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Exact and Numerical Treatment of a Special Kind of the Pantograph Model via Laplace Technique

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Abstract. The Pantograph is a device of practical application in electric trains, by which the current is collected. The mathematical problem of this device is generally given by the delay differential equation $\phi'(t) = \alpha y(t) + \beta\phi(\gamma t)$, where α and β are real constants and γ is a proportional delay parameter. In the literature, a special attention has been given to the particular case $\gamma = -1$. The objective of this paper is to extend the application of the Laplace transform (LT) combined with the Adomian decomposition method (ADM) to analyze the above model at such particular case of γ. The solution will be determined in exact form which agrees with the corresponding results in the literature. Various properties of the obtained exact solution are discussed in detail. Moreover, it will be declared that for sufficiently small values of α compared to β there exists an accurate approximate solution. The accuracy of the approximate solution is numerically validated. In addition, some numerical results are conducted for the behavior of the present solution at selected values of α and β.

1. Introduction

In electric trains, the current is collected through a certain device, called the Pantograph [1]. The process of such device is a mathematical problem which is governed by the delay differential equation $\phi'(t) = \alpha y(t) + \beta\phi(\gamma t)$, where α and β are real constants and γ is a delay parameter. The standard Pantograph model has been analyzed by several authors utilizing various techniques [2-8]. A special case of the Pantograph model is known as Ambartsumian equation which is of practical applications

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in Astronomy [9-19]. However, an interest is recently given to another special case of such model when $\gamma = -1$, given by [20]

$$\phi'(t) = \alpha\phi(t) + \beta\phi(-t), \tag{1}$$

subject to

$$\phi(0) = \lambda, \tag{2}$$

where α, β and λ are real constants. In Ref. [20], the standard series method (SSM) has been employed to solve the model given by Eqs. (1-2). Although other approaches can be implemented to solve the current model such as the Adomian decomposition method (ADM) [21-34], the regular perturbation method (if one of the constants α or β is small enough) [35,36] and the homotopy perturbation method [37-39] but the solution by these methods is regularly given in terms of infinite series. However, the author believe that the Laplace transform (LT) is capable of obtaining the exact solution in a direct manner. The LT was widely used to solve several scientific models with various applications [40-52].

In this paper, a hybrid approach will be developed to deal with the current problem. The hybrid approach is based on combining the LT and the ADM. At first, the LT will be applied to transform the model (1-2) to a difference equation. Then, the ADM will be used to solve the transformed difference equation and finally the exact solution is given by the inverse LT. The paper is structured as follows. In section 2, the LT is employed to transform the present problem to a difference equation. In addition, the ADM is used to establish the corresponding recurrence scheme. Section 3 focuses on determining a compact form for the Adomian-components of the transformed difference equation. Moreover, the solution is provided in exact form via applying the inverse LT on the such compact form. Furthermore, the properties of the obtained solution is analyzed in section 4 at several cases of the constants α and β. The paper is finally concluded in section 5.

2. The LT-decomposition method

Applying the LT on Eq. (1) gives

$$s\Phi(s) - \lambda = \alpha\Phi(s) - \beta\Phi(-s), \tag{3}$$

where $\Phi(s)$ and $\Phi(-s)$ are the LTs of $\phi(t)$ and $\phi(-t)$, respectively. The ADM [21] requires to put Eq. (3) in the canonical form:

$$\Phi(s) = \frac{\lambda}{s-\alpha} + \frac{\Phi(-s)}{s-\alpha}. \tag{4}$$

Eq. (4) is now a difference equation which has no a known solution. However, the ADM can be used to accomplish this target. The ADM assumes that $\Phi(s)$ can be decomposed as

$$\Phi(s) = \sum_{i=0}^{\infty} \Phi_i(s). \tag{5}$$

Substituting (5) into (4), it then follows

$$\sum_{i=0}^{\infty} \Phi_i(s) = \frac{\lambda}{s-\alpha} + \frac{1}{s-\alpha}\sum_{i=0}^{\infty} \Phi_i(-s). \tag{6}$$

