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Fuzzy Subalgebras and Ideals With Thresholds of Hilbert Algebras

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Abstract. The concepts of fuzzy subalgebras and ideals with thresholds of Hilbert algebras are presented, some of their features are explained, and their extensions are demonstrated using the theory of fuzzy sets as a foundation. We also talk about the connections between fuzzy subalgebras (ideals) with thresholds and their level subsets. The homomorphic images and inverse images of fuzzy subalgebras and ideals with thresholds in Hilbert algebras are also studied and some related properties are investigated.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [25]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 3, 6]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision-making [12–14]. The concept of Hilbert algebras was introduced in early 50-ties by Henkin [15] for

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some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [8] from algebraic point of view. Diego [8] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [4, 5] and Jun [16] and some of their filters forming deductive systems were recognized. Dudek [10] considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras. The concept of fuzzy sets with thresholds is presented in several articles as seen in [9, 17–21, 23, 24].

This study builds on the theory of fuzzy sets to introduce fuzzy subalgebras and ideals with thresholds of Hilbert algebras, explain some of their characteristics, and show how they may be extended. We also talk about the connections between fuzzy subalgebras (ideals) with thresholds and their level subsets. Additionally, the homomorphic images and inverse images of fuzzy subalgebras and ideals with thresholds in Hilbert algebras are investigated and some associated features are examined.

2. Preliminaries

Let's go through the idea of Hilbert algebras as it was introduced by Diego [8] in 1966 before we start.

Definition 2.1. *[8] A Hilbert algebra is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In [10], the following conclusion was established.

Lemma 2.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2. *[26] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a subalgebra of X if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 2.3. *[7] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an ideal of X if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

A *fuzzy set* [25] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.4. *[22] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy subalgebra of X if the following condition holds:*

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.5. *[11] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy ideal of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

3. Fuzzy subalgebras and ideals with thresholds

For $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, we introduce the concepts of fuzzy subalgebras and ideals with thresholds ε and δ of Hilbert algebras and investigate some related properties.

Definition 3.1. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is called a fuzzy subalgebra with thresholds ε and δ (FST_ε^δ) of X, where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$ if the following condition holds:*

$$(\forall x, y \in X)(\max\{\mu(x \cdot y), \varepsilon\} \geq \min\{\mu(x), \mu(y), \delta\}).$$

Example 3.1. *Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define a fuzzy set $\mu : A \to [0, 1]$ as follows:

$$\mu(1) = 0.9, \mu(x) = 0.6, \mu(y) = 0.8, \mu(z) = 0.4, \mu(0) = 0.2.$$

Then μ is a $FST_{0.8}^{0.9}$ of X.

Proposition 3.1. *If μ is a $FST_{\varepsilon}^{\delta}$ of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x \in X)(\max\{\mu(1), \varepsilon\} \geq \min\{\mu(x), \delta\}).$$

Proof. For all $x \in X$, we have

$$\max\{\mu(1), \varepsilon\} = \max\{\mu(x \cdot x), \varepsilon\} \geq \min\{\mu(x), \mu(x), \delta\} = \min\{\mu(x), \delta\}.$$

□

Definition 3.2. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is called a fuzzy ideal with thresholds ε and δ ($FIT_{\varepsilon}^{\delta}$) of X if the following conditions hold:*

(1) $(\forall x \in X)(\max\{\mu(1), \varepsilon\} \geq \min\{\mu(x), \delta\})$,
(2) $(\forall x, y \in X)(\max\{\mu(x \cdot y), \varepsilon\} \geq \min\{\mu(y), \delta\})$,
(3) $(\forall x, y_1, y_2 \in X)(\max\{\mu((y_1 \cdot (y_2 \cdot x)) \cdot x), \varepsilon\} \geq \min\{\mu(y_1), \mu(y_2), \delta\})$.

Example 3.2. *Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define a fuzzy set $\mu : A \to [0, 1]$ as follows:

$$\mu(1) = 0.9, \mu(x) = 0.5, \mu(y) = 0.4, \mu(z) = 0.3, \mu(0) = 0.2.$$

Then μ is a $FIT_{0.8}^{0.9}$ of X.

