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A Novel of Cubic Ideals in Γ-Semigroups

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Abstract. In this paper, we give the concepts of new types of cubic ideals in Γ-semigroups. We study properties and relationships between cubic (α, β)-ideals and ideals in semigroups. Furthermore, we proved some basic properties of cubic almost ideals in semigroups.

1. Introduction

The theory for dealing with uncertainty, fuzzy set theory, was discovered by Zadeh in 1965 [18], mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. The studies of cubic sets and cubic subgroups were presented by Jun et al in 2012 [10, 11]. Later V. Chinnadurai and K. Bharathivelan [2] studies cubic ideals in Γ-semigroups and proved basic properties of cubic ideals in Γ-semigroups.

The theory of ideal is structured important in semigroups and many researchers used knowledge of ideals in Γ-semigroups discussed in fuzzy semigroup such as Chinram et al. [3] discussed almost quasi-Γ-ideal and fuzzy almost quasi-Γ-ideals in Γ-semigroup, M. K. R. Marapureddy and PRV S. R. Doradla [14] discussed weak interior ideals of Γ-semigroups, S.K. Majumder and M. Mandal [9] discussed fuzzy generalized bi-ideal in Γ-semigroups. In the study of the concept of cubic ideals, many researchers expanded on this idea [4, 6, 7, 7, 8, 13, 15]. Recently, in 2021 [17] A. Simuen et al. discussed a novel of ideals and fuzzy ideals of Γ-semigroups.

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In this paper we extend new fuzzy ideals to cubic ideals of Γ-semigroups and we investigate the properties of new types cubic ideals. of Γ-semigroups.

2. Preliminaries

In this section, we review concepts basic definitions and the theorem used to prove all result in the next section.

A *sub-Γ-semigroup* of a Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ of $\mathcal{S}$ such that $\mathcal{K}\Gamma\mathcal{K} \subseteq \mathcal{K}$. A *left (right) ideal* of a Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ of $\mathcal{S}$ such that $\mathcal{S}\Gamma\mathcal{K} \subseteq \mathcal{K}$ ($\mathcal{K}\Gamma\mathcal{S} \subseteq \mathcal{K}$). By an *ideal* of a Γ-semigroup $\mathcal{S}$ it is both a left and a right ideal of $\mathcal{S}$. A *quasi-ideal* of a Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ of $\mathcal{S}$ such that $\mathcal{K}\Gamma\mathcal{S} \cap \mathcal{S}\Gamma\mathcal{K} \subseteq \mathcal{K}$. A sub-Γ-semigroup K of a Γ-semigroup $\mathcal{S}$ is called a *bi-ideal* of $\mathcal{S}$ if $\mathcal{K}\Gamma\mathcal{S}\Gamma\mathcal{K} \subseteq \mathcal{K}$.

Definition 2.1. *[17] Let $\mathcal{S}$ be a Γ-semigroup, $\mathcal{K}$ be a non-empty subset of $\mathcal{S}$, for all $e \in \mathcal{S}$ and $\alpha, \beta \in \Gamma$. Then $\mathcal{K}$ is said to be*

1. *a* left (right) almost ideal *of Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ such that $(e\Gamma\mathcal{K}) \cap \mathcal{K} \neq \emptyset$ $((\mathcal{K}\Gamma e) \cap \mathcal{K} \neq \emptyset)$*
2. *an* almost bi-ideal *of Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ such that $(\mathcal{K}\Gamma e\Gamma\mathcal{K}) \cap \mathcal{K} \neq \emptyset$.*
3. *an* almost quasi-ideal *of Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ such that $(e\Gamma\mathcal{K} \cap \mathcal{K}\Gamma e) \cap \mathcal{K} \neq \emptyset$.*
4. *a left α-ideal of a Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ such that $\mathcal{S}\alpha\mathcal{K} \subseteq \mathcal{K}$. A right β-ideal of a Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ such that $\mathcal{K}\beta\mathcal{S} \subseteq \mathcal{K}$.*
5. *an* (α, β)-ideal *of a Γ-semigroup $\mathcal{S}$ is a non-empty set $\mathcal{K}$ such that it is both a left α-ideal and a right β-ideal of $\mathcal{S}$.*

We see that for any $\zeta_1, \zeta_2 \in [0, 1]$, we have

$$\zeta_1 \vee \zeta_2 = \max\{\zeta_1, \zeta_2\} \quad \text{and} \quad \zeta_1 \wedge \zeta_2 = \min\{\zeta_1, \zeta_2\}.$$

A fuzzy set υ of a non-empty set $\mathcal{T}$ is function from $\mathcal{T}$ into unit closed interval $[0, 1]$ of real numbers, i.e., $\upsilon : \mathcal{T} \to [0, 1]$.

For any two fuzzy sets υ and ν of a non-empty set $\mathcal{T}$, define $\geq, =, \wedge$, and $\vee$ as follows:

1. $\upsilon \geq \nu \Leftrightarrow \upsilon(e) \geq \nu(e)$ for all $e \in \mathcal{T}$,
2. $\upsilon = \nu \Leftrightarrow \upsilon \geq \nu$ and $\nu \geq \upsilon$,
3. $(\upsilon \wedge \nu)(e) = \min\{\upsilon(e), \nu(e)\} = \upsilon(e) \wedge \nu(e)$ for all $e \in \mathcal{T}$,
4. $(\upsilon \vee \nu)(e) = \max\{\upsilon(e), \nu(e)\} = \upsilon(e) \vee \nu(e)$ for all $e \in \mathcal{T}$.

   For the symbol $\upsilon \leq \nu$, we mean $\nu \geq \upsilon$.

The following definitions are types of fuzzy subsemigroups on Γ-semigroups.

Definition 2.2. *[17] A fuzzy set υ of a Γ-semigroup $\mathcal{S}$ is said to be*

1. *a fuzzy subsemigroup of $\mathcal{S}$ if $\upsilon(e\gamma f) \geq \upsilon(e) \wedge \upsilon(f)$ for all $e, f \in \mathcal{S}$ and $\gamma \in \Gamma$,*
2. *a fuzzy left (right) ideal of $\mathcal{S}$ if $\upsilon(e\gamma f) \geq \upsilon(f)$ $(\upsilon(e\gamma f) \geq \upsilon(e))$ for all $e, f \in \mathcal{S}$ and $\gamma \in \Gamma$,*

(3) *a fuzzy ideal of* $\mathcal{S}$ *if it is both a fuzzy left ideal and a fuzzy right ideal of* $\mathcal{S}$,

(4) *a fuzzy bi-ideal of* $\mathcal{S}$ *if* υ *is a fuzzy subsemigroup of* $\mathcal{S}$ *and* $\upsilon(e\gamma f\beta h) \geq \upsilon(e) \wedge \upsilon(h)$ *for all* $e, f, h \in \mathcal{S}$ *and* $\gamma, \beta \in \Gamma$.

Now, we review the concept of interval valued fuzzy sets.

Let $CS[0,1]$ be the set of all closed subintervals of $[0,1]$, i.e.,

$$CS[0,1] = \{\breve{\omega} = [\omega_l, \omega_u] \mid 0 \leq \omega_l \leq \omega_u \leq 1\},$$

where ω_l is a lower interval value of $\breve{\omega}$ and ω_u is an upper interval value of $\breve{\omega}$.

We note that $[\omega, \omega] = \{\omega\}$ for all $\omega \in [0,1]$. For $\omega = 0$ or 1, we shall denote $[0,0]$ by $\breve{0}$ and $[1,1]$ by $\breve{1}$.

For $\breve{\omega} := [\omega_l, \omega_u]$ and $\breve{\zeta} := [\zeta_l, \zeta_u]$ in $CS[0,1]$, the operations "$\preceq$", "$=$", "$\curlywedge$", "$\curlyvee$" are defined as follows:

(1) $\breve{\omega} \preceq \breve{\zeta}$ if and only if $\omega_l \leq \zeta_l$ and $\omega_u \leq \zeta_u$

(2) $\breve{\omega} = \breve{\zeta}$ if and only if $\omega_l = \zeta_l$ and $\omega_u = \zeta_u$

(3) $\breve{\omega} \curlywedge \breve{\zeta} = [(\omega_l \wedge \zeta_l), (\omega_u \wedge \zeta_u)]$

(4) $\breve{\omega} \curlyvee \breve{\zeta} = [(\omega_l \vee \zeta_l), (\omega_u \vee \zeta_u)]$.

If $\breve{\omega} \succeq \breve{\zeta}$, we mean $\breve{\zeta} \preceq \breve{\omega}$.

Definition 2.3. *[19] An interval valued fuzzy set (shortly, IVF set) of a non-empty set* $\mathcal{T}$ *is a function* $\breve{\omega} : \mathcal{T} \to CS[0,1]$.

Next, we shall give definitions of various types of IVF subsemigroups.

Definition 2.4. *[1] An IVF set* $\breve{\omega}$ *of a* Γ*-semigroup* $\mathcal{S}$ *is said to be an IVF subsemigroup of* $\mathcal{S}$ *if* $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(e) \curlywedge \breve{\omega}(f)$ *for all* $e, f \in \mathcal{S}$ *and* $\alpha \in \Gamma$.

Definition 2.5. *[1] An IVF set* $\breve{\omega}$ *of a semigroup* $\mathcal{S}$ *is said to be an IVF left (right) ideal of* $\mathcal{S}$ *if* $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(e)$ $(\breve{\omega}(e\alpha f) \succsim \breve{\omega}(e))$ *for all* $e, f \in \mathcal{S}$ *and* $\alpha \in \Gamma$.

