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On Global Existence of the Fractional Reaction-Diffusion System's Solution

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Abstract. The purpose of this paper is to prove the global existence of solution for one of most significant fractional partial differential system called the fractional reaction-diffusion system. This will be carried out by combining the compact semigroup methods with some L^1-estimate methods. Our investigation can be applied to a wide class of fractional partial differential equations even if they contain nonlinear terms in their constructions.

1. Introduction

In this paper, we intend to study the following nonlinear parabolic system:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 (-\Delta)^{\alpha} u = f(u,v), & \text{in }]0,+\infty[\times \Omega, \\ \frac{\partial v}{\partial t} - d_2 (-\Delta)^{\beta} v = g(u,v), & \text{in }]0,+\infty[\times \Omega, \end{cases} \tag{1.1}$$

subject to the following boundary conditions:

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \text{ or } u = v = 0, \text{ in }]0,+\infty[\times \partial\Omega, \tag{1.2}$$

and the initial data:

$$u(0,\cdot) = u_0(\cdot), \; v(0,\cdot) = v_0(\cdot), \text{ in } \Omega, \tag{1.3}$$

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where Ω is a regular and bounded domain of $\mathbb{R}^n$, $(n \geq 1)$, with its boundary $\partial\Omega$, $u = u(t,x)$, $v = v(t,x)$ are two real-valued functions such that $x \in \Omega$ and $t > 0$, and where $(-\Delta)^{\delta}$ is a non local operator that accounts for the anomalous diffusion [1,2] so that $0 < \delta < 1$, $(\delta = \alpha$ or $\beta)$, and d_1, d_2 are two constants of diffusion assumed to be nonnegative, whereas f and g are two functions in which they "enough regular". It should be furthermore mentioned that the functions $u(0,\cdot)$ and $v(0,\cdot)$ are assumed to be continuous and nonnegative. Besides, the local existence of the solution (u,v) in times is classical, and moreover it is not negative if u_0 and v_0 are so.

It is worth mentioned that system (1.1)-(1.3) arises in many field of applied science such as physics, chemistry and various biological processes including population dynamics and others, see [3] and references therein. In this regard and in order to make system (1.1)-(1.3) more reality, we assume that the following hypothesis:

- The initial data u_0 and v_0 are nonnegative functions such that:

$$u_0, v_0 \in L^1(\Omega). \tag{1.4}$$

- The two functions f and g are a quasi-positives functions, i.e.,

$$f(0,v) \geq 0, \;\; g(u,0) \geq 0, \;\; \forall u, v \geq 0. \tag{1.5}$$

- It exists a nonnegative constant C independent of (ξ_1, ξ_2) such that:

$$f(\xi_1,\xi_2) + g(\xi_1,\xi_2) \leq C(\xi_1 + \xi_2), \forall (\xi_1,\xi_2) \in \mathbb{R}^2_+. \tag{1.6}$$

- In addition, we have:

$$f(\xi_1,\xi_2) \leq C(\xi_1 + \xi_2), \forall (\xi_1,\xi_2) \in \mathbb{R}^2_+. \tag{1.7}$$

The main question we want to address here is the existence of global solution for system (1.1)-(1.3). In fact, the subject of the global existence of fractional reaction-diffusion systems has received a lot of attention in the last decades and several outstanding results have been proved by some of the major experts in the field, see [4–10]. In the same context, replacing the anomalous diffusion operator by the standard Laplacian operator $(-\Delta)$ was firstly studied in one-dimensional space. This notion has been investigated by many authors by considering certain special forms of the nonlinear terms f and g. In particular, Alikakos showed in [11] the existence of global bounded solutions whenever $f(u,v) = -g(u,v) = -uv^{\sigma}$, for $1 < \sigma < \dfrac{n+2}{n}$. The extension of this result for $\sigma > 1$ is studied later by Masuda [12]. Following that, Haraux and Youkana generalized the result of Masuda via the functional of Lyapunov in reference [13]. Actually, they performed their generalization by putting $f(u,v) = g(u,v) = -u\Psi(v)$, where Ψ is a nonlinear function satisfying the condition:

$$\lim_{v \to +\infty} \frac{[\log(1 + \Psi(v))]}{v} = 0.$$

In the same regard, Barabanova generalized the result of Haraux and Youkana in [14] by concerning with the global existence of nonnegative solutions of a reaction-diffusion equation with exponential nonlinearity. Lately, it has been shown that there is also another very powerful method relying on compact semigroups could be used for examining the global existence of solutions for a reaction-diffusion equation [15–19]. For a better understanding, we send the reader to the works of Moumeni and Barrouk [20, 21].

Later on, a more general model was studied by Haraux and Kirane [22]. They took different diffusion coefficients for the two equations and for the general nonlinear terms. They proved the existence of global bounded solutions and investigated their asymptotic behavior. Equally, Hnaien et al. proved in reference [23] the existence of a local mild solution, global existence solution and asymptotic behavior of solutions for the system (1.1)-(1.3) when $f(u,v) = -\lambda uv$ and $g(u,v) = \lambda uv - \mu v$.