According to Eq. (6) we have the recurrence scheme:

$$\begin{aligned} \Phi_0(s) &= \frac{\lambda}{s-\alpha}, \\ \Phi_i(s) &= -\frac{\beta\Phi_{i-1}(-s)}{s-\alpha}, \quad i \geq 1. \end{aligned} \tag{7}$$

3. The exact solution

The algorithm (7) is used here to obtain a compacted form for the ADM-components. Regarding, Eq. (7) at $i = 1$ gives

$$\begin{aligned} \Phi_1(s) &= -\frac{\beta\Phi_0(-s)}{s-\alpha}, \\ &= \frac{\beta\lambda}{(s-\alpha)(s+\alpha)}, \\ &= \frac{\lambda\beta}{(s^2-\alpha^2)}. \end{aligned} \tag{8}$$

At $i = 2$, we have

$$\begin{aligned} \Phi_2(s) &= -\frac{\beta\Phi_1(-s)}{s-\alpha}, \\ &= -\frac{\lambda\beta^2}{(s^2-\alpha^2)(s-\alpha)}. \end{aligned} \tag{9}$$

Proceeding as above, other higher-order components are found as

$$\Phi_3(s) = -\frac{\lambda\beta^3}{(s^2-\alpha^2)^2}, \tag{10}$$

$$\Phi_4(s) = \frac{\lambda\beta^4}{(s^2-\alpha^2)^2(s-\alpha)}, \tag{11}$$

$$\Phi_5(s) = \frac{\lambda\beta^5}{(s^2-\alpha^2)^3}, \tag{12}$$

$$\Phi_6(s) = -\frac{\lambda\beta^6}{(s^2-\alpha^2)^3(s-\alpha)}, \tag{13}$$

$$\Phi_7(s) = -\frac{\lambda\beta^7}{(s^2-\alpha^2)^4}. \tag{14}$$

In view of the above calculations, it can be observed that the even-order components follow the formula:

$$\Phi_{2i}(s) = \frac{\lambda(-1)^i\beta^{2i}}{(s^2-\alpha^2)^i(s-\alpha)}, \qquad i \geq 0, \tag{15}$$

while the even-order components follow the formula:

$$\Phi_{2i+1}(s) = \frac{\lambda(-1)^i\beta^{2i+1}}{(s^2-\alpha^2)^{i+1}}, \qquad i \geq 0, \tag{16}$$

Therefore, the solution of the difference equation (5) becomes

$$\Phi(s) = \sum_{i=0}^{\infty}\left(\Phi_{2i}(s) + \Phi_{2i+1}(s)\right), \tag{17}$$

i.e.,

$$\Phi(s) = \frac{\lambda}{(s-\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i\beta^{2i}}{(s^2-\alpha^2)^i} + \frac{\lambda\beta}{(s^2-\alpha^2)}\sum_{i=0}^{\infty}\frac{(-1)^i\beta^{2i}}{(s^2-\alpha^2)^i}, \tag{18}$$

which can be rewritten as

$$\Phi(s) = \frac{\lambda}{(s-\alpha)}\sum_{i=0}^{\infty}\left(\frac{-\beta^2}{s^2-\alpha^2}\right)^i + \frac{\lambda\beta}{(s^2-\alpha^2)}\sum_{i=0}^{\infty}\left(\frac{-\beta^2}{s^2-\alpha^2}\right)^i. \tag{19}$$

Under the assumption $\left|\frac{\beta^2}{s^2-\alpha^2}\right| < 1$, i.e., $|s| > \sqrt{\alpha^2+\beta^2}$, the series in the right hand side of Eq. (19) can be summed. In this case, we obtain

$$\sum_{i=0}^{\infty}\left(\frac{-\beta^2}{s^2-\alpha^2}\right)^i = \frac{1}{1+\frac{\beta^2}{s^2-\alpha^2}} = \frac{s^2-\alpha^2}{s^2+\beta^2-\alpha^2}. \tag{20}$$