An IVF subset $\breve{\omega}$ *of* $\mathcal{S}$ *is called an IVF ideal of* $\mathcal{S}$ *if it is both an IVF left ideal and an IVF right ideal of* $\mathcal{S}$.

Definition 2.6. *[1] Let* $\mathcal{K}$ *be a subset of a non-empty set* $\mathcal{T}$. *An interval valued characteristic function (shortly, IVCF)* $\breve{\chi}_{\mathcal{K}}$ *of* $\mathcal{T}$ *is defined to be a function* $\breve{\chi}_{\mathcal{K}} : \mathcal{T} \to CS[0,1]$ *by*

$$\breve{\chi}_{\mathcal{K}}(e) = \begin{cases} \breve{1} & \text{if} \quad e \in \mathcal{K}, \\ \breve{0} & \text{if} \quad e \notin \mathcal{K} \end{cases}$$

for all $e \in \mathcal{T}$.

For two IVF subsets $\breve{\omega}$ and $\breve{\zeta}$ of a non-empty set $\mathcal{T}$, define

(1) $\breve{\omega} \sqsubseteq \breve{\zeta} \Leftrightarrow \breve{\omega}(e) \preceq \breve{\zeta}(e)$ for all $e \in \mathcal{T}$,

(2) $\breve{\omega} = \breve{\zeta} \Leftrightarrow \breve{\omega} \sqsubseteq \breve{\zeta}$ and $\breve{\zeta} \sqsubseteq \breve{\omega}$,

(3) $(\breve{\omega} \sqcap \breve{\zeta})(e) = \breve{\omega}(e) \curlywedge \breve{\zeta}(e)$ for all $e \in \mathcal{T}$.

(4) $(\breve{\omega} \sqcup \breve{\zeta})(e) = \breve{\omega}(e) \curlyvee \breve{\zeta}(e)$ for all $e \in \mathcal{T}$.

Definition 2.7. *[10] A* cubic set *(CB set) $\mathfrak{C}$ of a non-empty set $\mathcal{T}$ is a structure of the form*

$$\mathfrak{C} = \{\langle e, \breve{\omega}(e), \upsilon(r)\rangle \mid e \in \mathcal{T}\}$$

and denoted by $\mathfrak{C} = \langle \breve{\omega}, \upsilon\rangle$ where $\breve{\omega}$ is an IVF set and υ is a fuzzy set. In this case, we will use

$$\mathfrak{C}(e) = \langle \breve{\omega}(e), \upsilon(e)\rangle = \langle [\omega_l(e), \omega_u(e)], \upsilon(e)\rangle$$

for all $e \in \mathcal{T}$.

Definition 2.8. *[11] Let $\mathcal{T}$ be a semigroup and $\mathcal{K}$ be a non-empty set of $\mathcal{T}$, the characteristic CB set of $\mathcal{K}$ in $\mathcal{T}$ is defined to be the structure $\geq_{\mathcal{K}} = \{\langle e, \breve{\omega}_{\lambda_{\mathcal{K}}}(e), \upsilon_{\lambda_{\mathcal{K}}}(e)\rangle : e \in \mathcal{T}\}$ which is briefly denoted by $\geq_{\mathcal{K}} = \langle \breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}}\rangle$ where*

$$\breve{\omega}_{\lambda_{\mathcal{K}}}(e) = \begin{cases} \breve{1}, & \text{if} \quad e \in \mathcal{K}, \\ \breve{0}, & \text{if} \quad e \notin \mathcal{K} \end{cases}$$

and

$$\upsilon_{\lambda_{\mathcal{K}}}(e) = \begin{cases} 0, & \text{if} \quad e \in \mathcal{K}, \\ 1, & \text{if} \quad e \notin \mathcal{K}. \end{cases}$$

Definition 2.9. *[2] A CB set $\mathfrak{C} = \langle \breve{\omega}, \upsilon\rangle$ of a Γ-semigroup $\mathcal{S}$ is called*

(1) *a* CB subsemigroup *of $\mathcal{S}$ if $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(e) \curlywedge \breve{\omega}(f)$ and $\upsilon(e\alpha f) \leq \upsilon(e) \vee \upsilon(f)$ for all $e, f \in \mathcal{S}$ and $\alpha \in \Gamma$.*

(2) *a* CB left*(right)ideal of $\mathcal{S}$ if $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(f)$ $(\breve{\omega} e\alpha f) \succeq \breve{\omega}(e))$ and $\upsilon(e\alpha f) \leq \upsilon(f)(\upsilon(e\alpha f) \leq \upsilon(e))$ for all $e, f \in \mathcal{S}$ and $\alpha \in \Gamma$.*

A CB ideal *of $\mathcal{S}$ if it is both a CB left ideal and a CB right ideal of $\mathcal{S}$.*

For $e \in \mathcal{T}$, define $F_e = \{(y, z) \in \mathcal{T} \times \mathcal{T} \mid e = yz\}$.

Definition 2.10. *[11] Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon\rangle$ and $\mathfrak{D} = \langle \breve{\zeta}, \nu\rangle$ be two CB set in a semigroup $\mathcal{S}$. Then the CB product of $\mathfrak{C}$ and $\mathfrak{D}$ is a structure*

$$\mathfrak{C} \boxdot \mathfrak{D} = \{\langle e, (\breve{\omega} \Box \breve{\zeta})(e, (\upsilon \cdot \nu)(e)\rangle : e \in \mathcal{S}\}$$

which is briefly denoted by $\mathfrak{C} \boxdot \mathfrak{D} = \langle (\breve{\omega} \Box \breve{\zeta}), (\upsilon \cdot \nu)\rangle$ where $\breve{\omega} \Box \breve{\zeta}$ and $\upsilon \cdot \nu$ are defined as follows, respectively:

$$(\breve{\omega} \Box \breve{\zeta})(e) = \begin{cases} \mathrm{rsup}_{(y,z) \in F_e}\{\breve{\omega}(y) \curlywedge \breve{\zeta}(z)\} & \text{if} \quad F_e \neq \emptyset, \\ \breve{0}, & \text{if} \quad F_e = \emptyset, \end{cases}$$

and

$$(\upsilon \cdot \nu)(e) = \begin{cases} \inf_{(y,z)\in F_e}\{\upsilon(y) \vee \nu(z)\} & \text{if} \quad F_e \neq \emptyset, \\ 1, & \text{if} \quad F_e = \emptyset. \end{cases}$$

Definition 2.11. *[11] For two CB st $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ in a semigroup $\mathcal{S}$, we define*

$$\mathfrak{C} \sqsubseteq \mathfrak{D} \Leftrightarrow \breve{\omega} \precsim \breve{\rho} \quad \text{and} \quad \upsilon \geq \tau$$

Definition 2.12. *[11] Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ be two CB set in a semigroup $\mathcal{S}$. Then the intersection of $\mathfrak{C}$ and $\mathfrak{D}$ denoted by $\mathfrak{C}\bar{\sqcap}\mathfrak{D}$ is the CB set*

$$\mathfrak{C}\check{\sqcap}\mathfrak{D} = \langle \breve{\omega} \sqcap \check{\zeta}, \upsilon \vee \nu \rangle$$

where $(\mathfrak{C} \sqcap \mathfrak{D})(e) = \breve{\omega}(e) \curlywedge \breve{\rho}(e)$ and $(\upsilon \vee \tau)(e) = \upsilon(e) \vee \tau(e)$ for all $e \in \mathcal{S}$. And union of $\mathfrak{C}$ and $\mathfrak{D}$ denoted by $\mathfrak{C}\bar{\sqcup}\mathfrak{D}$ is the CB set

$$\mathfrak{C}\check{\sqcup}\mathfrak{D} = \langle \breve{\omega} \sqcup \breve{\rho}, \upsilon \wedge \tau \rangle$$

where $(\breve{\omega} \sqcup \breve{\rho})(e) = \breve{\omega}(e) \curlyvee \breve{\rho}(e)$ and $(\upsilon \wedge \tau)(e) = \upsilon(e) \wedge \tau(e)$ for all $e \in \mathcal{S}$.

3. New Types of Cubic Ideals

In this section, we define cubic fuzzy (α, β)-ideal and study basic properties of it.

Definition 3.1. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB set of a Γ-semigroup $\mathcal{S}$ and $\alpha, \beta \in \Gamma$. Then $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is called*

(1) *a* CB left α-ideal *of $\mathcal{S}$ if $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(f)$ and $\upsilon(e\alpha f) \leq \upsilon(f)$ for all $e, f \in \mathcal{S}$.*
(2) *a* CB right β-ideal *of $\mathcal{S}$ if $\breve{\omega}(e\beta f) \succsim \breve{\omega}(e)$ and $\upsilon(e\beta f) \leq \upsilon(f)$ for all $e, f \in \mathcal{S}$.*
(3) *a* CB (α, β)-ideal *of $\mathcal{S}$ if it is both a CB left α-ideal and a CB right β-ideal of $\mathcal{S}$.*
(4) *a* CB α-ideal *of $\mathcal{S}$ if it is a CB (α, α)-ideal of $\mathcal{S}$.*

Theorem 3.1. *Let $\mathcal{K}$ be a nonempty subset of Γ-semigroup $\mathcal{S}$. Then $\mathcal{K}$ is a left α-ideal (right β-ideal, (α, β)-ideal) of $\mathcal{S}$ if and only if $\geq_{\mathcal{K}} = \langle \breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}} \rangle$ is a CB left α-ideal (right β-ideal, (α, β)-ideal) of S.*

Proof. Suppose that $\mathcal{K}$ is a left α-ideal of $\mathcal{S}$ and $e, f \in \mathcal{S}$.