The remainder of this paper is organized as follows. In Section 2, we present some definitions and preliminaries. In Section 3, we provide some results related to the compactness of a proposed operator. In Section 4, we prove the existence of a local mild solution, positivity and global existence of solution for particular system. Finally, the global existence of solutions for system (1.1)-(1.3) are studied in Section 5, followed by Section 6 that abbreviates the work.

2. Preliminaries

In this part, some preliminaries and overview of the local existence and global existence of solution for fractional reaction-diffusion system are illustrated. This will pave the way to introduce our findings later on.

Definition 2.1. *Let $F(u,v) \in X$, where X is a Banach space. The function F is locally Lipschitz if for all $t_1 \geq 0$ and all constant $k > 0$, there exist a constant $L(k, t_1) > 0$ such that:*

$$\|F(u_1, v_1) - F(u_2, v_2)\| \leq L\,|(u_1, v_1) - (u_2, v_2)|\,,$$

is satisfied $\forall (u_1, v_1), (u_2, v_2) \in \mathbb{R} \times \mathbb{R}$ with $|(u_1, v_1)| \leq k$, $|(u_2, v_2)| \leq k$ and $t \in [0, t_1]$ such that $t > 0$.

Lemma 2.1. *Let A be m-dissipative operator in the Banach space X and $S(t)$ be a semigroup engendered by A. Let F be a function locally Lipschitz. Then, for all $u_0 \in X$, there exists $T_{\max} = T(u_0)$ such that the system:*

$$\begin{cases} u \in C([0,T], D(A)) \cap C^1([0,T], X), \\ \dfrac{du}{dt} - Au = F(u(s)), \\ u(0) = u_0 \end{cases} \tag{2.1}$$

admits a unique local solution u verifying

$$u(t) = S(t)\,u_0 + \int_0^t S(t-s)\,F(u(s))\,ds, \ \forall t \in [0, T_{\max}].$$

Next, some further preliminaries associated with the existence of global solution for the fractional reaction-diffusion system (1.1)-(1.3) will be recalled.

Theorem 2.1. *[5] Consider the following classical boundary-eigenvalue system for the fractional power of the Laplacian in Ω with homogeneous Neumann boundary condition:*

$$\begin{cases} (-\Delta)^{\alpha}\varphi_k = \lambda_k^{\alpha}\varphi_k, & \text{in } \Omega, \\ \dfrac{\partial \varphi_k}{\partial \eta} = 0, & \text{on } \Omega, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ and

$$D\left((-\Delta)^{\alpha}\right) = \left\{ u \in L^2(\Omega),\ \frac{\partial u}{\partial \eta} = 0,\ \left\|(-\Delta)^{\alpha} u\right\|_{L^2(\Omega)} < +\infty \right\},$$

such that:

$$\left\|(-\Delta)^{\alpha} u\right\|_{L^2(\Omega)}^2 = \sum_{k=1}^{+\infty} \left|\lambda_k^{\alpha} \langle u, \varphi_k\rangle\right|^2.$$

Then this system has a countable system of eigenvalues of the Laplacian operator in $L^2(\Omega)$ with homogeneous Neumann boundary condition in which $0 < c \le \lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots$ so that $\lambda_j \to \infty$ as $j \to \infty$, and φ_k is the corresponding eigenvectors for $k = 1, 2, \ldots, +\infty$.

Thus, based on what we have mentioned above, we can infer that, for $u \in D\left((-\Delta)^{\alpha}\right)$, we have:

$$(-\Delta)^{\alpha} u = \sum_{k=1}^{+\infty} \lambda_k^{\alpha} \langle u, \varphi_k\rangle \varphi_k.$$

In addition, with using integration by parts, we can have the following formula:

$$\int_{\Omega} u(x)(-\Delta)^{\alpha} v(x)\,dx = \int_{\Omega} v(x)(-\Delta)^{\alpha} u(x)\,dx, \tag{2.2}$$

for $u, v \in D\left((-\Delta)^{\alpha}\right)$.

Lemma 2.2. *Let $\theta \in C_0^{\infty}(Q)$ such that $\theta \ge 0$. Then, there is a nonnegative function $\Phi \in C^{1,2}(Q)$ that represents a solution of the system:*

$$\begin{cases} -\Phi_t - d\Delta\Phi = \theta & \text{on } Q, \\ \Phi(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ \Phi(T,x) = 0 & \text{on } \Omega. \end{cases} \tag{2.3}$$

Actually, in accordance with Ladyzenskaya and Solonnikov in [24], we observe that system (2.3) possesses a unique nonnegative solution. Moreover, for all $q \in]1, +\infty[$, we note that there exists a nonnegative constant c independent of θ such that:

$$\|\Phi\|_{L^p(Q)} \le c\,\|\theta\|_{L^q(Q)}.$$

Besides, for all $\omega_0 \in L^1(\Omega)$ and $h \in L^1(Q)$, we can have the following equalities:

$$\int_Q \left(S(t)\,\omega_0(x)\right)\theta\,dx\,dt = \int_{\Omega} \omega_0(x)\,\Phi(0,x)\,dx, \tag{2.4}$$

and

$$\int_Q \left(\int_0^t S(t-s)\,h(s,x)\,ds\right)\theta\,dx\,dt = \int_Q h(s,x)\,\Phi(s,x)\,dx\,ds. \tag{2.5}$$

3. Compactness of operator

In this section, we will provide a result connected with a compactness of operator L that define the solution of system (2.1) in the case where the initial value equals zero $(u(0) = 0)$, i.e.,

$$L(F)(t) = u(t) = \int_0^t S(t-s) F(u(s))\, ds, \ \forall t \in [0, T].$$

From this point of view, we will first recall the Dunford–Pettis Theorem, which can be found with its proof in [25]. This will help us to derive our first result in this work.