Proposition 3.2. *If μ is $FIT_{\varepsilon}^{\delta}$ of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(\max\{\mu_A((y \cdot x) \cdot x), \varepsilon) \geq \min\{\mu(1), \mu(y), \delta\}).$$

Proof. Let $x, y \in X$. Then

$$\max\{\mu((y \cdot x) \cdot x), \varepsilon\} = \max\{\mu((1 \cdot (y \cdot x)) \cdot x), \varepsilon\} \geq \min\{\mu(1), \mu(y), \delta\}.$$

□

Lemma 3.1. *If μ is a $FIT_{\varepsilon}^{\delta}$ of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(x \leq y \ \Rightarrow \min\{\mu(1), \mu(x), \delta\} \leq \max\{\mu(y), \varepsilon\}). \tag{3.1}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{aligned}\max\{\mu(y), \varepsilon\} &= \max\{\mu(1 \cdot y), \varepsilon\} \\ &= \max\{\mu(((x \cdot y) \cdot (x \cdot y)) \cdot y), \varepsilon\} \\ &\geq \min\{\mu(x \cdot y), \mu(x), \delta\} \\ &= \min\{\mu(1), \mu(x), \delta\}.\end{aligned}$$

□

Theorem 3.1. *Every $FIT_{\varepsilon}^{\delta}$ of a Hilbert algebra $X = (X, \cdot, 1)$ is a $FST_{\varepsilon}^{\delta}$.*

Proof. Assume that μ is a $\mathrm{FIT}_{\varepsilon}^{\delta}$ of X. Let $x, y \in X$. Then

$$\max\{\mu(x \cdot y), \varepsilon\} \geq \min\{\mu(y), \delta\} \geq \min\{\mu(x), \mu(y), \delta\}.$$

Hence, μ is a $\mathrm{FST}_{\varepsilon}^{\delta}$ of X. □

Theorem 3.2. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is such that $\mu(x) \leq \varepsilon$ for all $x \in X$, then it is a $FIT_{\varepsilon}^{\delta}$ (resp., $FST_{\varepsilon}^{\delta}$) of X.*

Proof. Let $x \in X$. Then $\max\{\mu(1), \varepsilon\} = \varepsilon \geq \mu(x) \geq \min\{\mu(x), \delta\}$. Let $x, y \in X$. Then $\max\{\mu(x \cdot y), \varepsilon\} = \varepsilon \geq \mu(y) \geq \min\{\mu(y), \delta\}$. Let $x, y_1, y_2 \in X$. Then $\max\{\mu((y_1 \cdot (y_2 \cdot x)) \cdot x), \varepsilon\} = \varepsilon \geq \mu(y_1) \geq \min\{\mu(y_1), \mu(y_2), \delta\}$. Hence, μ is a $\mathrm{FIT}_{\varepsilon}^{\delta}$ of X.

Let $x, y \in X$. Then $\max\{\mu(x \cdot y), \varepsilon\} = \varepsilon \geq \mu(x) \geq \min\{\mu(x), \mu(y), \delta\}$. Hence, μ is a $\mathrm{FST}_{\varepsilon}^{\delta}$ of X. □

By Theorem 3.2, we get the following theorem.

Theorem 3.3. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is such that $\mu(x) \geq \delta$ for all $x \in X$, then it is a $FIT_{\varepsilon}^{\delta}$ (resp., $FST_{\varepsilon}^{\delta}$) of X.*

Theorem 3.4. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$ and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then μ is a $FST_{\varepsilon}^{\delta}$ of X if and only if for all $t \in (\varepsilon, \delta], U(\mu, t) := \{x \in X \mid \mu(x) \geq t\}$ is a subalgebra of X if it is nonempty.*

Proof. Assume that μ is a $\mathrm{FST}_{\varepsilon}^{\delta}$ of X. Let $t \in (\varepsilon, \delta]$ be such that $U(\mu, t) \neq \emptyset$ and let $x, y \in U(\mu, t)$. Then $\mu(x) \geq t$, $\mu(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{\mu(x), \mu(y), \delta\}$. Since μ is a $\mathrm{FST}_{\varepsilon}^{\delta}$ of X, we have $\max\{\mu(x \cdot y), \varepsilon\} \geq \min\{\mu(x), \mu(y), \delta\} \geq t > \varepsilon$. So, $\max\{\mu(x \cdot y), \varepsilon\} = \mu(x \cdot y)$. Since $\max\{\mu(x \cdot y), \varepsilon\} \geq t$, we get $\mu(x \cdot y) \geq t$. Hence, $x \cdot y \in U(\mu, t)$. Therefore, $U(\mu, t)$ is a subalgebra of X.

Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(\mu, t)$ is a subalgebra of X if it is nonempty. Let $x, y \in X$. Then $\mu(x), \mu(y) \in [0, 1]$. Choose $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus $x, y \in U(\mu, t) \neq \emptyset$. By the assumption, we have $U(\mu, t)$ is a subalgebra of X. So, $x \cdot y \in U(\mu, t)$, that is, $\mu(x \cdot y) \geq t = \min\{\mu(x), \mu(y)\}$. Thus $\max\{\mu(x \cdot y), \varepsilon\} \geq \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{\mu(x), \mu(y), \delta\}$. Hence, μ is a $\mathrm{FST}_{\varepsilon}^{\delta}$ of X. □

The following theorem can be proved similarly to Theorem 3.4.

Theorem 3.5. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$ and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. Then μ is a $FIT_{\varepsilon}^{\delta}$ of X if and only if for all $t \in (\varepsilon, \delta], U(\mu, t)$ is an ideal of X if it is nonempty.*

Definition 3.3. *Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then the fuzzy set β in Y defined by*

$$\beta(y) = \begin{cases} \sup\limits_{t \in f^{-1}(y)} \{\mu(t)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f. Similarly, if α is a fuzzy set in Y, then the fuzzy set $f^{-1}(\alpha) = \alpha \circ f$ in X (i.e., the fuzzy set is defined by $f^{-1}(\alpha)(x) = \alpha(f(x))$ for all $x \in X$) is called the pre-image of α under f.

Definition 3.4. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to have the* sup *property if for any nonempty subset T of X, there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$.*

Definition 3.5. *Let X and Y be any two nonempty sets and let $f : X \to Y$ be any function. A fuzzy set μ in X is said to be f-invariant if*

$$(\forall x, y \in X)(f(x) = f(y) \Rightarrow \mu(x) = \mu(y)).$$

Lemma 3.2. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras and let $f : X \to Y$ be a surjective homomorphism. Let μ be an f-invariant fuzzy set in X with* sup *property. For any $x, y \in Y$, there exist $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ such that $\beta(x) = \mu(x_0), \beta(y) = \mu(y_0)$, and $\beta(x \star y) = \mu(x_0 \cdot y_0)$.*

Proof. Let $x, y \in Y$. Since f is surjective, we have $f^{-1}(x), f^{-1}(y)$, and $f^{-1}(x \cdot y)$ are nonempty subsets of X. Since μ has sup property, there exist elements $x_0 \in f^{-1}(x), y_0 \in f^{-1}(y)$, and $z_0 \in f^{-1}(x \star y)$ such that

$$\beta(x) = \sup_{s \in f^{-1}(x)} \{\mu(s)\} = \mu(x_0),$$

$$\beta(y) = \sup_{s \in f^{-1}(y)} \{\mu(s)\} = \mu(y_0),$$

$$\beta(x \star y) = \sup_{s \in f^{-1}(x \star y)} \{\mu(s)\} = \mu(z_0).$$

Since

$$f(z_0) = x \star y = f(x_0) \star f(y_0) = f(x_0 \cdot y_0),$$

and μ is f-invariant, it follows that

$$\beta(x \star y) = \mu(z_0) = \mu(x_0 \cdot y_0).$$

$\square$

Theorem 3.6. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras and let $f : X \to Y$ be a surjective homomorphism. Then the following statements hold:*

(1) if μ is an f-invariant FST_ε^δ of X with sup *property, then β is a FST_ε^δ of Y,*
(2) if μ is an f-invariant FIT_ε^δ of X with sup *property, then β is a FIT_ε^δ of Y.*

Proof. (1) Assume that μ is an f-invariant $\mathrm{FST}_\varepsilon^\delta$ of X with sup property. Let $a, b \in Y$. Then by Lemma 3.2, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0)$, $\beta(b) = \mu(b_0)$, and $\beta(a \star b) = \mu(a_0 \cdot b_0)$. Thus $\max\{\beta(a \star b), \varepsilon\} = \max\{\mu(a_0 \cdot b_0), \varepsilon\} \geq \min\{\mu(a_0), \mu(b_0), \delta\} = \min\{\beta(a), \beta(b), \delta\}$. Hence, β is a $\mathrm{FST}_\varepsilon^\delta$ of Y.