If $f \in \mathcal{K}$, then $e\alpha f \in \mathcal{K}$. Thus $\breve{\omega}_{\lambda_{\mathcal{K}}}(f) = \breve{\omega}_{\lambda_{\mathcal{K}}}(e\alpha f) = \check{1}$ and $\upsilon_{\lambda_{\mathcal{K}}}(f) = \upsilon_{\lambda_{\mathcal{K}}}(e\alpha f) = 0$. Hence $\breve{\omega}_{\lambda_{\mathcal{K}}}(e\alpha f) \succsim \breve{\omega}_{\lambda_{\mathcal{K}}}(f)$ and $\upsilon_{\lambda_{\mathcal{K}}}(e\alpha f) \leq \upsilon_{\lambda_{\mathcal{K}}}(f)$.

If $f \notin \mathcal{K}$, then $e\alpha f \in \mathcal{K}$. Thus $\breve{\omega}_{\lambda_{\mathcal{K}}}(f) = \check{0}, \breve{\omega}_{\lambda_{\mathcal{K}}}(e\alpha f) = \check{1}$ and $\upsilon_{\lambda_{\mathcal{K}}}(f) = 1, \upsilon_{\lambda_{\mathcal{K}}}(e\alpha f) = 0$. Hence, $\breve{\omega}_{\lambda_{\mathcal{K}}}(e\alpha f) \succsim \breve{\omega}_{\lambda_{\mathcal{K}}}(f)$ and $\upsilon_{\lambda_{\mathcal{K}}}(e\alpha f) \leq \upsilon_{\lambda_{\mathcal{K}}}(f)$.

Therefore $\geq_{\mathcal{K}} = \langle \breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}} \rangle$ is a CB left α-ideal of $\mathcal{S}$.

Conversely, assume that $\geq_{\mathcal{K}} = \langle \breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}} \rangle$ is a CB left α-ideal of $\mathcal{S}$ and $e, f \in \mathcal{S}$ with $f \in \mathcal{K}$. Then $\breve{\omega}_{\lambda_{\mathcal{K}}}(f) = \check{1}$ and $\upsilon_{\lambda_{\mathcal{K}}}(f) = 0$. By assumption, $\breve{\omega}_{\lambda_{\mathcal{K}}}(e\alpha f) \succsim \breve{\omega}_{\lambda_{\mathcal{K}}}(f)$ and $\upsilon_{\lambda_{\mathcal{K}}}(e\alpha f) \leq \upsilon_{\lambda_{\mathcal{K}}}(f)$. Thus, $e\alpha f \in \mathcal{K}$. Hence, $\mathcal{K}$ is a left α-ideal of $\mathcal{S}$. $\square$

Theorem 3.2. *The intersection and union of any two CB left α-ideals (right β-ideals, (α,β)-ideals) of a Γ-semigroup $\mathcal{S}$ is a CB left α-ideal (right β-ideal, (α,β)-ideal) of $\mathcal{S}$.*

Proof. Let $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ and $\mathfrak{D} = \langle\breve{\rho}, \tau\rangle$ be CB left α-ideals of $\mathcal{S}$ and
let $e, f \in \mathcal{S}$. Then

$$(\breve{\omega} \sqcap \breve{\rho})(e\alpha f) = \breve{\omega}(e\alpha f) \curlywedge \breve{\rho}(e\alpha f) \succsim \breve{\omega}(f) \curlywedge \breve{\rho}(f) = (\breve{\omega} \sqcap \breve{\rho})(f)$$

and

$$(\upsilon \cap \tau)(e\alpha f) = \upsilon(e\alpha f) \vee \tau(e\alpha f) \leq \upsilon(v) \vee \tau(f) = (\upsilon \cap \tau)(f).$$

Similarly,

$$(\breve{\omega} \sqcup \breve{\rho})(e\alpha f) = \breve{\omega}(e\alpha f) \curlyvee \breve{\rho}(e\alpha f) \succsim \breve{\omega}(f) \curlyvee \breve{\rho}(f) = (\breve{\omega} \sqcap \breve{\rho})(f)$$

and

$$(\upsilon \cup \tau)(e\alpha f) = \upsilon(e\alpha f) \wedge \tau(e\alpha f) \leq \upsilon(v) \wedge \tau(f) = (\upsilon \cup \tau)(f).$$

Thus, $\mathfrak{C}\breve{\sqcap}\mathfrak{D}$ and $\mathfrak{C}\breve{\sqcup}\mathfrak{D}$ are CB left α-ideals of $\mathcal{S}$. □

Theorem 3.3. *Let $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ be a CB set of a Γ-semigroup $\mathcal{S}$ and $\mathfrak{C}_{(\check{n},m)} = (\breve{\omega}_{\check{n}}, \upsilon_m)$ be CB point with $\breve{\omega}_{\check{n}} = \{f \in \mathcal{S} \mid \breve{\omega}_{\check{n}}(f) \succsim \check{n}\}$ and $\upsilon_m = \{f \in \mathcal{S} \mid \upsilon_m(f) \leq m\}$. Then $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ is a CB left α-ideal (right β-ideal, (α,β)-ideal) of $\mathcal{S}$ if and only if $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ is a nonempty set and $\mathfrak{C}_{(\check{n},m)}$ is a left α-ideal (right β-ideal, (α,β)-ideal) of $\mathcal{S}$ for all $(\check{n}, m) \in (0,1] \times [0,1)$.*

Proof. Suppose that $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ is a CB left α-ideal of $\mathcal{S}$ and $\alpha \in \Gamma$. Then $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(f)$ and $\upsilon(e\alpha f) \leq \upsilon(f)$ for all $e, f \in \mathcal{S}$. Let $(\check{n}, m) \in (0,1] \times [0,1)$ be such that $\mathfrak{C}_{(\check{n},m)} \neq \emptyset$. Let $f \in \mathfrak{C}_{(\check{n},m)}$ and $f \in \mathcal{S}$. Then $\breve{\omega}(f) \succsim \check{1}$ and $\upsilon(f) \leq m$. Thus $\breve{\omega}(e\alpha f) \succsim \breve{\omega}(f) \succsim \check{n}$ and $\upsilon(e\alpha f) \leq \upsilon(f) \leq m$ so $e\alpha f \in \mathfrak{C}_{(\check{n},m)}$. Hence, $\mathfrak{C}_{(\check{n},m)} = (\breve{\omega}_{\check{n}}, \upsilon_m)$ is a left α-ideal of $\mathcal{S}$.

Conversely, assume that $\mathfrak{C}_{(\check{n},m)} = (\breve{\omega}_{\check{n}}, \upsilon_m)$ is a left α-ideal of $\mathcal{S}$ if $(\check{n}, m) \in (0,1] \times [0,1)$ and $\mathfrak{C}_{(\check{n},m)} \neq \emptyset$. Let $e, f \in \mathcal{S}$ and $\check{n} = \breve{\omega}(f), m = \upsilon(f)$. By assumption $\breve{\omega}(f) \succsim \check{n}$ and $\upsilon(f) \leq m$. Then $f \in \mathfrak{C}_{(\check{n},m)}$. Thus $\mathfrak{C}_{(\check{n},m)} \neq \emptyset$. Hence, $\mathfrak{C}_{(\check{n},m)}$ is a left α-ideal of $\mathcal{S}$. Since $f \in \mathfrak{C}_{(\check{n},m)}$ and $e \in \mathcal{S}$, we have $e\alpha f \in \mathfrak{C}_{(\check{n},m)}$. Thus, $\breve{\omega}(e\alpha f) \geq \check{n} = \breve{\omega}(f)$ and $\upsilon(e\alpha f) \leq m = \upsilon(f)$. Hence, $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ is a CB left α-ideal of $\mathcal{S}$. □

Next, we will define the (α,β)-product.

For CB $\mathfrak{C} = \langle\breve{\omega}, \upsilon\rangle$ and $\mathfrak{D} = \langle\breve{\rho}, \tau\rangle$, define product $\breve{\omega} \circ_\alpha \breve{\rho}$ and $\upsilon \circ_\alpha \tau$ as follows: For $e \in \mathcal{S}$

$$(\breve{\omega}\Box_\alpha\breve{\rho})(e) = \begin{cases} \mathrm{rsup}_{(y,z)\in F_{e_\alpha}}\{\breve{\omega}(y) \curlywedge_\alpha \breve{\rho}(z)\} & \text{if} \quad Fe_\alpha \neq \emptyset, \\ \breve{0}, & \text{if} \quad F_{e_\alpha} = \emptyset, \end{cases}$$

and

$$(\upsilon \cdot_\alpha \tau)(e) = \begin{cases} \inf_{(y,z)\in F_e}\{\upsilon(y) \vee_\alpha \tau(z)\} & \text{if} \quad Fe_\alpha \neq \emptyset, \\ 1, & \text{if} \quad F_{e_\alpha} = \emptyset. \end{cases}$$

where $F_{e_\alpha} = \{(y,z) \in \mathcal{S} \times \mathcal{S} \mid e = yz\}$, for $e \in \mathcal{S}$ and $\alpha \in \Gamma$.

Next, we define CB (α,β)-bi-ideal and study basic properties of it.