Theorem 3.1 (Dunford–Pettis)**.** *Let $\mathcal{F}$ be a bounded set in $L^1(\Omega)$. Then $\mathcal{F}$ has compact closure in the weak topology $\sigma\left(L^1, L^\infty\right)$ if and only if $\mathcal{F}$ is equiintegrable, i.e.,*

(*a*)

$$\left\{\forall \varepsilon > 0, \exists \delta > 0, \ \textit{such that} \int_A |f| < \varepsilon, \ \forall A \subset \Omega, \ \textit{measurable with } |A| < \delta, \ \forall f \in \mathcal{F}\right\},$$

(*b*)

$$\left\{\forall \varepsilon > 0, \ \exists \omega \subset \Omega, \ \textit{measurable with } |\omega| < \infty, \textit{such that} \int_{\Omega\backslash\omega} |f| < \varepsilon, \ \forall f \in \mathcal{F}\right\}.$$

Theorem 3.2. *If for all $t > 0$, the operator $S(t)$ is compact, then L is compact in $L^1([0, T], X)$.*

Proof. The proof of this result consists of two steps.

Step 1: We show that $S(\lambda) L : F \to S(\lambda) L(F)$ is compact in $L^1([0, T], X)$, i.e., we show that the set $\{S(\lambda) L(F)(t) : \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0, T], X), \forall t \in [0, T]$. To this aim, we notice that due to $S(t)$ is compact, then the operator $t \to S(t)$ is continuous over $]0, +\infty[$ in $\mathcal{L}(X)$. Therefore, we have:

$$\forall \varepsilon > 0, \forall \delta > 0, \ \exists \eta > 0. \ \forall 0 \leq h \leq \eta, \ \forall t \geq \delta, \ \ \|S(t+h) - S(t)\|_{\mathcal{L}(X)} \leq \varepsilon.$$

Now, if one chooses $\lambda = \delta$, we obtain:

$$\begin{aligned} S(\lambda)u(t+h) - S(\lambda) u(t) &= \int_0^{t+h} S(\lambda + t + h - s) F(u(s))\, ds - \int_0^t S(\lambda + t - s) F(u(s))\, ds \\ &= \int_t^{t+h} S(\lambda + t + h - s) F(u(s))\, ds + \int_0^t \left(S(\lambda + t + h - s) - S(\lambda + t - s)\right) F(u(s))\, ds, \end{aligned}$$

for $0 \leq t \leq T - h$. Consequently, based on the inequality:

$$\|S(\lambda) u(t+h) - S(\lambda) u(t)\|_X \leq \int_t^{t+h} \|F(u(s))\|_X\, ds + \varepsilon \int_0^t \|F(u(s))\|_X\, ds,$$

we can define $v(t)$ by:

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq T, \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, we have:

$$\|S(\lambda) v(t+h) - S(\lambda) v(t)\|_1 \leq (h + \varepsilon T) \|F(u(s))\|_1,$$

which implies that $\{S(\lambda)v : \|F\|_1 \leq 1\}$ is equiintegrable. Hence, we infer $\{S(\lambda)L(F)(t) : \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0,T],X)$, and so $S(\lambda)L$ is compact.

Step 2: We show that $S(\lambda)L$ converges towards L when λ goes towards 0 in $L^1([0,T],X)$. For this purpose, we observe:

$$S(\lambda)u(t) - u(t) = \int_0^t S(\lambda + t - s) F(u(s))\, ds - \int_0^t S(t-s) F(u(s))\, ds.$$

So, for $t \geq \delta$, we can have:

$$\|S(\lambda)u(t) - u(t)\| \leq \int_\delta^t \|S(\lambda+s) - S(s)\|_{\mathcal{L}(X)} \|F(u(s))\|\, ds + 2\int_{t-\delta}^t \|F(u(s))\|\, ds.$$

Immediately, if we choose $0 < \lambda < \eta$, we get:

$$\|S(\lambda)u(t) - u(t)\| \leq \varepsilon \int_\delta^t \|F(u(s))\|\, ds + 2\int_{t-\delta}^t \|F(u(s))\|\, ds.$$