(2) Assume that μ is an f-invariant $\mathrm{FIT}_\varepsilon^\delta$ of X with sup property. Since $f(1_X) = 1_Y$, we have $f^{-1}(1_Y) \neq \emptyset$. By Lemma 3.2, there exists $x_1 \in f^{-1}(1_Y)$ such that $\mu(x_1) = \beta(1_Y)$. Thus $f(x_1) = 1_Y = f(1_X)$. Since μ is f-invariant, we have $\mu(x_1) = \mu(1_X)$. So, $\beta(1_Y) = \mu(1_X)$. Let $y \in Y$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. By Lemma 3.2, there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$. Thus $\max\{\beta(1_Y), \varepsilon\} = \max\{\mu(1_X), \varepsilon\} \geq \min\{\mu(x), \delta\} = \min\{\beta(y), \delta\}$. Now, let $a, b \in Y$. Then by Lemma 3.2, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\mu(a_0) = \beta(a)$, $\mu(b_0) = \beta(b)$, and $\mu(a_0 \cdot b_0) = \beta(a \star b)$. Thus $\max\{\beta(a \star b), \varepsilon\} = \max\{\mu(a_0 \cdot b_0), \varepsilon\} \geq \min\{\mu(b_0), \delta\} = \min\{\beta(b), \delta\}$. Next, let $a, b, y \in Y$. By Lemma 3.2, there exist $a_0 \in f^{-1}(a)$, $b_0 \in f^{-1}(b)$ and $x \in f^{-1}(y)$ such that $\mu(a_0) = \beta(a)$, $\mu(b_0) = \beta(b)$, $\mu(y) = \beta(x)$ and $\mu((a_0 \cdot (b_0 \cdot x)) \cdot x) = \beta((a \star (b \star x)) \star x)$. Thus $\max\{\beta((a \star (b \star x)) \star x), \varepsilon\} = \max\{\mu((a_0 \cdot (b_0 \cdot x)) \cdot x) \cdot x), \varepsilon\} \geq \min\{\mu(a_0), \mu(b_0), \delta\} = \min\{\beta(a), \beta(b), \delta\}$. Hence, β is a $\mathrm{FIT}_\varepsilon^\delta$ of Y. □

Theorem 3.7. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras and let $f : X \to Y$ be a homomorphism. Then the following statements hold:*

(1) if α is a FST_ε^δ of Y, then $f^{-1}(\alpha)$ is a FST_ε^δ of X,
(2) if α is a FIT_ε^δ of Y, then $f^{-1}(\alpha)$ is a FIT_ε^δ of X.

Proof. (1) Assume that α is a $\mathrm{FST}_\varepsilon^\delta$ of Y. Let $x, y \in X$. Then

$$\begin{aligned}
\max\{f^{-1}(\alpha)(x \cdot y), \varepsilon\} &= \max\{(\alpha \circ f)(x \cdot y), \varepsilon\} \\
&= \max\{(\alpha(f(x \cdot y)), \varepsilon\} \\
&= \max\{(\alpha(f(x) \star f(y)), \varepsilon\} \\
&\geq \min\{\alpha(f(x)), \alpha(f(y)), \delta\} \\
&= \min\{(\alpha \circ f)(x), (\alpha \circ f)(y), \delta\} \\
&= \min\{f^{-1}(\alpha)(x), f^{-1}(\alpha)(y), \delta\}.
\end{aligned}$$

Hence, $f^{-1}(\alpha)$ is a $\mathrm{FST}_\varepsilon^\delta$ of X.