Inserting (20) into (19) and simplifying, then

$$\Phi(s) = \frac{\lambda(s+\alpha)}{s^2+\beta^2-\alpha^2} + \frac{\lambda\beta}{s^2+\beta^2-\alpha^2}, \tag{21}$$

or

$$\Phi(s) = \lambda\left(\frac{s}{s^2+\beta^2-\alpha^2} + \frac{\alpha+\beta}{s^2+\beta^2-\alpha^2}\right). \tag{22}$$

Applying the inverse LT on Eq. (22) (see Ref. [52]), we directly get

$$\begin{aligned}\phi(t) &= \lambda\left[L^{-1}\left\{\frac{s}{s^2+\beta^2-\alpha^2}\right\} + L^{-1}\left\{\frac{\alpha+\beta}{s^2+\beta^2-\alpha^2}\right\}\right],\\ &= \lambda\left[\cos\left(\sqrt{\beta^2-\alpha^2}\,t\right) + \frac{\alpha+\beta}{\sqrt{\beta^2-\alpha^2}}\sin\left(\sqrt{\beta^2-\alpha^2}\,t\right)\right],\end{aligned} \tag{23}$$

or

$$\phi(t) = \lambda\left[\cos\left(\sqrt{\beta^2-\alpha^2}\,t\right) + \sqrt{\frac{\beta+\alpha}{\beta-\alpha}}\sin\left(\sqrt{\beta^2-\alpha^2}\,t\right)\right], \quad |\alpha| < |\beta|, \tag{24}$$

which is the same obtained expression in Ref. [20] using a direct SSM.

4. Properties of solution

In this section, we introduce some properties of the obtained exact solution (24). Some of these properties were addressed in Ref. [20] but represented here just for adding some materials and observations as follows.

4.1. **Symmetry&behaviour of the solution.** The solution (24) is symmetrical with respect to the signs of α and β. This property can be shown by re-expressing the solution as a function in α and β in addition to the independent variable t, as

$$\phi(a,b,t) = \lambda\left[\cos\left(\sqrt{\beta^2-\alpha^2}\,t\right) + \sqrt{\frac{\beta+\alpha}{\beta-\alpha}}\sin\left(\sqrt{\beta^2-\alpha^2}\,t\right)\right], \quad |\alpha| < |\beta|. \tag{25}$$

It can be easily verified from (25) that

$$\phi(\alpha,\beta,t) = \phi(-\alpha,-\beta,t). \tag{26}$$

In figure 1, the coincidence of the curves $\phi(1,2,t) = \phi(-1,-2,t)$ and $\phi(1,-2,t) = \phi(-1,2,t)$ is shown. Moreover, the influences of β and α on the behaviour of the exact solution (24) are depicted in figure 2 and figure 3, respectively.

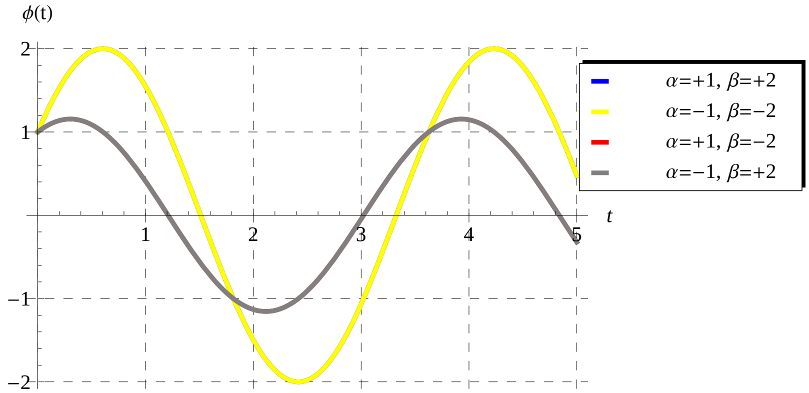

Figure 1. Symmetry property of the exact solution Eq. (24) where $\phi(1, 2, t) = \phi(-1, -2, t)$ and $\phi(1, -2, t) = \phi(-1, 2, t)$.