Besides, for $0 \leq t < \delta$, we can have:

$$\|S(\lambda)u(t) - u(t)\| \leq 2\int_0^t \|F(u(s))\|\, ds.$$

As $F \in L^1(0,T,X)$, we gain:

$$\|S(\lambda)u(t) - u(t)\| \leq (\varepsilon T + 2\delta)\|F(u(s))\|_1.$$

Thus, as $\lambda \to 0$, then $S(\lambda)u \to u$ in $L^1([0,T],X)$, where the operator L is a uniform limit with compact linear operator between two Banach spaces, which confirms that L is compact in $L^1([0,T],X)$. □

Remark 3.1. *The semigroup $S(t)$ generated by the operator $d(-\Delta)^\delta$ is compact in $L^1(\Omega)$.*

4. Study of a particular system

This section is divided into three subsections so that the first one aims to deals with the local existence of solution for a first-order system derived from system (1.1)-(1.3), then the positivity of such solution will be discussed, followed by exploring the global existence of the solution of the derived system. Thus, in order to achieve this objective, we first convert system (1.1)-(1.3) to an abstract first-order system in the Banach space $X = L^1(\Omega) \times L^1(\Omega)$. To this aim, we define the functions u_{n_0} and v_{n_0} by:

$$u_{n_0} = min(u_0, n), \text{ and } v_{n_0} = min(v_0, n),$$

for all $n > 0$. It is clear that u_{n_0} and v_{n_0} verify (1.4), i.e.,

$$u_{n_0}, v_{n_0} \in L^1(\Omega) \text{ and } u_{n_0} \geq 0,\ v_{n_0} \geq 0.$$

Thus, based on the previous assumptions, we can formulate the first-order system derived from system (1.1)-(1.3) as:

$$\begin{cases} \frac{\partial w_n}{\partial t} - Aw_n = F(w_n) & \text{in } [0,T[\times\Omega, \\ \frac{\partial w_n}{\partial \eta} = 0, \text{ or } w_n = 0 & \text{in } [0,T[\times\partial\Omega, \\ w_n(0,\cdot) = w_{n_0}(\cdot) & \text{in } \Omega. \end{cases} \tag{4.1}$$

4.1. **Local existence of solution for system (4.1).** In this subsection, we intend to discuss the local existence of solution for system (4.1). In this connection, we let $w_n = (u_n, v_n)$, $w_{n_0} = (u_{n_0}, v_{n_0})$ and $F = (f, g)$. Besides, we suppose A is an operator defined as:

$$A = \begin{pmatrix} d_1(-\Delta)^\alpha & 0 \\ 0 & d_2(-\Delta)^\beta \end{pmatrix},$$

where $D(A) := \left\{ w_n \in L^1(\Omega) \times L^1(\Omega) : \left((-\Delta)^\alpha u_n, (-\Delta)^\beta v_n \right) \in L^1(\Omega) \times L^1(\Omega) \right\}$. In view of the above assumptions, system (4.1) can be returned to the shape of system (2.1). Thus, if (u_n, v_n) is a solution of system (4.1), then it verifies the following integral equations:

$$\begin{cases} u_n(t) = S_1(t)\, u_{n_0} + \int_0^t S_1(t-s)\, f(u_n(s), v_n(s))\, ds, \\ v_n(t) = S_2(t)\, v_{n_0} + \int_0^t S_2(t-s)\, g(u_n(s), v_n(s))\, ds, \end{cases} \tag{4.2}$$

where $S_1(t)$ and $S_2(t)$ are the semigroups of contractions in $L^1(\Omega)$ generated by the operator $d_1(-\Delta)^\alpha$ and $d_2(-\Delta)^\beta$.

Theorem 4.1. *There exists $T_M > 0$ such that (u_n, v_n) is a local solution of (4.1), for all $t \in [0, T_M]$.*

Proof. Due to $S_1(t)$ and $S_2(t)$ are semigroups of contraction and as F is locally Lipschitz for $0 \leq u_{n_0}, v_{n_0} \leq n$, then $\exists T_M > 0$ such that (u_n, v_n) is a local solution of system (4.1) on $[0, T_M]$. □

Theorem 4.2. *Let $u_{n_0}, v_{n_0} \in L^1(\Omega)$, then there exists a maximal time $T_{\max} > 0$ and a unique mild solution $(u_n, v_n) \in C\left([0, T_{\max}), L^1(\Omega) \times L^1(\Omega)\right)$ of system (4.1) subject to either*

$$T_{\max} = +\infty,$$

or

$$T_{\max} < +\infty \text{ and } \lim_{t \to T_{\max}} \left(\|u_n(t)\|_\infty + \|v_n(t)\|_\infty \right) = +\infty.$$

Proof. For arbitrary $T > 0$, we define the Banach space as:

$$E_T := \left\{ (u_n, v_n) \in C\left([0, T], L^1(\Omega) \times L^1(\Omega)\right) : \|(u_n, v_n)\| \leq 2\,\|(u_{n_0}, v_{n_0})\| = R \right\},$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}$ and $\|\cdot\|$ is the norm of E_T defined by:

$$\|(u_n, v_n)\| := \|u_n\|_{L^\infty([0,T], L^\infty(\Omega))} + \|v_n\|_{L^\infty([0,T], L^\infty(\Omega))}.$$