(2) Assume that α is a $\mathrm{FIT}_{\varepsilon}^{\delta}$ of Y. Let $x \in X$. Then

$$\begin{aligned}
\max\{f^{-1}(\alpha)(1_X), \varepsilon\} &= \max\{(\alpha \circ f)(1_X), \varepsilon\} \\
&= \max\{(\alpha(f(1_X)), \varepsilon\} \\
&= \max\{\alpha(1_Y), \varepsilon\} \\
&\geq \min\{\alpha(f(x)), \delta\} \\
&= \min\{(\alpha \circ f)(x), \delta\} \\
&= \min\{f^{-1}(\alpha)(x), \delta\}.
\end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned}
\max\{f^{-1}(\alpha)(x \cdot y), \varepsilon\} &= \max\{(\alpha \circ f)(x \cdot y), \varepsilon\} \\
&= \max\{\alpha(f(x \cdot y)), \varepsilon\} \\
&\geq \min\{\alpha(f(y)), \varepsilon\} \\
&= \min\{(\alpha \circ f)(y), \delta\} \\
&= \min\{f^{-1}(\alpha)(y), \delta\}.
\end{aligned}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}
\max\{f^{-1}(\alpha)((y_1 \cdot (y_2 \cdot x)) \cdot x), \varepsilon\} &= \max\{(\alpha \circ f)((y_1 \cdot (y_2 \cdot x)) \cdot x), \varepsilon\} \\
&= \max\{\alpha(f((y_1 \cdot (y_2 \cdot x)) \cdot x)), \varepsilon\} \\
&= \max\{\alpha(f(y_1) \star (f(y_2) \star f(x)) \star f(x)), \varepsilon\} \\
&\geq \min\{\alpha(f(y_1)), \alpha(f(y_2)), \delta\} \\
&= \min\{(\alpha \circ f)(y_1), (\alpha \circ f)(y_2), \delta\} \\
&= \min\{f^{-1}(\alpha)(y_1), f^{-1}(\alpha)(y_2), \delta\}.
\end{aligned}$$

Hence, $f^{-1}(\alpha)$ is a $\mathrm{FIT}_{\varepsilon}^{\delta}$ of X. □

Definition 3.6. *Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then the fuzzy set η in Y defined by*

$$\eta(y) = \begin{cases} \inf\limits_{t \in f^{-1}(y)} \{\mu(t)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f.

Definition 3.7. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to have the* inf *property if for any nonempty subset T of X, there exists $t_0 \in T$ such that $\mu(t_0) = \inf_{t \in T} \mu(t)$.*

The following lemma can be proved similarly to Lemma 3.2.

Lemma 3.3. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras and let $f : X \to Y$ be a surjective homomorphism. Let μ be an f-invariant fuzzy set in X with* inf *property. For any $x, y \in Y$, there exist $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ such that $\eta(x) = \mu(x_0), \eta(y) = \mu(y_0)$, and $\eta(x \star y) = \mu(x_0 \cdot y_0)$.*

The following theorem can be proved similarly to Theorem 3.6.

Theorem 3.8. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras and let $f : X \to Y$ be a surjective homomorphism. Then the following statements hold:*

(1) if μ is an f-invariant FST_ε^δ of X with inf *property, then η is a FST_ε^δ of Y,*
(2) if μ is an f-invariant FIT_ε^δ of X with inf *property, then η is a FIT_ε^δ of Y.*

4. Conclusion

In the present paper, we have introduced the concepts of fuzzy subalgebras and ideals with thresholds of Hilbert algebras. The relationship between fuzzy subalgebras (ideals) and their level subsets is described. Moreover, the homomorphic images and inverse images of fuzzy subalgebras and ideals with thresholds in Hilbert algebras are also studied and some related properties are investigated.