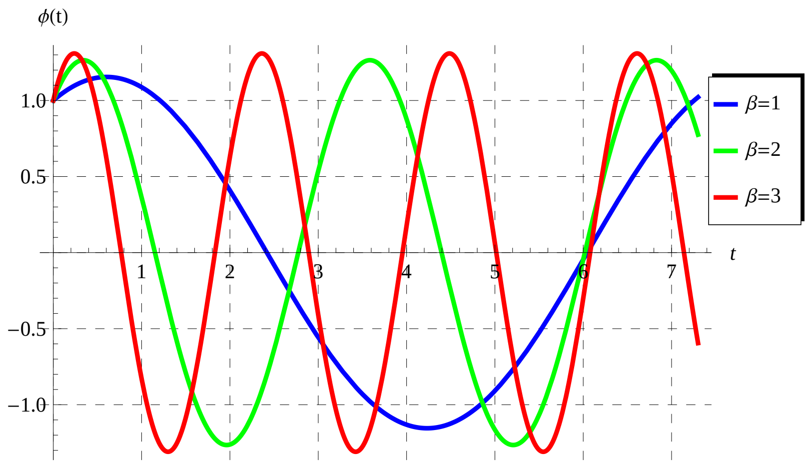

Figure 2. Variation of the exact solution in Eq. (24) against t at different values of β when $\lambda = 1$ and $\alpha = -1/2$.

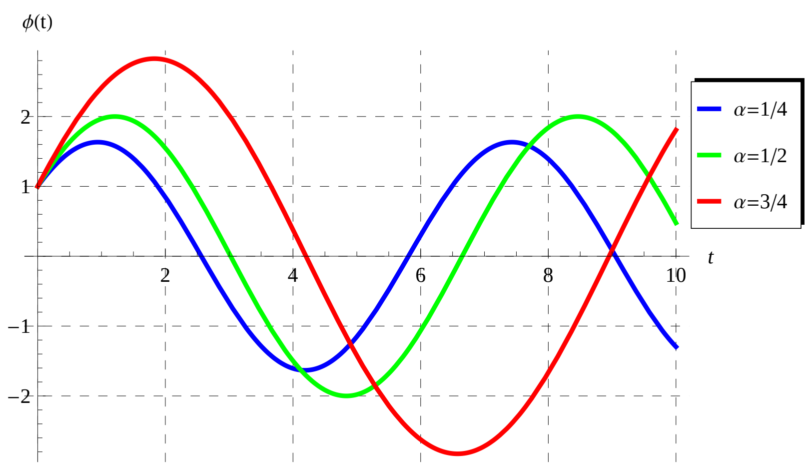

Figure 3. Variation of the exact solution in Eq. (24) against t at different values of α when $\lambda = 1$ and $\beta = 1$.

4.2. **Periodicity.** The trigonometric functions involved in (24) are periodic and hence the present exact solution is of periodic nature with periodicity P, given by

$$P = \frac{2\pi}{\sqrt{\beta^2 - \alpha^2}}, \qquad |\alpha| < |\beta|, \tag{27}$$

which has been also mentioned in Ref. [20]. However, after taking a deep insight at the periodicity (27) for sufficiently small values of α compared to β one can detect that $P \approx \frac{2\pi}{\beta}$. The solution in such case is given by the following theorem

Theorem 1. *For sufficiently small values of α compared to β, i.e., $|\alpha| << |\beta|$, the approximate solution of Eqs. (1-2) takes the form:*

$$\phi(t) \approx \lambda \left[\cos(\beta t) + \left(1 + \frac{\alpha}{\beta}\right)\sin(\beta t)\right], \tag{28}$$

wit periodicity $P \approx \frac{2\pi}{\beta}$.