Next, for every $(u_n, v_n) \in E_T$, we define $\Psi(u_n, v_n) := (\Psi_1(u_n, v_n), \Psi_2(u_n, v_n))$ as:

$$\Psi_1(u_n, v_n) = S_1(t)\, u_{n_0} + \int_0^t S_1(t-s)\, f(u_n(s), v_n(s))\, ds,$$
$$\Psi_2(u_n, v_n) = S_2(t)\, v_{n_0} + \int_0^t S_2(t-s)\, g(u_n(s), v_n(s))\, ds,$$

for $t \in [0, T]$. Now, we will prove the local existence of solution for the considered system by the Banach fixed point theorem. To this aim, we let $\Psi : E_T \to E_T$ and $(u_n, v_n) \in E_T$. This leads to infer the inequality:

$$\begin{aligned} \|\Psi_1(u_n, v_n)\|_\infty &\leq \|u_{n_0}\|_\infty + C(\|u_n\|_\infty + \|v_n\|_\infty)\, T \\ &\leq \|u_{n_0}\|_\infty + C\left(\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty\right) T, \text{ (by maximum principle).} \end{aligned}$$

Similarly, we have:

$$\|\Psi_2(u_n, v_n)\|_\infty \leq \|v_{n_0}\|_\infty + C\left(\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty\right) T.$$

This immediately implies:

$$\begin{aligned} \|\Psi(u_n, v_n)\| &\leq \|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty + 2C\left(\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty\right) T \\ &\leq 2\left(\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty\right), \text{ by choosing } T \text{ such that } T \leq \frac{\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty}{CR}. \end{aligned}$$

Therefore, we gain $\Psi(u_n, v_n) \in E_T$ for $T \leq \frac{\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty}{CR}$. Now, to complete the proof, we need to show that Ψ is a contraction map. In this regard, we have:

$$\begin{aligned} \|\Psi_1(u_n, v_n) - \Psi_1(\tilde{u}_n, \tilde{v}_n)\|_\infty &\leq L \int_0^t \|(u_n, v_n) - (\tilde{u}_n, \tilde{v}_n)\|_\infty\, d\tau \\ &\leq LT\left(\|\tilde{v}_n - v_n\|_\infty + \|\tilde{u}_n - u_n\|_\infty\right), \end{aligned}$$

for $(u_n, v_n), (\tilde{u}_n, \tilde{v}_n) \in E_T$. Similarly, we obtain:

$$\|\Psi_2(u_n, v_n) - \Psi_2(\tilde{u}_n, \tilde{v}_n)\|_\infty \leq LT\left(\|\tilde{v}_n - v_n\|_\infty + \|\tilde{u}_n - u_n\|_\infty\right).$$

Actually, the above two estimates imply:

$$\begin{aligned} \|\Psi(u_n, v_n) - \Psi(\tilde{u}_n, \tilde{v}_n)\|_\infty &\leq 2LT\left(\|\tilde{v}_n - v_n\|_\infty + \|\tilde{u}_n - u_n\|_\infty\right) \\ &\leq \frac{1}{2}\|(u_n, v_n) - (\tilde{u}_n, \tilde{v}_n)\|, \end{aligned}$$

where $T \leq \max\left(\frac{\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty}{CR}, \frac{1}{4L}\right)$. This exactly shows the contraction result. Hence, by the Banach fixed point theorem, system (4.1) admits a unique mild solution $(u_n, v_n) \in E_T$. In general, this solution can be extended on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup\{T > 0 : (u_n, v_n) \text{ is a solution to (4.1) in } E_T\}.$$

□

However, with the aim of showing the global existence of solution for the system at hand, we need the fact that such solution should be positive, and this what we aim to address in the next subsection.

4.2. **Positivity of solution for system (4.1).** In what follow, we intend to prove the positivity of solution for system (4.1). This would help us to discuss the global existence of such solution. In this respect, we introduce next another result.

Lemma 4.1. *Let (u_n, v_n) be the solution of system (4.1) such that:*

$$u_{n_0}(x) \geq 0, \;\; v_{n_0}(x) \geq 0, \; x \in \Omega.$$

Then

$$u_n(t,x) \geq 0 \text{ and } v_n(t,x) \geq 0, \; \forall (t,x) \in]0,T[\times \Omega.$$

Proof. Let $\bar{u}_n(t,x) = 0$ in $]0,T[\times \Omega \implies \dfrac{\partial \bar{u}_n}{\partial t} = 0$ and $(-\Delta)^{\alpha} \bar{u}_n = 0$. Then, according to the hypothesis (1.5), we can have:

$$\frac{\partial u_n}{\partial t} - d_1 (-\Delta)^{\alpha} u_n - f(u_n, v_n) = 0 \geq \frac{\partial \bar{u}_n}{\partial t} - d_1 (-\Delta)^{\alpha} \bar{u}_n - f(\bar{u}_n, v_n),$$

and

$$u_n(0,x) = u_{n_0}(x) \geq 0 = \bar{u}_n(0,x).$$

Hence, by the comparison theorem, we obtain:

$$u_n(t,x) \geq \bar{u}_n(t,x),$$

which implies $u_n(t,x) \geq 0$. In a similar manner, we can gain $v_n(t,x) \geq 0$, and this completes the proof. □

4.3. **Global existence of solution for system (4.1).** To prove the global existence of the solution of system (4.1) for all nonnegative t, it is enough, according to Routh [26], to find an estimate of the solution for all $t \geq 0$ in $L^1(\Omega)$. In this regard, we introduce the next lemma.