Definition 3.2. *Let $\mathfrak{C}=\langle\breve{\omega},\upsilon\rangle$ be a CB set of a Γ-semigroup $\mathcal{S}$ and $\alpha,\beta\in\Gamma$. Then $\mathfrak{C}=\langle\breve{\omega},\upsilon\rangle$ is called a CB (α,β)-bi-ideal of $\mathcal{S}$ if $\breve{\omega}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega}\gtrsim\breve{\omega}$ and $\upsilon\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon\leq\upsilon$ where $\geq_\mathcal{S}=\langle\breve{\omega}_{\lambda_\mathcal{S}},\upsilon_{\lambda_\mathcal{S}}\rangle$ is CB set mapping every element of $\mathcal{S}$ to $\langle\check{1},0\rangle$.*

Theorem 3.4. *Let $\mathcal{K}$ be a nonempty subset of Γ-semigroup $\mathcal{S}$. Then $\mathcal{K}$ is an (α,β)-bi-ideal of $\mathcal{S}$ if and only if characteristic function $\geq_\mathcal{K}=\langle\breve{\omega}_{\lambda_\mathcal{K}},\upsilon_{\lambda_\mathcal{K}}\rangle$ is a CB (α,β)-bi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is an (α,β)-bi-ideal of $\mathcal{S}$ and $\mathcal{K}\alpha\mathcal{S}\beta\mathcal{K}\subseteq\mathcal{K}$.

If $e\in\mathcal{K}\alpha\mathcal{S}\beta\mathcal{K}$, then $\breve{\omega}_{\lambda_\mathcal{K}}(e)=(\breve{\omega}_{\lambda_\mathcal{K}}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega}_{\lambda_\mathcal{K}})(e)=\check{1}$ and
$\upsilon_{\lambda_\mathcal{K}}(e)=(\upsilon_{\lambda_\mathcal{K}}\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon_{\lambda_\mathcal{K}})(e)=0$.
Hence, $(\breve{\omega}_{\lambda_\mathcal{K}}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega}_{\lambda_\mathcal{K}})(e)\gtrsim\breve{\omega}_{\lambda_\mathcal{K}}(e)$ and $(\upsilon_{\lambda_\mathcal{K}}\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon_{\lambda_\mathcal{K}})\leq\upsilon_{\lambda_\mathcal{K}}(e)$

If $e\in\mathcal{K}\alpha\mathcal{S}\beta\mathcal{K}$, then $\breve{\omega}_{\lambda_\mathcal{K}}(e)=(\breve{\omega}_{\lambda_\mathcal{K}}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega}_{\lambda_\mathcal{K}})(e)=\check{0}$ and
$\upsilon_{\lambda_\mathcal{K}}(e)=(\upsilon_{\lambda_\mathcal{K}}\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon_{\lambda_\mathcal{K}})(e)=1$.
Hence, $(\breve{\omega}_{\lambda_\mathcal{K}}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega}_{\lambda_\mathcal{K}})(e)\gtrsim\breve{\omega}_{\lambda_\mathcal{K}}(e)$ and $(\upsilon_{\lambda_\mathcal{K}}\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon_{\lambda_\mathcal{K}})\leq\upsilon_{\lambda_\mathcal{K}}(e)$.

Therefore, $\geq_\mathcal{K}=(\breve{\omega}_{\lambda_\mathcal{K}},\upsilon_{\lambda_\mathcal{K}})$ is a CB (α,β)-bi-ideal of $\mathcal{S}$.

Conversely, assume that $\geq_\mathcal{K}=(\breve{\omega}_{\lambda_\mathcal{K}},\upsilon_{\lambda_\mathcal{K}})$ is a CB (α,β)-bi-ideal of $\mathcal{S}$. and
$e\in\mathcal{K}\alpha\mathcal{S}\beta\mathcal{K}$. Then $(\breve{\omega}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{K}}\circ_\beta\breve{\omega})(e)=\check{1}$ and $(\upsilon\circ_\alpha\geq_S\circ_\beta\upsilon)(e)=0$.
By assumption, $(\breve{\omega}_{\lambda_\mathcal{K}}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega}_{\lambda_\mathcal{K}})(e)\gtrsim\breve{\omega}_{\lambda_\mathcal{K}}(e)$ and $(\upsilon_{\lambda_\mathcal{K}}\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon_{\lambda_\mathcal{K}})\leq\upsilon_{\lambda_\mathcal{K}}(e)$.
Thus, $e\in\mathcal{K}$. Hence, $\mathcal{K}$ is an (α,β)-bi-ideal of $\mathcal{S}$. □

Theorem 3.5. *The intersection of any two CB (α,β)-bi-ideals of a Γ-semigroup $\mathcal{S}$ is a CB (α,β)-bi-ideal of $\mathcal{S}$.*

Proof. Let $\mathfrak{C}=\langle\breve{\omega},\upsilon\rangle$ and $\mathfrak{D}=\langle\breve{\rho},\tau\rangle$ be CB (α,β)-bi-ideals of $\mathcal{S}$ and $e\in\mathcal{S}$. Then

$$((\breve{\omega}\sqcap\breve{\rho})\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta(\breve{\omega}\sqcap\breve{\rho}))(e)\gtrsim(\breve{\omega}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\omega})(e)\curlywedge(\breve{\rho}\circ_\alpha\breve{\omega}_{\lambda_\mathcal{S}}\circ_\beta\breve{\rho})(e)\gtrsim(\breve{\omega}\sqcap\breve{\rho})(e)$$

and

$$((\upsilon\cap\tau)\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta(\upsilon\cap\tau))(e)\leq(\upsilon\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\upsilon)(u)\vee(\tau\circ_\alpha\upsilon_{\lambda_\mathcal{S}}\circ_\beta\tau)(u)\leq(\upsilon\cap\tau)(e).$$

Thus, $\mathfrak{C}\breve{\sqcap}\mathfrak{D}$ is a CB α-bi-ideals of $\mathcal{S}$. □

Next, we define CB (α,β)-quasi-ideal and study basic properties of it.

Definition 3.3. *Let $\mathfrak{C}=\langle\breve{\omega},\upsilon\rangle$ be a CB set of a Γ-semigroup S and $\alpha,\beta\in\Gamma$. Then $\mathfrak{C}=\langle\breve{\omega},\upsilon\rangle$ is called a* CB (α,β)-quasi-ideal *of $\mathcal{S}$ if $\breve{\omega}_{\lambda_\mathcal{S}}\circ_\alpha\breve{\omega}\sqcap\breve{\omega}\circ_\beta\breve{\omega}_{\lambda_\mathcal{S}}\precsim\breve{\omega}$ and $\upsilon_{\lambda_\mathcal{S}}\circ_\alpha\upsilon\cup\upsilon\circ_\beta\upsilon_{\lambda_\mathcal{S}}\geq\upsilon$.*

Theorem 3.6. *If $\mathfrak{C}=\langle\breve{\omega},\upsilon\rangle$ and $\mathfrak{D}=\langle\breve{\rho},\eta\rangle$ is a CB left α-ideal and a CB right α-ideal of $\mathcal{S}$ respectively, then $\mathfrak{C}\breve{\sqcap}\mathfrak{D}$ is a CB α-quasi-ideal of S.*

Proof. Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ and $\mathfrak{D} = \langle \breve{\rho}, \eta \rangle$ is a CB left α-ideal and a CB right α-ideal of $\mathcal{S}$ respectively. Then $\breve{\rho} \circ_\alpha \breve{\omega} \precsim \breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega} \precsim \breve{\omega}$ and $\breve{\rho} \circ_\alpha \breve{\omega} \precsim \breve{\rho} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}} \precsim \breve{\rho}$. Thus, $\breve{\rho} \circ_\alpha \breve{\omega} \precsim \breve{\omega} \cap \breve{\rho}$. So,

$$\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha (\breve{\omega} \sqcap \breve{\rho}) \sqcap (\breve{\omega} \sqcap \breve{\rho}) \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}} \sqsubseteq \breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha (\breve{\omega} \sqcap \breve{\rho}) \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}} \precsim \breve{\omega} \cap \breve{\rho}.$$

Thus, $\breve{\omega} \sqcap \breve{\rho}$ is a CB α-quasi-ideal of $\mathcal{S}$. Similarly, we can show that $\upsilon \cap \eta$ is a CS α-quasi-ideal of S. Hence, $\mathfrak{C} \breve{\sqcap} \mathfrak{D}$ is a CB α-quasi-ideal of $\mathcal{S}$. □

Theorem 3.7. *Every CB (α, β)-quasi-ideal of Γ-semigroup $\mathcal{S}$ is intersection of a CB left α-ideal and a CB right β-ideal of $\mathcal{S}$.*

Proof. Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB (α, β)-quasi-ideal of $\mathcal{S}$.
Consider $\breve{\rho} = \breve{\omega} \sqcup (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega})$ and $\tau = \upsilon \cup (\upsilon_{\lambda_\mathcal{S}} \circ_\alpha \upsilon)$ where $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$, $\breve{\varpi} = \breve{\omega} \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}})$ and $\nu = \upsilon \cup (\upsilon \circ_\beta \upsilon_{\lambda_\mathcal{S}})$ where $\mathfrak{K} = \langle \breve{\varpi}, \nu \rangle$. Then

$$\begin{aligned} \breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\rho} &= \breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha (\breve{\omega} \sqcup (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega})) = (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega}) \sqcup (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega})) \\ &= (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega}) \sqcup ((\geq_\mathcal{S} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) \circ_\alpha \breve{\omega}) = (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega}) \sqcup (\geq_\mathcal{S} \circ_\alpha \breve{\omega}) \\ &\precsim \breve{\omega} \sqcup (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega}) = \breve{\rho}. \end{aligned}$$

