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Some Results on Conditionally Sequential Absorbing Maps in Multiplicative Metric Space

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Abstract. This paper aims to prove two general fixed point theorems in multiplicative metric space (MMS) by using reciprocally continuous mappings and conditionally sequential absorbing mappings. Further our outcomes are validated by discussing two appropriate examples.

1. Introduction

One of the most exciting areas of contemporary mathematics is fixed point theory, which is also interesting topic of the analysis. Further this topic has became a platform due to its wide applications in pure and applied mathematics. In this connection S. Young Cho et al [1] proved a common fixed point theorem over a complete metric space. Later,many researchers generated results in diffeent spaces. In this process Monika Verma et al [2] generalized [1] for multiplicative metric space.Furthermore some results can be witnessed like [3], [4], [5], [6], [7] [8] and [9] in MMS. Using the conditions conditionally sequential absorption and reciprocally continuous mappings, the goal of this research is to derive two common fixed point theorems for MMS. Further two suitable examples are discussed to validate our theorems.

2. Preliminaries

Definition 2.1 Let X be a non empty set and $d : X \times X \to R^+$ then (X, d) is said to be MMS if satisfying the following conditions:

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(i) $d(\zeta,\eta)\geq 1$ for all $\zeta,\eta\in X$ and $d(\zeta,\eta)=1$ if and only if $x=y$

(ii) $d(\zeta,\eta)=d(\zeta,\eta)$ for all $\zeta,\eta\in X$

(iii) $d(\zeta,\eta)\leq d(\zeta,\beta).d(\beta,\eta)$ for all $\zeta,\eta,\beta\in X$ (multiplicative triangle inequality).

Then (X,d) is called MMS.

The pair of mapping (G,J) of a MMS (X,d) is said to be

Definition 2.2 Compatible if $\lim\limits_{j\to\infty} d(GJ\eta_j, JG\eta_j)=1$, whenever η_j is a sequence in X such that $G\eta_j=J\eta_j=\zeta$ for some $\zeta\in X$.

Definition 2.3 Weakly compatible if $G\zeta=J\zeta$ for some $\zeta\in X$ such that $IJ\zeta=JI\zeta$.

Definition 2.4 If there is a coincidence point where the mappings commute then it is said to be Occasionally Weakly Compatible (OWC).

Example 2.4.1 Let (X,d) be a MMS and $\forall\eta,\zeta\in X$ we have $(\eta,\zeta)=e^{|\eta-\zeta|}$.

Now the self mappings G,J are defined on $X=[0,\infty)$ and given below

$G(\eta)=\frac{\eta+1}{2}$ and $J(\eta)=\frac{\eta^2+1}{2}$ for all $\eta\in X$.

From above $\eta=0,1$ are coincidence points for the mappings G,J.

At $\eta=0$

$G(0)=J(0)=\frac{1}{2}$,

$GJ(0)=G(\frac{1}{2})=\frac{3}{4}$,

$JG(0)=J(\frac{1}{2})=\frac{5}{8}$.

Therefore $GJ(0)\neq JG(0)$.

And also $GJ(1)=JG(1)=1$.

Resulting that the maps G, J are OWC but not weakly compatible.

Definition 2.5 Conditionally sequentially absorbing if whenever a sequence (ζ_j) satisfying $\{(\zeta_j)\colon \lim\limits_{j\to\infty} G\zeta_j=\lim\limits_{j\to\infty} J\zeta_j\}\neq\emptyset$ then there exists another sequence (η_j) in X with $\lim\limits_{j\to\infty} G\eta_j=\lim\limits_{j\to\infty} J\eta_j=u$ for some $u\in X$ such that $\lim\limits_{j\to\infty} d(G\eta_j, GJ\eta_j)=1$ and $\lim\limits_{j\to\infty} d(J\eta_j, JG\eta_j)=1$.

Example 2.5.1 Lt (X,d) be an MMS and $\forall\eta,\zeta\in X$ we have $d(\eta,\zeta)=e^{|\eta-\zeta|}$.

Now the self mappings G,J are defined on $X=[0,\infty)$ and given below

$$G(\eta)=\begin{cases} sin\eta & \text{if } 0\leq\eta<\frac{\pi}{2}\\ 2\eta^2 & \text{if } 0\frac{\pi}{2}\leq\eta\leq\pi;\end{cases}$$

$$J(\eta)=\begin{cases} cos\eta & \text{if } 0\leq\eta<\frac{\pi}{2}\\ \pi\eta & \text{if } 0\frac{\pi}{2}\leq\eta\leq\pi;\end{cases}$$

From above $\eta=0,\frac{\pi}{2}$ are coincidence points for the mappings G,J.

At $\eta=0$

$G(0)=J(0)=0$,

$GJ(0)=G(0)=0$,

$JG(0)=J(0)=0$.

Therefore $GJ(0) = JG(0)$.
And also $GJ(\frac{\pi}{2}) = G(\frac{\pi^2}{2}) = \frac{\pi^4}{2}$,
$JG(\frac{\pi}{2}) = G(\pi^2) = 2\pi^4$,
Therefore $GJ(\frac{\pi}{2}) \neq JG(\frac{\pi}{2})$.