Proof. To start the proof, we rewrite the solution (24) in the form

$$\phi(t) = \lambda \left[\cos\left(\beta\sqrt{1 - \left(\frac{\alpha}{\beta}\right)^2} t\right) + \sqrt{\frac{\alpha/\beta + 1}{1 - \alpha/\beta}} \sin\left(\beta\sqrt{1 - \left(\frac{\alpha}{\beta}\right)^2} t\right)\right], \tag{29}$$

or

$$\phi(t) = \lambda \left[\cos\left(\beta\sqrt{1 - \epsilon^2} t\right) + \sqrt{\frac{1 + \epsilon}{1 - \epsilon}} \sin\left(\beta\sqrt{1 - \epsilon^2} t\right)\right], \tag{30}$$

where $\epsilon = \frac{\alpha}{\beta} << 1$. Expanding the expressions $\sqrt{1 - \epsilon^2}$ and $\frac{1+\epsilon}{1-\epsilon}$ as power series and neglecting higher-order terms of ϵ , we have

$$\sqrt{1 - \epsilon^2} = 1 - \frac{\epsilon^2}{2} - \frac{\epsilon^4}{8} - \frac{\epsilon^6}{16} - \cdots \approx 1, \tag{31}$$

$$\frac{1 + \epsilon}{1 - \epsilon} = 1 + \epsilon + \frac{\epsilon^2}{2} + \frac{\epsilon^3}{2} + \frac{3\epsilon^4}{8} + \cdots \approx 1 + \epsilon. \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (30), we obtain the approximate solution:

$$\phi(t) \approx \lambda \left[\cos(\beta t) + \left(1 + \frac{\alpha}{\beta}\right)\sin(\beta t)\right]. \tag{33}$$

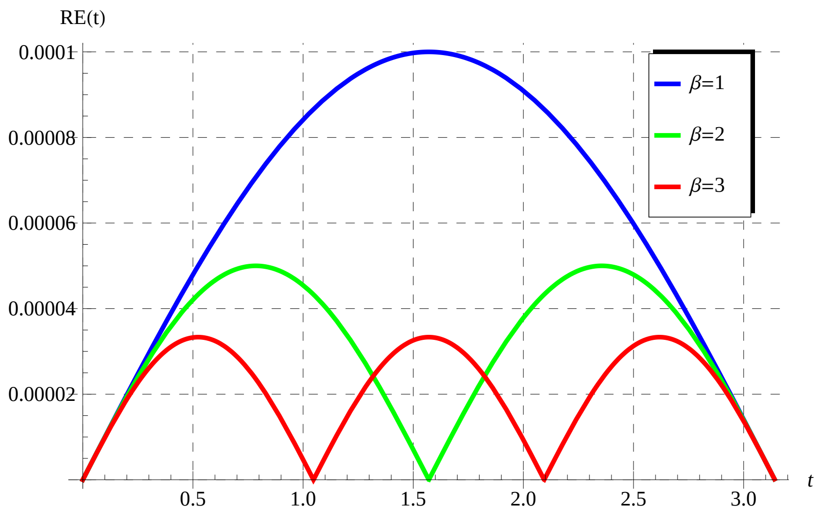

Figure 4. Plots of the absolute residual error in Eq. (37) at different values of β when $\lambda = 1$ and $\alpha = 1/100$.

The periodicity P in Eq. (27) can be also written in terms of ϵ as

$$\begin{aligned} P &= \frac{2\pi}{\sqrt{\beta^2 - \alpha^2}}, \\ &= \frac{2\pi}{\beta} \times \frac{1}{\sqrt{1-\epsilon^2}}, \\ &= \frac{2\pi}{\beta}\left(1 + \frac{\epsilon^2}{2} + \frac{3\epsilon^4}{8} + \dots\right), \quad (34) \\ &\approx \frac{2\pi}{\beta}, \quad (35) \end{aligned}$$

which completes the proof. □

4.2.1. *Behaviour of the approximate solution.* In order to check the accuracy of the approximate solution (28), some numerical results are conducted in this section. Simulation of the approximate solution is accomplished here through calculating the absolute residual error for solution (24), given by

$$RE(t) = \left|\phi'(t) - \alpha\phi(t) - \beta\phi(-t)\right|. \quad (36)$$

Substituting Eq. (24) into Eq. (36) gives

$$RE(t) = \left|\frac{2\lambda\alpha^2}{\beta}\sin(\beta t)\right|. \quad (37)$$

The accuracy of the approximate solution is clearly verified in figures 4, 5, and 6 through plotting the residual (37) and comparing the approximate solution (28) with the exact one.