And

$$\begin{aligned} \breve{\varpi} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}} &= (\breve{\omega} \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}})) \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}} = (\breve{\omega} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) \\ &= (\breve{\omega} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) \sqcup \breve{\omega} \circ_\beta (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) = (\breve{\omega} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}}) \\ &\precsim \breve{\omega} \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}}) = \breve{\varpi}. \end{aligned}$$

Simlarly, we can show that $\upsilon_{\lambda_\mathcal{S}} \circ_\alpha \tau \geq \tau$ and $\nu \circ_\beta \upsilon_{\lambda_\mathcal{S}} \geq \nu$.
Thus $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ and $\mathfrak{K} = \langle \breve{\varpi}, \nu \rangle$ is a CB left α-ideal and a CB right β-ideal of S. Now,

$$\breve{\omega} \sqsubseteq (\breve{\omega} \sqcup (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega})) \sqcap (\breve{\omega} \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}})) = \breve{\rho} \sqcap \breve{\varpi}$$

and

$$\breve{\rho} \cap \breve{\varpi} = (\breve{\omega} \sqcup (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\alpha \breve{\omega})) \sqcap (\breve{\omega} \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}})) = \breve{\omega} \cap ((\geq_\mathcal{S} \circ_\alpha \breve{\omega}) \sqcup (\breve{\omega} \circ_\beta \breve{\omega}_{\lambda_\mathcal{S}})) \precsim \breve{\omega} \sqcap \breve{\omega} = \breve{\omega}.$$

Hence, $\breve{\omega} = \breve{\rho} \sqcap \breve{\varpi}$. Simlarly, we can show that $\upsilon = \tau \cap \nu$. □

Theorem 3.8. *Let $\mathcal{K}$ be a nonempty subset of Γ-semigroup $\mathcal{S}$. Then $\mathcal{K}$ is a (α, β)-quasi-ideal of $\mathcal{S}$ if and only if characteristic function $\geq_\mathcal{K} = (\breve{\omega}_{\lambda_\mathcal{K}}, \upsilon_{\lambda_\mathcal{K}})$ is a CB (α, β)-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is a (α, β)-quasi-ideal of $\mathcal{S}$ and $e \in \mathcal{S}$.

If $f \in (\mathcal{S}\alpha\mathcal{K}) \cap (\mathcal{K}\beta\mathcal{S})$, then $e \in \mathcal{K}$. Thus $\breve{\omega}_{\lambda_\mathcal{K}}(e) = \check{1}$ and $\upsilon_{\lambda_\mathcal{K}}(e) = 0$.
Hence $((\breve{\omega}_{\lambda_\mathcal{K}} \circ_\alpha \breve{\omega}_{\lambda_\mathcal{S}}) \curlywedge (\breve{\omega}_{\lambda_\mathcal{S}} \circ_\beta \breve{\omega}_{\lambda_\mathcal{K}}))(f) \precsim \breve{\omega}_{\lambda_\mathcal{K}}(u)$ and
$((\upsilon_{\lambda_\mathcal{K}} \circ_\alpha \upsilon_{\lambda_\mathcal{S}}) \vee (\upsilon_{\lambda_\mathcal{S}} \circ_\beta \upsilon_{\lambda_\mathcal{K}}))(u) \geq \upsilon_{\lambda_\mathcal{K}}(f)$.

If $f \notin (\mathcal{S}\alpha\mathcal{K}) \cap (\mathcal{K}\beta\mathcal{S})$, then $e \in \mathcal{K}$. Thus $\breve{\omega}_{\lambda_\mathcal{K}}(e) = \check{0}$ and $\upsilon_{\lambda_\mathcal{K}}(e) = 1$.
Hence, $\breve{\omega}_{\lambda_\mathcal{K}}))(f) \precsim \breve{\omega}_{\lambda_\mathcal{K}}(u)$ and $((\upsilon_{\lambda_\mathcal{K}} \circ_\alpha \upsilon_{\lambda_\mathcal{S}}) \vee (\upsilon_{\lambda_\mathcal{S}} \circ_\beta \upsilon_{\lambda_\mathcal{K}}))(u) \geq \upsilon_{\lambda_\mathcal{K}}(f)$.

Therefore $\geq_\mathcal{K} = (\breve{\omega}_{\lambda_\mathcal{K}}, \upsilon_{\lambda_\mathcal{K}})$ is a CB (α, β)-quasi-ideal of $\mathcal{S}$.

Conversely, assume that $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB (α,β)-quasi-ideal of $\mathcal{S}$ and $f \in (\mathcal{S}\alpha\mathcal{K}) \cap (\mathcal{K}\beta\mathcal{S})$. Then $((\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{S}}}) \curlywedge (\breve{\omega}_{\lambda_{\mathcal{S}}} \circ_\beta \breve{\omega}_{\lambda_{\mathcal{K}}}))(f) = \breve{1}$ and $((\upsilon_{\lambda_{\mathcal{K}}} \circ_\alpha \upsilon_{\lambda_{\mathcal{S}}}) \vee (\upsilon_{\lambda_{\mathcal{S}}} \circ_\beta \upsilon_{\lambda_{\mathcal{K}}}))(f) = 0$. By assumption, $((\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{S}}}) \curlywedge (\breve{\omega}_{\lambda_{\mathcal{S}}} \circ_\beta \breve{\omega}_{\lambda_{\mathcal{K}}}))(f) \precsim \breve{\omega}_{\lambda_{\mathcal{K}}}(f)$ and $((\upsilon_{\lambda_{\mathcal{K}}} \circ_\alpha \upsilon_{\lambda_{\mathcal{S}}}) \vee (\upsilon_{\lambda_{\mathcal{S}}} \circ_\beta \upsilon_{\lambda_{\mathcal{K}}}))(f) \geq \upsilon_{\lambda_{\mathcal{K}}}(f)$ Thus $e \in \mathcal{K}$. Hence, $\mathcal{K}$ is a (α,β)-quasi-ideal of $\mathcal{S}$. □

4. New Types of Cubic Almost Ideals

Definition 4.1. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB set of a Γ-semigroup $\mathcal{S}$ and $\alpha, \beta \in \Gamma$ is said to be*

(1) *a CB almost left α-ideal of $\mathcal{S}$ if $(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap \breve{\omega} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon) \cup \upsilon \neq 1$.*
(2) *a CB almost right β-ideal of $\mathcal{S}$ if $(\breve{\omega} \circ_\beta \breve{\omega}_n) \sqcap \breve{\omega} \neq \breve{0}$ and $(\upsilon \circ_\beta \upsilon_m) \cup \upsilon \neq 1$.*
(3) *a CB almost (α,β)-ideal of $\mathcal{S}$ if it is both a CB almost left α-ideal and a CB almost right β-ideal of $\mathcal{S}$.*

Theorem 4.1. *If $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost left α-ideal (right β-ideal, (α,β)-ideal) of a Γ-semigroup $\mathcal{S}$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB set of $\mathcal{S}$ such that $\mathfrak{C} \sqsubseteq \mathfrak{D}$, then $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB left almost α-ideal (right β-ideal, (α,β)-ideal) of S.*

Proof. Suppose that $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost left α-ideal of $\mathcal{S}$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB set of $\mathcal{S}$ such that $\mathfrak{C} \sqsubseteq \mathfrak{D}$. Then $(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap \breve{\omega} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon) \cup \upsilon \neq 1$. Thus $(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap \breve{\omega} \precsim (\breve{\rho}_n \circ_\alpha \breve{\rho}) \sqcap \breve{\rho} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon) \cup \upsilon \geq (\tau_m \circ_\alpha \tau) \cup \tau \neq 0$. Hence, $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB left almost α-ideal of $\mathcal{S}$. □

Theorem 4.2. *Let $\mathcal{K}$ be a nonempty subset of Γ-semigroup $\mathcal{S}$. Then $\mathcal{K}$ is an almost left α-ideal (right β-ideal, (α,β)-ideal) of $\mathcal{S}$ if and only if characteristic function $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost left α-ideal (right β-ideal, (α,β)-ideal) of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is an almost left α-ideal of $\mathcal{S}$. Then $e\alpha\mathcal{K} \cap \mathcal{K} \neq \emptyset$ for all $e \in \mathcal{S}$. Thus there exists $r \in e\alpha\mathcal{K}$ and $r \in \mathcal{K}$. So $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}})(r) = \breve{\omega}_{\lambda_{\mathcal{K}}}(r) = \breve{1}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}})(r) = \upsilon_{\lambda_{\mathcal{K}}}(r) = 0$. Hence, $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap \breve{\omega}_{\lambda_{\mathcal{K}}} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \cup \upsilon_{\lambda_{\mathcal{K}}} \neq 1$. Therefore, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost left α-ideal of S.

Conversely, assume that $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost left α-ideal of $\mathcal{S}$ and $e \in \mathcal{S}$. Then $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap \breve{\omega}_{\lambda_{\mathcal{K}}} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \cup \upsilon_{\lambda_{\mathcal{K}}} \neq 1$. Thus there exists $r \in \mathcal{S}$ such that $((\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \curlywedge \breve{\omega}_{\lambda_{\mathcal{K}}})(r) \neq \breve{0}$ and $((\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \vee \upsilon_{\lambda_{\mathcal{K}}})(r) \neq 1$. Hence, $r \in e\alpha\mathcal{K} \cap \mathcal{K}$ implies $e\alpha\mathcal{K} \cap \mathcal{K} \neq \emptyset$. Therefore $\mathcal{K}$ is an almost left α-ideal of $\mathcal{S}$. □

Next, we review definition of supp($\mathfrak{C}$) and we study properties between supp(ξ) and CB almost left α-ideal (right β-ideal, (α,β)-ideal) of Γ-semigroups.

Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB set of a non-empty of $\mathcal{S}$. Then the *support* of $\mathfrak{C}$ instead of $\text{supp}(\mathfrak{C}) = \{e \in \mathcal{S} \mid \breve{\omega}(e) \neq \breve{0} \,\text{and}\, \upsilon(e) \neq 0\}$.

Theorem 4.3. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB set of a non-empty of a Γ-semigroup $\mathcal{S}$. Then $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost left α-ideal (right β-ideal, (α, β)-ideal) of $\mathcal{S}$ if and only if* $\mathrm{supp}(\mathfrak{C})$ *is an almost left α-ideal (right β-ideal, (α, β)-ideal) of $\mathcal{S}$.*

Proof. Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB almost left α-ideal of $\mathcal{S}$ and $e \in \mathcal{S}$. Then
$(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap \breve{\omega} \neq \breve{0}$and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_\mathcal{K}}) \cup \upsilon_{\lambda_\mathcal{K}} \neq 1$. Thus there exists $r \in \mathcal{S}$ such that $((\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_\mathcal{K}}) \curlywedge \breve{\omega}_{\lambda_\mathcal{K}})(r) \neq \breve{0}$ and $((\upsilon_m \circ_\alpha \upsilon_{\lambda_\mathcal{K}}) \vee \upsilon_{\lambda_\mathcal{K}})(r) \neq 1$. So there exists $k \in \mathcal{S}$ such that $r = u\alpha k$, $\breve{\omega}(r) \neq \breve{0}$, $\upsilon(r) \neq 0$ and $\breve{\omega}(k) \neq \breve{0}$, $\upsilon(k) \neq 0$. It implies that $r, k \in \mathrm{supp}(\mathfrak{C})$. Thus, $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}})(r) \neq \breve{0}$, $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}})(r) \neq 1$ and $\breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq \breve{0}$, $\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq 1$. Hence, $(\breve{\omega}_m \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \sqcap \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \cup \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq 1$. Therefore, $\mathrm{supp}(\mathfrak{C})$ is a CB almost left α-ideal of $\mathcal{S}$. This show that $\mathrm{supp}(\mathfrak{C})$ is an almost left α-ideal of $\mathcal{S}$.

Conversely, let $\mathrm{supp}(\mathfrak{C})$ be an almost left α-ideal of $\mathcal{S}$. Then by Theorem 4.2, $\geq_{\mathrm{supp}(\mathfrak{C})}$ is a CB almost left α-ideal of $\mathcal{S}$. Thus $(\breve{\omega}_m \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \sqcap \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \cup \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq 1$. So there exists $r \in \mathcal{S}$ such that
$((\breve{\omega}_m \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \curlywedge \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}})(r) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \vee \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}(r) \neq 1$.
It implies that $((\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}))(r) \neq \breve{0}$, $((\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}))(r) \neq 1$ and
$\breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}(r) \neq \breve{0}$, $\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}(r) \neq 1$. Thus there exists $k \in \mathcal{S}$ such that $r = u\alpha k$, $\breve{\omega}_n(r) \neq \breve{0}$, $\upsilon_n(r) \neq 0$ and $\breve{\omega}_n(k) \neq \breve{0}$, $\upsilon_m(k) \neq 0$. Hence, $(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap \breve{\omega} \neq 0$ and $(\upsilon_m \circ_\alpha \upsilon) \cup \upsilon \neq 1$. Therefore, $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB almost left α-ideal of $\mathcal{S}$. □

Definition 4.2. *An almost ideal $\mathcal{I}$ of a Γ-semigroup $\mathcal{S}$ is called a* minimal *if for every almost ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we have $\mathcal{J} = \mathcal{I}$.*

Definition 4.3. *A CB almost left α-ideal (right β-ideal, (α, β)-ideal) $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ of a Γ-semigroup $\mathcal{S}$ is* minimal *if for all CB almost left α-ideal (right β-ideal, (α, β)-ideal) $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ of $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \mathfrak{C}$, then* $\mathrm{supp}(\mathfrak{D}) = \mathrm{supp}(\mathfrak{C})$.

Theorem 4.4. *Let $\mathcal{K}$ be a nonempty subset of a Γ-semigroup $\mathcal{S}$ Then $\mathcal{K}$ is a minimal almost left α-ideal (right β-ideal, (α, β)-ideal) if and only if $\geq_\mathcal{K} = (\breve{\omega}_{\lambda_\mathcal{K}}, \upsilon_{\lambda_\mathcal{K}})$ is a minimal CB almost left α-ideal (right β-ideal, (α, β)-ideal) of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is a minimal almost left α-ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an almost left α-ideal of $\mathcal{S}$. Thus by Theorem 4.2, $\geq_\mathcal{K} = (\breve{\omega}_{\lambda_\mathcal{K}}, \upsilon_{\lambda_\mathcal{K}})$ is a CB left α-ideal of $\mathcal{S}$. Let $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ be a CB left α-ideal of $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \mathfrak{C}$ Then by Theorem 4.3, $\mathrm{supp}(\mathfrak{D})$ is an almost left α-ideal of $\mathcal{S}$. Thus $\mathrm{supp}(\mathfrak{D}) \sqsubseteq \mathrm{supp}(\geq_\mathcal{K}) = \mathcal{K}$. By assumption, $\mathrm{supp}(\mathfrak{D}) = \mathcal{K} = \mathrm{supp}(\geq_\mathcal{K})$. Thus, $\geq_\mathcal{K} = (\breve{\omega}_{\lambda_\mathcal{K}}, \upsilon_{\lambda_\mathcal{K}})$ is a minimal CB almost left α-ideal of $\mathcal{S}$.

Conversely, suppose that $\geq_\mathcal{K} = (\breve{\omega}_{\lambda_\mathcal{K}}, \upsilon_{\lambda_\mathcal{K}})$ is a minimal CB almost left α-ideal of $\mathcal{S}$. Then by Theorem 4.2, $\mathcal{K}$ is an almost left α-ideal of $\mathcal{S}$. Let $\mathcal{J}$ be an almost left α-ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then by Theorem 4.2, $\geq_\mathcal{J} = (\breve{\omega}_{\lambda_\mathcal{J}}, \upsilon_{\lambda_\mathcal{J}})$ is a CS left α-ideal of $\mathcal{S}$ such that $\geq_\mathcal{J} \sqsubseteq \geq_\mathcal{K}$. Thus, $\mathcal{J} = \mathrm{supp}(\geq_\mathcal{J}) = \mathrm{supp}(\geq_\mathcal{K}) = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal almost left α-ideal of $\mathcal{S}$. □

Corollary 4.1. *Let $\mathcal{S}$ be a Γ-semigroup Then $\mathcal{S}$ has no proper almost left α-ideal (right β-ideal, (α, β)-ideal) if and only if for any CB almost left α-ideal (right β-ideal, (α, β)-ideal) $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ of $\mathcal{S}$,* $\text{supp}(\mathfrak{C}) = \mathcal{S}$.

Next, we define CB almost (α, β)-quasi-ideals and we study properties of it.

Definition 4.4. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB set of a Γ-semigroup $\mathcal{S}$ and $\alpha, \beta \in \Gamma$ is said to be* CB almost (α, β)-quasi-ideal *of $\mathcal{S}$ if $(\breve{\omega} \circ_\alpha \breve{\omega}_n) \sqcap (\breve{\omega}_n \circ_\beta \breve{\omega}) \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_m) \vee (\upsilon_m \circ_\beta \upsilon) \neq 1$.*

Theorem 4.5. *If $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-quasi-ideal of a Γ-semigroup $\mathcal{S}$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB set of $\mathcal{S}$ such that $\mathfrak{C} \sqsubseteq \mathfrak{D}$, then $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB (α, β)-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-quasi-ideal of a Γ-semigroup $\mathcal{S}$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB set of $\mathcal{S}$ such that $\mathfrak{C} \sqsubseteq \mathfrak{D}$. Then $(\breve{\omega} \circ_\alpha \breve{\omega}_n) \sqcap (\breve{\omega}_n \circ_\beta \breve{\omega}) \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_m) \cup (\upsilon_m \circ_\beta \upsilon) \neq 1$. Thus, $(\breve{\omega} \circ_\alpha \breve{\omega}_n) \sqcap (\breve{\omega}_n \circ_\beta \breve{\omega}) \precsim (\breve{\rho} \circ_\alpha \breve{\rho}_n) \sqcap (\breve{\rho}_n \circ_\beta \breve{\rho}) \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_m) \cup (\upsilon_m \circ_\beta \upsilon) \geq (\tau \circ_\alpha \tau_m) \cup (\tau_m \circ_\beta \tau) \neq 1$. Hence, $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB (α, β)-quasi-ideal of $\mathcal{S}$. □

Theorem 4.6. *Let $\mathcal{K}$ be a nonempty subset of Γ-semigroup $\mathcal{S}$. Then $\mathcal{K}$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$ if and only if characteristic function $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost (α, β)-quasi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$. Then $(\mathcal{K}\alpha e) \cap (e\beta\mathcal{K}) \cap \mathcal{K} \neq \emptyset$ for all $e \in \mathcal{S}$. Thus there exists $v \in (\mathcal{K}\alpha e) \cap (e\beta\mathcal{K})$ and $v \in \mathcal{K}$. So $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \curlywedge (\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\beta \breve{\omega}_n)(v) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \vee (\upsilon_{\lambda_{\mathcal{K}}} \circ_\beta \upsilon_m)(v) \neq 1$.
Hence, $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap (\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\beta \breve{\omega}_n) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \cup (\upsilon_{\lambda_{\mathcal{K}}} \circ_\beta \upsilon_m) \neq 1$. Therefore, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost (α, β)-quasi-ideal of $\mathcal{S}$.