Lemma 4.2. *Let (u_n, v_n) be the solution of system (4.1), then there exists $M(t)$, which depends only on t, such that for all $0 \leq t \leq T_M$, we have:*

$$\|u_n + v_n\|_{L^1(\Omega)} \leq M(t).$$

Based on this estimate, we confirm that the solution (u_n, v_n) given by Theorem **4.1** *is a global solution.*

Proof. First of all, it should be noted that we can write system (4.1) in the form:

$$\begin{cases} \frac{\partial u_n}{\partial t} - d_1 (-\Delta)^{\alpha} u_n = f(u_n, v_n), & \text{in } [0,T[\times \Omega, \\ \frac{\partial v_n}{\partial t} - d_2 (-\Delta)^{\beta} v_n = g(u_n, v_n), & \text{in } [0,T[\times \Omega, \\ \frac{\partial u_n}{\partial \eta} = \frac{\partial v_n}{\partial \eta} = 0, \text{ or } u_n = v_n = 0, & \text{in } [0,T[\times \partial\Omega, \\ u_n(0,x) = u_{n_0}(x), \; v_n(0,x) = v_{n_0}(x), & \text{in } \Omega. \end{cases} \tag{4.3}$$

With the use of the first and second equations of system (4.3), we can obtain:

$$\frac{\partial}{\partial t}(u_n+v_n)-d_1(-\Delta)^{\alpha}u_n-d_2(-\Delta)^{\beta}v_n=f(u_n,v_n)+g(u_n,v_n).$$

By taking into account assumption (1.6), we have:

$$\frac{\partial}{\partial t}(u_n+v_n)-d_1(-\Delta)^{\alpha}u_n-d_2(-\Delta)^{\beta}v_n\leq C(u_n+v_n).$$

Now, let us integrate the above inequality over Ω, and then use the integration by parts performed on formula (2.2) to get $\int_{\Omega}(-\Delta)^{\alpha}u_n(x)\,dx=0$ and $\int_{\Omega}(-\Delta)^{\beta}v_n(x)\,dx=0$. This would gives:

$$\int_{\Omega}\frac{\partial}{\partial t}(u_n+v_n)\,dx\leq C\int_{\Omega}(u_n+v_n)\,dx \quad \text{or} \quad \frac{\frac{\partial}{\partial t}\int_{\Omega}(u_n+v_n)\,dx}{\int_{\Omega}(u_n+v_n)\,dx}\leq C.$$

By integrate the above inequality over $[0,t]$, we get:

$$\ln\int_{\Omega}(u_n+v_n)\,dx\Big|_0^t\leq Ct \quad \text{or} \quad \ln\frac{\int_{\Omega}(u_n+v_n)\,dx}{\int_{\Omega}(u_{n_0}+v_{n_0})\,dx}\leq Ct,$$

which implies:

$$\frac{\int_{\Omega}(u_n+v_n)\,dx}{\int_{\Omega}(u_{n_0}+v_{n_0})\,dx}\leq \exp(Ct),$$

i.e.,

$$\begin{aligned}&\Rightarrow \int_{\Omega}(u_n+v_n)\,dx\leq \exp(Ct)\int_{\Omega}(u_{n_0}+v_{n_0})\,dx\\&\Rightarrow \int_{\Omega}(u_n+v_n)\,dx\leq \exp(Ct)\int_{\Omega}(u_0+v_0)\,dx,\ \text{as if } u_{n_0}\leq u_0,\ v_{n_0}\leq v_0.\end{aligned}$$

Now, let us assume that $M(t)=\exp(Ct)\|u_0+v_0\|_{L^1(\Omega)}$. Then, due to u_n and v_n are positives, we gain:

$$\|u_n+v_n\|_{L^1(\Omega)}\leq M(t),\ 0\leq t\leq T_M,$$

which completes the proof. □

In the following content, we provide a further result that aims to show the existence of the solution's estimate (u_n,v_n) for system (4.1) in $L^1(Q)$.