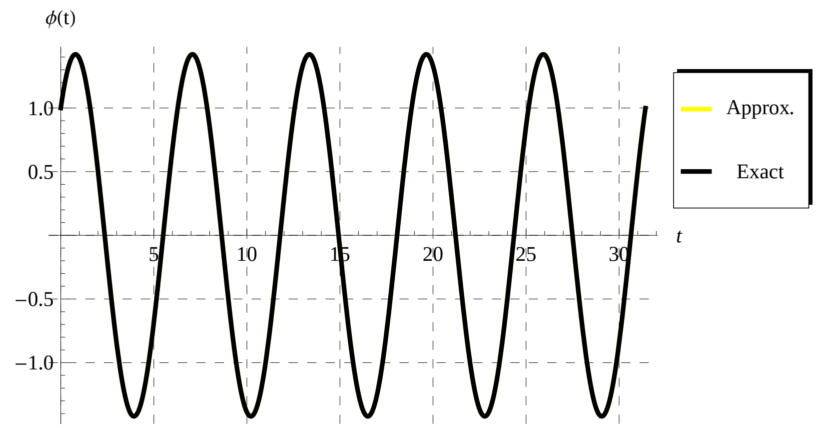

Figure 5. Comparisons between the approximate solution in Eq. (28) and the exact solution in Eq. (24) at $\lambda = 1$, $\alpha = 1/100$, and $\beta = 1$.

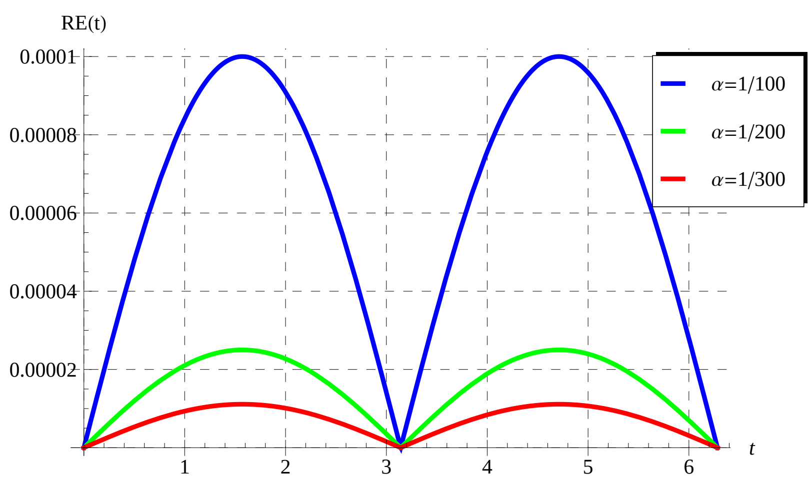

Figure 6. Plots of the absolute residual error in Eq. (37) at different values of α when $\lambda = 1$ and $\beta = 1$.