Conversely, assume that $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost (α, β)-quasi-ideal of $\mathcal{S}$ and $e \in \mathcal{S}$. Then $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap (\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\beta \breve{\omega}_n) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \cup (\upsilon_{\lambda_{\mathcal{K}}} \circ_\beta \upsilon_m) \neq 1$.
Thus there exists $r \in \mathcal{S}$ such that $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap (\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\beta \breve{\omega}_n)(r) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathcal{K}}}) \cup (\upsilon_{\lambda_{\mathcal{K}}} \circ_\beta \upsilon_m)(r) \neq 1$.
Hence, $r \in (\mathcal{K}\alpha e) \cap (e\beta\mathcal{K}) \cap \mathcal{K}$ implies $(\mathcal{K}\alpha e) \cap (e\beta\mathcal{K}) \cap \mathcal{K} \neq \emptyset$. Therefore, $\mathcal{K}$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$. □

Next, we study properties between $\text{supp}(\mathfrak{C})$ and CB almost (α, β)-quasi-ideal of Γ-semigroups.

Theorem 4.7. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB sets of a non-empty of a Γ-semigroup $\mathcal{S}$. Then $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-quasi-ideal of $\mathcal{S}$ if and only if* $\text{supp}(\mathfrak{C})$ *is an almost (α, β)-quasi-ideal of $\mathcal{S}$.*

Proof. Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB almost (α, β)-quasi-ideal of $\mathcal{S}$ and $e \in \mathcal{S}$. Then
$(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap (\breve{\omega} \circ_\beta \breve{\omega}_n) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon) \cup (\upsilon \circ_\beta \upsilon_m) \neq 1$. Thus there exists $r \in \mathcal{S}$ such that $(\breve{\omega}_n \circ_\alpha \breve{\omega}) \curlywedge (\breve{\omega} \circ_\beta \breve{\omega}_n)(r) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon) \vee (\upsilon \circ_\beta \upsilon_m)(r) \neq 1$. So there exists $k_1, k_2 \in \mathcal{S}$ such that $r = k_1 \alpha u = u \beta k_2$, $\breve{\omega}(r) \neq \breve{0}$, $\upsilon(r) \neq 0$ and $\breve{\omega}(k_1) \neq \breve{0}$, $\upsilon(k_1) \neq 0$. It implies that $r, k_1, k_2 \in \text{supp}(\mathfrak{C})$. Thus $((\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m) \curlywedge (\breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}}))(r) \neq 0$ and $\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \neq \breve{0}$. Similalry $((\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_m) \vee (\upsilon_m \circ_\beta \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}}))(r) \neq 1$ and

$\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq 1$. Hence, $(\breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m) \sqcap (\breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \sqcap \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq \breve{0}$ and $(\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_m) \sqcup (\upsilon_m \circ_\beta \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \neq 1$. Therefore, $\geq_{\mathrm{supp}(\mathfrak{C})}$ is a CB almost (α, β)-quasi-ideal of S. This show that $\mathrm{supp}(\mathfrak{C})$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$.

Conversely, let $\mathrm{supp}(\mathfrak{C})$ be an almost (α, β)-quasi-ideal of $\mathcal{S}$. Then by Theorem 4.6, $\mathrm{supp}(\mathfrak{C})$ is a CB (α, β)-quasi-ideal of $\mathcal{S}$. Thus $(\breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_n) \sqcap (\breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \sqcap \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq \breve{0}$ and $(\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_m) \cup (\upsilon_m \circ_\beta \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \cup \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \neq 1$. So there exists $r \in \mathcal{S}$ such that $((\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \curlywedge (\breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\beta \breve{\omega}_n))(r) \neq \breve{0}$ and $((\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \vee (\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\beta \upsilon_m))(r) \neq 1$. It implies that $(\breve{\omega}_n \circ_\alpha \breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \curlywedge (\breve{\omega}_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \circ_\beta \breve{\omega}_n)(r) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}}) \vee (\upsilon_{\lambda_{\mathrm{supp}(\mathfrak{C})}} \circ_\beta \upsilon_m)(r) \neq 1$. Thus there exist $k_1, k_2 \in \mathcal{S}$ such that $r = k_1 \alpha e = e \beta k_2$, $\breve{\omega}_n(r) \neq \breve{0}$, $\upsilon_m(r) \neq 1$ and $\breve{\omega}_n(k) \neq \breve{0}$, $\upsilon_m(k) \neq 1$. Hence, $(\breve{\omega}_n \circ_\alpha \breve{\omega}) \sqcap (\breve{\omega} \circ_\beta \breve{\omega}_n) \neq \breve{0}$ and $(\upsilon_m \circ_\alpha \upsilon) \cup (\upsilon \circ_\beta \upsilon_m) \neq 1$. Therefore, $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB almost (α, β)-quasi-ideal of $\mathcal{S}$. □

Definition 4.5. *A CB almost (α, β)-quasi-ideal $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ of a Γ-semigroup $\mathcal{S}$ is* minimal *if for all CS almost (α, β)-quasi-ideal $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ of $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \mathfrak{C}$, then* $\mathrm{supp}(\mathfrak{D}) = \mathrm{supp}(\mathfrak{C})$.

Theorem 4.8. *Let $\mathcal{K}$ be a nonempty subset of a Γ-semigroup $\mathcal{S}$ Then $\mathcal{K}$ is a minimal almost (α, β)-quasi-ideal if and only if $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a minimal CB almost (α, β)-quasi-ideal of S.*

Proof. Suppose that $\mathcal{K}$ is a minimal almost (α, β)-quasi-ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$. Thus by Theorem 4.6, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CS (α, β)-quasi-ideal of $\mathcal{S}$. Let $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ be a CB (α, β)-quasi-ideal of $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \geq_{\mathcal{K}}$. Then by Theorem 4.7, $\mathrm{supp}(\mathfrak{D})$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$. Thus $\mathrm{supp}(\mathfrak{D}) \sqsubseteq \mathrm{supp}(\geq_{\mathcal{K}}) = \mathcal{K}$. By assumption, $\mathrm{supp}(\mathfrak{D}) = \mathcal{K} = \mathrm{supp}(\geq_{\mathcal{K}})$. Thus, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a minimal CB almost (α, β)-quasi-ideal of $\mathcal{S}$.

Conversely, suppose that $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a minimal CB almost (α, β)-quasi-ideal of $\mathcal{S}$. Then by Theorem 4.7, $\mathcal{K}$ is an almost (α, β)-quasi-ideal of $\mathcal{S}$.
Let $\mathcal{J}$ be an almost (α, β)-quasi-ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then by Theorem 4.7, $\geq_{\mathcal{J}} = (\breve{\omega}_{\lambda_{\mathcal{J}}}, \upsilon_{\lambda_{\mathcal{J}}})$ is a CB (α, β)-quasi-ideal of S such that $\geq_{\mathcal{J}} \subseteq \geq_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathrm{supp}(\geq_{\mathcal{J}}) = \mathrm{supp}(\geq_{\mathcal{K}}) = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal almost (α, β)-quasi-ideal of $\mathcal{S}$. □

Corollary 4.2. *Let $\mathcal{S}$ be a Γ-semigroup Then $\mathcal{S}$ has no proper almost (α, β)-quasi-ideal if and only if for any CB almost (α, β)-quasi-ideal $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ of S,* $\mathrm{supp}(\mathfrak{C}) = \mathcal{S}$.

Next, we define CB almost (α, β)-bi-ideals and we study properties of it.

Definition 4.6. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a CB set of a Γ-semigroup $\mathcal{S}$ and $\alpha, \beta \in \Gamma$ is said to be* CB almost (α, β)-bi-ideal *of $\mathcal{S}$ if $(\breve{\omega} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}) \sqcap \breve{\omega} \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_m \circ_\beta \upsilon) \cup \upsilon \neq 0$.*

Theorem 4.9. *If $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-bi-ideal of a Γ-semigroup $\mathcal{S}$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB set of $\mathcal{S}$ such that $\mathfrak{C} \sqsubseteq \mathfrak{D}$, then $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB (α, β)-bi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-bi-ideal of a Γ-semigroup $\mathcal{S}$ and $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB set of $\mathcal{S}$ such that $\mathfrak{C} \sqsubseteq \mathfrak{D}$. Then $(\breve{\omega} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}) \curlywedge \breve{\omega} \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_m \circ_\beta \upsilon) \vee \upsilon \neq 0$. Thus,

$(\breve{\omega} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}) \curlywedge \breve{\omega} \precsim (\breve{\rho} \circ_\alpha \breve{\rho}_n \circ_\beta \breve{\rho}) \curlywedge \breve{\omega} \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_m \circ_\beta \upsilon) \vee \upsilon \geq (\tau \circ_\alpha \tau_m \circ_\beta \tau) \vee \tau \neq 0$. Hence, $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ is a CB (α, β)-bi-ideal of $\mathcal{S}$. □

Theorem 4.10. *Let $\mathcal{K}$ be a nonempty subset of Γ-semigroup $\mathcal{S}$. Then $\mathcal{K}$ is an almost (α, β)-bi-ideal of $\mathcal{S}$ if and only if characteristic function $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is an almost (α, β)-bi-ideal of $\mathcal{S}$. Then $\mathcal{K}\alpha e\beta\mathcal{K} \cap \mathcal{K} \neq \emptyset$. for all $e \in \mathcal{S}$. Thus there exists $f \in \mathcal{K}\alpha e\beta\mathcal{K}$ and $f \in \mathcal{K}$. So $(\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\mathcal{K}}})(f) = \breve{\omega}_{\lambda_{\mathcal{K}}}(f) = \check{1}$ and $(\upsilon_{\lambda_{\mathcal{K}}} \circ_\alpha \upsilon_m \circ_\beta \upsilon_{\lambda_{\mathcal{K}}})(f) = \upsilon_{\lambda_{\mathcal{K}}}(f) = 0$. Hence, $(\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap \breve{\omega}_{\lambda_{\mathcal{K}}} \neq \breve{0}$ and $(\upsilon_{\lambda_{\mathcal{K}}} \circ_\alpha \upsilon_m \circ_\beta \upsilon_{\lambda_{\mathcal{K}}}) \vee \upsilon_{\lambda_{\mathcal{K}}} \neq 1$. Therefore, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$.