Lemma 4.3. *For any solution (u_n,v_n) of system (4.1), there is a constant $K(t)$, which depends only on t, such that:*

$$\|u_n+v_n\|_{L^1(Q)}\leq K(t)\|u_0+v_0\|_{L^1(\Omega)}.$$

Proof. In order to prove this result, we multiply the first equation of system (4.2) by θ, and then integrate the result over Q. Accordingly, by using (2.4) and (2.5), we obtain:

$$\begin{aligned}\int_Q u_n\theta dxdt &= \int_Q S_1(t)\,u_{n_0}(x)\,\theta dxdt+\int_Q\left(\int_0^t S_1(t-s)\,f(u_n,v_n)\,ds\right)\theta dxdt\\ &= \int_\Omega u_{n_0}(x)\,\Phi(0,x)\,dx+\int_Q f(u_n,v_n)\,\Phi(s,x)\,dxds.\end{aligned}$$

Moreover, we find:

$$\int_Q v_n\theta dxdt=\int_\Omega v_{n_0}(x)\,\Phi(0,x)\,dx+\int_Q g(u_n,v_n)\,\Phi(s,x)\,dxds.$$

This yields:

$$\begin{aligned}\int_Q(u_n+v_n)\,\theta dxdt &= \int_\Omega\left(u_{n_0}(x)+v_{n_0}(x)\right)\Phi(0,x)\,dx+\int_Q\left(f(u_n,v_n)+g(u_n,v_n)\right)\Phi(s,x)\,dxds\\ &\le \int_\Omega\left(u_0(x)+v_0(x)\right)\Phi(0,x)\,dx+\int_Q C\,(u_n+v_n)\,\Phi(s,x)\,dxds.\end{aligned}$$

Consequently, using Holder inequality implies:

$$\begin{aligned}\int_Q(u_n+v_n)\,\theta dxdt &\le \|u_0+v_0\|_{L^1(\Omega)}\,.\,\|\Phi(0,x)\|_{L^\infty(Q)}+C\,\|u_n+v_n\|_{L^1(Q)}\,.\,\|\Phi\|_{L^\infty(Q)}\\ &\le \left(\|u_0+v_0\|_{L^1(\Omega)}+C\,\|u_n+v_n\|_{L^1(Q)}\right).\,\|\Phi\|_{L^\infty(Q)}\\ &\le \max(1,C)\left(\|u_0+v_0\|_{L^1(\Omega)}+\|u_n+v_n\|_{L^1(Q)}\right).\,\|\Phi\|_{L^\infty(Q)}\\ &\le k_1(t)\left(\|u_0+v_0\|_{L^1(\Omega)}+\|u_n+v_n\|_{L^1(Q)}\right).\,\|\theta\|_{L^\infty(Q)}\,,\end{aligned}$$

where $k_1(t)\ge\max(c,cC)$. Now, since θ is arbitrary in $C_0^\infty(Q)$, then we have:

$$\|u_n+v_n\|_{L^1(Q)}\le k_1()\left(\|u_0+v_0\|_{L^1(\Omega)}+\|u_n+v_n\|_{L^1(Q)}\right).$$

Thus, by taking $k(t)=\frac{k_1(t)}{1-k_1(t)}$, we obtain:

$$\|u_n+v_n\|_{L^1(Q)}\le k(t)\,\|u_0+v_0\|_{L^1(\Omega)}\,,$$

which finishes the proof. □

5. Global existence of solution for system (1.1)-(1.3)

In this section, we will provide one of the main results of this work. In particular, with the help of using the four assumptions (1.4)-(1.7), we will explore the global existence of solution for system (1.1)-(1.3).

Theorem 5.1. *Suppose that the hypotheses (1.4)-(1.7) are satisfied. Then there exists a solution (u, v) of system (1.1)-(1.3) of the form:*

$$\begin{cases} u(t) = S_1(t) u_0 + \int_0^t S_1(t-s) f(u(s), v(s))\, ds, \ \forall t \in [0, T[, \\ v(t) = S_2(t) v_0 + \int_0^t S_2(t-s) g(u(s), v(s))\, ds, \ \forall t \in [0, T[, \end{cases} \tag{5.1}$$

where $u, v \in C\left([0, +\infty[, L^1(\Omega)\right)$, $f(u, v), g(u, v) \in L^1(Q)$ such that $Q = (0, T) \times \Omega$ for all $T > 0$, and where $S_1(t)$ and $S_2(t)$ are the semigroups of contractions in $L^1(\Omega)$ generated by $d_1(-\Delta)^\alpha$ and $d_2(-\Delta)^\beta$.

Proof. To prove this result, we define the operator L by:

$$L : (w_0, h) \to S_d(t) w_0 + \int_0^t S_d(t-s) h(s)\, ds,$$

where $S_d(t)$ is the semigroup of contraction generated by the operator $d(-\Delta)^\delta$. According to Theorem 3.2 and as $S_d(t)$ is compact, then the operator L is an adding of two compact operators in $L^1(Q)$, and so L is compact in $L^1(Q) \times L^1(Q)$. Therefore, there exists a subsequence $\left(u_{n_j}, v_{n_j}\right)$ of (u_n, v_n) such that $\left(u_{n_j}, v_{n_j}\right)$ converges towards (u, v) in $L^1(Q) \times L^1(Q)$. Let us now show that $\left(u_{n_j}, v_{n_j}\right)$ is a solution of system (4.2), i.e.,

$$\begin{cases} u_{n_j}(t, x) = S_1(t) u_{n_0} + \int_0^t S_1(t-s) f\left(u_{n_j}(s), v_{n_j}(s)\right) ds, \\ v_{n_j}(t, x) = S_2(t) v_{n_0} + \int_0^t S_2(t-s) g\left(u_{n_j}(s), v_{n_j}(s)\right) ds, \end{cases} \tag{5.2}$$

