2022), 63–83. https://doi.org/10.30495/maca.2022.1949822.1046.

[12] K. Ravibabu, C.S. Rao, C.R. Naidu, Applications to Integral Equations with Coupled Fixed Point Theorems in A_b-Metric Space, Thai J. Math. Spec. Iss. (2018), 148-167.

[13] D. Guo, V. Lakshmikantham, Coupled Fixed Points of Nonlinear Operators With Applications, Nonlinear Anal.: Theory Methods Appl. 11 (1987), 623–632. https://doi.org/10.1016/0362-546x(87)90077-0.

[14] T.G. Bhaskar, V. Lakshmikantham, Fixed Point Theorems in Partially Ordered Metric Spaces and Applications, Nonlinear Anal.: Theory Methods Appl. 65 (2006), 1379–1393. https://doi.org/10.1016/j.na.2005.10.017.

[15] M. Abbas, M. Ali Khan, S. Radenović, Common Coupled Fixed Point Theorems in Cone Metric Spaces for ω-Compatible Mappings, Appl. Math. Comput. 217 (2010), 195–202. https://doi.org/10.1016/j.amc.2010.05.042.

[16] A. Aghajani, M. Abbas, E.P. Kallehbasti, Coupled Fixed Point Theorems in Partially Ordered Metric Spaces and Application, Math. Commun. 17 (2012), 497-509. https://hrcak.srce.hr/93280.

[17] E. Karapinar, Coupled Fixed Point on Cone Metric Spaces, Gazi Univ. J. Sci. 1 (2011), 51-58. https://dergipark.org.tr/en/download/article-file/83133.

[18] W. Long, B.E. Rhoades, M. Rajović, Coupled Coincidence Points for Two Mappings in Metric Spaces and Cone Metric Spaces, Fixed Point Theory Appl. 2012 (2012), 66. https://doi.org/10.1186/1687-1812-2012-66.

[19] J.G. Mehta, M.L. Joshi, On Coupled Fixed Point Theorem in Partially Ordered Complete Metric Space, Int. J. Pure Appl. Sci. Technol. 1 (2010), 87-92.

[20] W. Shantanawi, Some Common Coupled Fixed Point Results in Cone Metric Spaces, Int. J. Math. Anal. 4 (2010), 2381-2388.