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References

[1] B. Ahmad, A. Kharal, On Fuzzy Soft Sets, Adv. Fuzzy Syst. 2009 (2009), 586507. https://doi.org/10.1155/2009/586507.
[2] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/s0165-0114(86)80034-3.
[3] M. Atef, M.I. Ali, T.M. Al-shami, Fuzzy Soft Covering-Based Multi-Granulation Fuzzy Rough Sets and Their Applications, Comput. Appl. Math. 40 (2021), 115. https://doi.org/10.1007/s40314-021-01501-x.
[4] D. Busneag, A Note on Deductive Systems of a Hilbert Algebra, Kobe J. Math. 2 (1985), 29-35. https://cir.nii.ac.jp/crid/1570291224860431872.
[5] D. Busneag, Hilbert Algebras of Fractions and Maximal Hilbert Algebras of Quotients, Kobe J. Math. 5 (1988), 161-172. https://cir.nii.ac.jp/crid/1570572702603831808.
[6] N. Cağman, S. Enginoğlu, F. Citak, Fuzzy Soft Set Theory and Its Application, Iran. J. Fuzzy Syst. 8 (2011), 137-147.
[7] I. Chajda, R. Halas, Congruences and Ideals in Hilbert Algebras, Kyungpook Math. J. 39 (1999), 429-429.
[8] A. Diego, Sur les Algébres de Hilbert, Collect. Logique Math. Ser. A (Ed. Hermann, Paris), 21 (1966), 1-52.
[9] N. Dokkhamdang, A. Kesorn, A. Iampan, Generalized Fuzzy Sets in UP-Algebras, Ann. Fuzzy Math. Inform. 16 (2018), 171-190.
[10] W.A. Dudek, On Fuzzification in Hilbert Algebras, Contrib. Gen. Algebra, 11 (1999), 77-83.
[11] W.A. Dudek, Y.B. Jun, On Fuzzy Ideals in Hilbert Algebra, Novi Sad J. Math. 29 (1999), 193-207.
[12] H. Garg, K. Kumar, An Advanced Study on the Similarity Measures of Intuitionistic Fuzzy Sets Based on the Set Pair Analysis Theory and Their Application in Decision Making, Soft Comput. 22 (2018), 4959–4970. https://doi.org/10.1007/s00500-018-3202-1.

[13] H. Garg, K. Kumar, Distance Measures for Connection Number Sets Based on Set Pair Analysis and Its Applications to Decision-Making Process, Appl. Intell. 48 (2018), 3346–3359. https://doi.org/10.1007/s10489-018-1152-z.

[14] H. Garg, S. Singh, A Novel Triangular Interval Type-2 Intuitionistic Fuzzy Set and Their Aggregation Operators, Iran. J. Fuzzy Syst. 15 (2018), 69-93. https://doi.org/10.22111/ijfs.2018.4159.

[15] L. Henkin, An Algebraic Characterization of Quantifiers, Fund. Math. 37 (1950), 63-74. https://eudml.org/doc/213228.

[16] Y.B. Jun, Deductive Systems of Hilbert Algebras, Math. Japon. 43 (1996), 51-54. https://cir.nii.ac.jp/crid/1571417124616097792.

[17] Y.B. Jun, Fuzzy Subalgebras With Thresholds in BCK/BCI-Algebras, Commun. Korean Math. Soc. 22 (2007), 173–181. https://doi.org/10.4134/CKMS.2007.22.2.173.

[18] Y.B. Jun, On (α, β)-Fuzzy Subalgebras of BCK/BCI-Algebras, Bull. Korean Math. Soc. 42 (2005), 703-711. https://doi.org/10.4134/BKMS.2005.42.4.703.

[19] F.F. Kareem, M.M. Abed, Generalizations of Fuzzy k-ideals in a KU-algebra with Semigroup, J. Phys.: Conf. Ser. 1879 (2021), 022108. https://doi.org/10.1088/1742-6596/1879/2/022108.

[20] A. Khan, Y.B. Jun, T. Mahmood, Generalized Fuzzy Interior Ideals in Abel Grassmann's Groupoids, Int. J. Math. Math. Sci. 2010 (2010), 838392. https://doi.org/10.1155/2010/838392.

[21] A. Khan, M. Shabir, (α, β)-Fuzzy Interior Ideals in Ordered Semigroups, Lobachevskii J. Math. 30 (2009), 30-39. https://doi.org/10.1134/s1995080209010053.

[22] K.H. Kim, On T-Fuzzy Ideals in Hilbert Algebras, Sci. Math. Japon. 70 (2009), 7-15.

[23] A.B. Saeid, Redefined Fuzzy Subalgebra (With Thresholds) of BCK/BCI-Algebras, Iran. J. Math. Sci. Inform. 4 (2009), 19-24. https://doi.org/10.7508/ijmsi.2009.02.002.

[24] M. Siripitukdet, A. Ruanon, Fuzzy Interior Ideals With Thresholds $(s, t]$ in Ordered Semigroups, Thai J. Math. 11 (2013), 371-382.

[25] L.A. Zadeh, Fuzzy Sets, Inform. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[26] J. Zhan, Z. Tan, Intuitionistic Fuzzy Deductive Systems in Hibert Algebra, Southeast Asian Bull. Math. 29 (2005), 813-826.