4.3. **Polynomial solution at $\beta = \alpha$.** It is noted from the expression in Eq. (24) that the exact solution is not valid when $\beta = \alpha$. However, the solution of such case can be obtained through calculating the limit of Eq. (24) as $\beta \to \alpha$. To do so, we suppose that $\beta - \alpha = \sigma$ and thus $\sigma \to 0$ as $\alpha \to \beta$. Accordingly, the solution (24) becomes

$$\phi(t) = \lambda \lim_{\alpha \to \beta} \left[\cos\left(\sqrt{\beta^2 - \alpha^2}\, t\right) + \sqrt{\frac{\alpha + \beta}{\beta - \alpha}} \sin\left(\sqrt{\beta^2 - \alpha^2}\, t\right) \right], \tag{38}$$

or equivalently

$$\phi(t) = \lambda \lim_{\sigma \to 0} \left[\cos\left(\sqrt{2\alpha\sigma}\ t\right) + \sqrt{\frac{2\alpha}{\sigma}} \sin\left(\sqrt{2\alpha\sigma}\ t\right) \right],$$

$$\begin{aligned} &= \lambda\left[1+\sqrt{2\alpha}\lim_{\sigma\to 0}\left(\frac{\sin\left(\sqrt{2\alpha\sigma}\ t\right)}{\sqrt{\sigma}}\right)\right], \\ &= \lambda\left[1+\sqrt{2\alpha}.\sqrt{2\alpha}\ t\right], \\ &= \lambda\left(1+2\alpha t\right). \end{aligned} \tag{39}$$

It may be important to mention that Eq. (39) is the exact solution of the differential-difference equation:

$$\phi'(t)=\alpha\left(\phi(t)+\phi(-t)\right), \quad \phi(0)=\lambda. \tag{40}$$

4.4. **Hyperbolic solution at $\alpha > \beta$.** In order to obtain the solution in terms of the hyperbolic functions, Eq. (24) can be rewritten as

$$\phi(t)=\lambda\left[\cos\left(j\sqrt{\alpha^2-\beta^2}t\right)-j\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\sin\left(j\sqrt{\alpha^2-\beta^2}t\right)\right], \tag{41}$$

where $j=\sqrt{-1}$ is the imaginary number. Therefore, Eq. (41) is transformed into the form:

$$\phi(t)=\lambda\left[\cosh\left(\sqrt{\alpha^2-\beta^2}t\right)+\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\sinh\left(\sqrt{\alpha^2-\beta^2}t\right)\right], \quad \alpha>\beta. \tag{42}$$

This is also the corresponding obtained solution in Ref. [20] utilizing the SSM.

4.5. **Constant solution at $\beta=-\alpha$.** The solution of this case can be obtained by direct substitutions into Eq. (24). So, we observe from Eq. (24), at $\beta=-\alpha$, that

$$\begin{gathered} \cos\left(\sqrt{\beta^2-\alpha^2}t\right)=\cos(0)=1, \\ \sqrt{\frac{\alpha+\beta}{\beta-\alpha}}\sin\left(\sqrt{\beta^2-\alpha^2}t\right)=0, \end{gathered} \tag{43}$$

and consequently the solution (24) reduces to

$$\phi(t)=\lambda. \tag{44}$$

which is the constant solution of the following differential-difference equation:

$$\phi'(t)=\alpha\left(\phi(t)-\phi(-t)\right), \quad \phi(0)=\lambda. \tag{45}$$

4.6. **Periodic solution at a special case:** $\alpha = 0$**.** In this case, the present model (1-2) reduces to the differential-difference equation:

$$\phi'(t) = \beta\phi(-t), \quad \phi(0) = \lambda. \tag{46}$$

Although it is simpler, the exact solution may be not available in the literature However, the current study gives the solution directly from Eq. (24) by setting $\alpha = 0$, hence

$$\phi(t) = \lambda\left(\cos\beta t + \sin\beta t\right). \tag{47}$$

5. Conclusions

In this paper, a hybrid approach based on the LT and the ADM is applied to solve a special kind of the Pantograph delay functional-differential equation. The exact solution was obtained in a direct manner if compared with the series method in the literature. At a specific constrain of the constants involved in the present model, an accurate analytic approximation was determined. It was also shown that the obtained exact solution enjoyed several interesting properties such symmetry, periodicity, and others. In addition, several types of exact solutions were proved at special cases and expressed as hyperbolic, linear, and constant functions. The present method may deserve further extensions to analyze more complex delay models.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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