Conversely, assume that $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$ and $u \in \mathcal{S}$. Then $(\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\mathcal{K}}}) \sqcap \breve{\omega}_{\lambda_{\mathcal{K}}} \neq 0$ and $(\upsilon_{\lambda_{\mathcal{K}}} \circ_\alpha \upsilon_m \circ_\beta \upsilon_{\lambda_{\mathcal{K}}}) \cup \upsilon_{\lambda_{\mathcal{K}}} \neq 1$. Thus there exists $e \in \mathcal{S}$ such that $((\breve{\omega}_{\lambda_{\mathcal{K}}} \circ_\alpha \breve{\omega}_n \circ_\beta \breve{\omega}_{\lambda_{\mathcal{K}}}) \curlywedge \breve{\omega}_{\lambda_{\mathcal{K}}})(e) \neq 0$ and
$((\upsilon_{\lambda_{\mathcal{K}}} \circ_\alpha \upsilon_m \circ_\beta \upsilon_{\lambda_{\mathcal{K}}}) \vee \upsilon_{\lambda_{\mathcal{K}}})(e) \neq 1$. Hence, $r \in \mathcal{K}\alpha e\beta\mathcal{K} \cap \mathcal{K}$ implies that $\mathcal{K}\alpha e\beta\mathcal{K} \cap \mathcal{K} \neq \emptyset$. Therefore, $\mathcal{K}$ is an almost (α, β)-bi-ideal of $\mathcal{S}$. □

Next, we study properties between $\text{supp}(\mathfrak{C})$ and CB almost (α, β)-bi-ideal of Γ-semigroups.

Theorem 4.11. *Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ be a fuzzy sets of a non-empty of a Γ-semigroup $\mathcal{S}$. Then $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$ if and only if* $\text{supp}(\mathfrak{C})$ *is an almost (α, β)-bi-ideal of $\mathcal{S}$.*

Proof. Let $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$ and $u \in \mathcal{S}$. Then
$(\breve{\omega} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}) \sqcap \breve{\omega} \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_n \circ_\beta \upsilon) \cup \upsilon \neq 1$. Thus there exists $r \in \mathcal{S}$ such that $((\breve{\omega} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}) \curlywedge \breve{\omega})(r) \neq 0$ and $((\upsilon \circ_\alpha \upsilon_n \circ_\beta \upsilon) \vee \upsilon)(r) \neq 0$. So there exists $k_1, k_2 \in \mathcal{S}$ such that $r = k_1\alpha\beta k_2$, $\breve{\omega}(r) \neq \breve{0}$, $\upsilon(r) \neq 1$. It implies that $r, k_1, k_2 \in \text{supp}(\mathfrak{C})$. Thus $(\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}})(r) \neq 0$ and $\check{\geq}_{\text{supp}(\mathfrak{C})} \neq \breve{0}$.
Similarly $(\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_m \circ_\beta \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}})(r) \neq 0$ and $\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \neq 0$.
Hence, $(\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \sqcap \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \neq \breve{0}$ and $(\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_m \circ_\beta \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}}) \vee \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}}) \neq 0$. Therefore $\geq_{\text{supp}(\mathfrak{C})}$ is a CB almost (α, β)-bi-ideal of S. This show that $\text{supp}(\mathfrak{C})$ is an almost (α, β)-bi-ideal of $\mathcal{S}$.

Conversely, let $\text{supp}(\mathfrak{C})$ is an almost (α, β)-bi-ideal of $\mathcal{S}$. Then by Theorem 4.10, $\geq_{\text{supp}(\mathfrak{C})}$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$. Thus $(\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}}) \sqcap \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \neq \breve{0}$ and $(\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_n \circ_\beta \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}}) \cup \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \neq 0$. So there exists $r \in \mathcal{S}$ such that $((\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}}) \curlywedge \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}})(r) \neq \breve{0}$ and $((\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_n \circ_\beta \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}}) \vee \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}})(r) \neq 0$.
It implies that $(\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}})(r) \neq \breve{0}$ and $\breve{\omega}_{\lambda_{\text{supp}(\mathfrak{C})}}(r) \neq \breve{0}$. Similarly
$(\upsilon_{\lambda_{\text{supp}(\mathfrak{C})}} \circ_\alpha \upsilon_n \circ_\beta \upsilon_{\lambda_{\text{supp}(\mathfrak{C})}})(r) \neq 0$ and $\geq^{n}_{\text{supp}}(r) \neq 0$. Thus there exist $k_1, k_2 \in \mathcal{S}$ such that $r = k_1\alpha u\beta k_2$, $\breve{\omega}(r) \neq 0$, $\upsilon(r) \neq 0$ and $\breve{\omega}(k) \neq 0$, $\upsilon(k) \neq 0$. Hence, $(\breve{\omega} \circ_\alpha \breve{\omega}_m \circ_\beta \breve{\omega}) \sqcap \breve{\omega} \neq \breve{0}$ and $(\upsilon \circ_\alpha \upsilon_n \circ_\beta \upsilon) \cup \upsilon \neq 0$. Therefore, $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ is a CB almost (α, β)-bi-ideal of $\mathcal{S}$. □

Definition 4.7. *A CB almost (α, β)-bi-ideal $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ of a Γ-semigroup $\mathcal{S}$ is* minimal *if for all CB almost (α, β)-bi-ideal $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ of $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \mathfrak{C}$, then* $\text{supp}(\mathfrak{D}) = \text{supp}(\mathfrak{C})$.

Theorem 4.12. *Let $\mathcal{K}$ be a nonempty subset of a Γ-semigroup $\mathcal{S}$ Then $\mathcal{K}$ is a minimal almost (α,β)-bi-ideal if and only if $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a minimal CB almost (α,β)-bi-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is a minimal almost (α,β)-bi-ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an almost (α,β)-bi-ideal of $\mathcal{S}$. Thus by Theorem 4.10, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a CB (α,β)-bi-ideal of $\mathcal{S}$. Let $\mathfrak{D} = \langle \breve{\rho}, \tau \rangle$ be a CB (α,β)-bi-ideal of $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \geq_{\mathcal{K}}$. Then by Theorem 4.11, $\mathrm{supp}(\mathfrak{D})$ is an almost (α,β)-bi-ideal of S. Thus, $\mathrm{supp}(\mathfrak{D}) \sqsubseteq \mathrm{supp}(\geq_{\mathcal{K}}) = \mathcal{K}$. By assumption, $\mathrm{supp}(\mathfrak{D}) = \mathcal{K} = \mathrm{supp}(\geq_{\mathcal{K}})$. Thus, $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a minimal CB almost (α,β)-bi-ideal of $\mathcal{S}$.

Conversely, suppose that $\geq_{\mathcal{K}} = (\breve{\omega}_{\lambda_{\mathcal{K}}}, \upsilon_{\lambda_{\mathcal{K}}})$ is a minimal CB almost (α,β)-bi-ideal of $\mathcal{S}$. Then by Theorem 4.10, $\mathcal{K}$ is an almost (α,β)-bi-ideal of $\mathcal{S}$. Let $\mathcal{J}$ be an almost (α,β)-bi-ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then by Theorem 4.10, $\geq_{\mathcal{J}} = (\breve{\omega}_{\lambda_{\mathcal{J}}}, \upsilon_{\lambda_{\mathcal{J}}})$ is a CB (α,β)-bi-ideal of S such that $\geq_{\mathcal{J}} \subseteq \geq_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathrm{supp}(\geq_{\mathcal{J}}) = \mathrm{supp}(\geq_{\mathcal{K}}) = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal almost (α,β)-bi-ideal of $\mathcal{S}$. $\square$

Corollary 4.3. *Let $\mathcal{S}$ be a Γ-semigroup Then $\mathcal{S}$ has no proper almost (α,β)-bi-ideal if and only if for any CB almost (α,β)-bi-ideal $\mathfrak{C} = \langle \breve{\omega}, \upsilon \rangle$ of $\mathcal{S}$, $\mathrm{supp}(\mathfrak{C}) = \mathcal{S}$.*

5. Conclusion

In this article, we give the concept of a new cubic ideals and cubic almost ideals in a Γ-semigroups. We study properites of new cubic ideals and cubic almost ideals. We hope that the study of this topic are useful mathematical tools. In the future we study a new hesitant fuzzy ideal and hesitant fuzzy almost ideals in semigroups or algebric system.

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