That is, it is enough to show that (u, v) verifies (5.1). In this regard, it should be clearly noted that if $j \to +\infty$, then we gain $u_{n_0} \to u_0$ and $v_{n_0} \to v_0$, and so

$$f\left(u_{n_j}, v_{n_j}\right) \to f(u, v),\ g\left(u_{n_j}, v_{n_j}\right) \to g(u, v) \ \text{ a.e.} \tag{5.3}$$

Thus to show that (u, v) verifies (5.1), it remains to show that:

$$f\left(u_{n_j}, v_{n_j}\right) \to f(u, v),\ \ g\left(u_{n_j}, v_{n_j}\right) \to g(u, v),$$

in $L^1(Q)$ when $j \to +\infty$. To this aim, we integrate the equations of system (4.1) over Q coupled with take (2.2) into account to obtain:

$$-d_1 \int_Q (-\Delta)^\alpha u_{n_j} dx dt = 0,\ -d_2 \int_Q (-\Delta)^\beta v_{n_j} dx dt = 0.$$

Consequently, we have:

$$\begin{aligned} \int_\Omega u_{n_j} dx - \int_\Omega u_{n_0} dx &= \int_Q f\left(u_{n_j}, v_{n_j}\right) dx dt, \\ \int_\Omega v_{n_j} dx - \int_\Omega v_{n_0} dx &= \int_Q g\left(u_{n_j}, v_{n_j}\right) dx dt, \end{aligned}$$

such that:

$$-\int_Q f\left(u_{n_j}, v_{n_j}\right) dxdt \leq \int_\Omega u_0 dx, \tag{5.4}$$

and

$$-\int_Q g\left(u_{n_j}, v_{n_j}\right) dxdt \leq \int_\Omega v_0 dx. \tag{5.5}$$

Now, let us assume:

$$\begin{aligned} N_n &= C\left(u_{n_j}+v_{n_j}\right)-f\left(u_{n_j}, v_{n_j}\right), \\ M_n &= C\left(u_{n_j}+v_{n_j}\right)-f\left(u_{n_j}, v_{n_j}\right)-g\left(u_{n_j}, v_{n_j}\right). \end{aligned}$$

As a result, it is clear that, according to the two assumptions (1.6) and (1.7) that can be respectively used in of (5.4) and (5.5), the terms N_n and M_n are positives. This would lead to the following assertions:

$$\begin{aligned} \int_Q N_n dxdt &\leq C\int_Q \left(u_{n_j}+v_{n_j}\right) dxdt + \int_\Omega u_0 dx, \\ \int_Q M_n dxdt &\leq C\int_Q \left(u_{n_j}+v_{n_j}\right) dxdt + \int_\Omega \left(u_0+v_0\right) dx. \end{aligned}$$

Consequently, Lemma 4.3 implies:

$$\int_Q N_n dxdt < +\infty, \quad \int_Q M_n dxdt < +\infty,$$

which immediately gives:

$$\int_Q \left|f\left(u_{n_j}, v_{n_j}\right)\right| dxdt \leq C\int_Q \left(u_{n_j}+v_{n_j}\right) dxdt + \int_Q N_n dxdt < +\infty,$$

and

$$\int_Q \left|g\left(u_{n_j}, v_{n_j}\right)\right| dxdt \leq C\int_Q \left(u_{n_j}+v_{n_j}\right) dxdt + \int_Q M_n dxdt < +\infty.$$

Now, we assume $h_n = N_n + C\left(u_{n_j}+v_{n_j}\right)$ and $\Psi_n = M_n + C\left(u_{n_j}+v_{n_j}\right)$. Clearly, one can observe that h_n and Ψ_n are positives in $L^1(Q)$, and

$$\left|f\left(u_{n_j}, v_{n_j}\right)\right| \leq h_n \text{ a.e, and } \quad \left|g\left(u_{n_j}, v_{n_j}\right)\right| \leq \Psi_n \text{ a.e.}$$

Let us now combine this result with (5.3), and then apply the convergence theorem dominated by Lebesgue to obtain:

$$\begin{matrix} f\left(u_{n_j}, v_{n_j}\right) \to f(u,v) \\ g\left(u_{n_j}, v_{n_j}\right) \to g(u,v) \end{matrix} \quad \text{in } L^1(Q).$$

By passing in the limit $j \to +\infty$ of (5.2) in $L^1(Q)$, we obtain:

$$\begin{cases} u(t) = S_1(t)u_0 + \int_0^t S_1(t-s) f(u(s), v(s)) ds, \\ v(t) = S_2(t)v_0 + \int_0^t S_2(t-s) g(u(s), v(s)) ds. \end{cases}$$

Hence, (u,v) satisfies (5.1), and consequently (u,v) is the solution of system (1.1)-(1.3). □

6. Conclusions

In this paper, the global existence of solution for the fractional reaction-diffusion system has been discussed and proved as well. The compact semigroup methods and some L^1-estimates have been utilized for this purpose. Several theoretical results have been consequently inferred and derived.

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