Resulting that the maps G, J are not weakly compatible.

Let $(p_j) = \frac{\sqrt{4}}{j}$, for all $j \geq 1$.

Then

$$\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} G(\frac{\sqrt{4}}{j}) = \lim_{j\to\infty} sin(\frac{\sqrt{4}}{j}) = 0 \tag{2.1}$$

and

$$\lim_{j\to\infty} Jp_j = \lim_{j\to\infty} J(\frac{\sqrt{4}}{j}) = \lim_{j\to\infty} 1 - cos(\frac{\sqrt{4}}{j}) = 1 - 1 = 0. \tag{2.2}$$

From (2.1) and (2.2), we get

$$\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j \tag{2.3}$$

From (2.3) implies
$\{(p_j) : \lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j\} \neq \emptyset$.
Then $\exists$ another sequence $q_j = \frac{\pi}{2} + \frac{5}{j}$, for all $j \geq 1$.

$$\lim_{j\to\infty} Gq_j = \lim_{j\to\infty} G(\frac{\pi}{2} + \frac{5}{j}) = \lim_{j\to\infty} = 2(\frac{\pi}{2} + \frac{5}{j})^2 = \frac{\pi^2}{2} \tag{2.4}$$

and

$$\lim_{j\to\infty} Jq_j = \lim_{j\to\infty} J(\frac{\pi}{2} + \frac{5}{j}) = \lim_{j\to\infty} \pi(\frac{\pi}{2} + \frac{5}{j}) = \frac{\pi^2}{2}. \tag{2.5}$$

From (2.4) and (2.5), we get

$$\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j = \frac{\pi^2}{2}. \tag{2.6}$$

Now $\lim_{j\to\infty} GJ(q_j) = GJ(\frac{\pi}{2} + \frac{5}{j}) = \lim_{j\to\infty} G(\pi(\frac{\pi}{2} + \frac{5}{j})) =$
$\lim_{j\to\infty} G(\frac{\pi^2}{2} + \frac{5\pi}{j}) = \frac{\pi^3}{2}$
and
$\lim_{j\to\infty} JG(q_j) = JG(\frac{\pi}{2} + \frac{5}{j}) = \lim_{j\to\infty} J(2((\frac{\pi}{2} + \frac{5}{j})^2)) = \frac{\pi^3}{2}$.
Therefore $\lim_{j\to\infty} d(Gq_j, GJq_j) = 1$ and $\lim_{j\to\infty} d(Jq_j, JGq_j) = 1$.
Hence the pair(G, J) is conditionally sequentially absorbing but not weakly compatible.

Definition 2.6 Reciprocally continuous whenever (η_j) is a sequence in X such that $\lim_{j\to\infty} G\eta_j = \lim_{j\to\infty} J\eta_j = \zeta$ for some $\zeta \in X$ such that $\lim_{j\to\infty} d(G\zeta, GJ\eta_j) = 1$ and $\lim_{j\to\infty} d(J\zeta, JG\eta_j) = 1$.

Example 2.5.1 Lt (X, d) be a Multiplicative metric space and $\forall \eta, \zeta \in X$ we have $d(\eta, \zeta) = e^{|\eta-\zeta|}$.
Now the self mappings G, J are defined on $X = [0, \infty)$ and given below

$$G(\eta) = \begin{cases} \pi cos\eta & \text{if } 0 \leq \eta < \frac{\pi}{2} \\ \eta^2 & \text{if } 0\frac{\pi}{2} \leq \eta \leq \pi; \end{cases}$$

$$J(\eta) = \begin{cases} \pi sec\eta & \text{if } 0 \leq \eta < \frac{\pi}{2} \\ \pi\eta & \text{if } 0\frac{\pi}{2} \leq \eta \leq \pi; \end{cases}$$

From above $\eta = 0, \pi$ are coincidence points for the mappings G, J.

From above at $\eta = 0$

$G(0) = J(0) = \pi$,

$GJ(0) = G(\pi) = \pi^3$,

$JG(0) = J(\pi) = \pi^3$.

Therefore $GJ(0) = JG(0)$.

And also $GJ(\pi) = G(\pi^3) = \pi^7$,

$JG(\pi) = J(\pi^3) = \pi^9$.

Therefore $GJ(\pi) \neq JG(\pi)$

Resulting that the maps G, J are not weakly compatible.

Let $(r_j) = \pi - \frac{3}{j^3}$ for all $j \geq 1$.

Then

$$\lim_{j\to\infty} Gr_j = \lim_{j\to\infty} G(\pi - \frac{3}{j^3}) = \lim_{j\to\infty} (\pi - \frac{3}{j^3})^2 = \pi^2 \tag{2.7}$$

and

$$\lim_{j\to\infty} Jr_j = \lim_{j\to\infty} J(\pi - \frac{3}{j^3}) = \lim_{j\to\infty} \pi(\pi - \frac{3}{j^3}) = \pi^2. \tag{2.8}$$

From (2.7) and (2.8), we get

$$\lim_{j\to\infty} Gr_j = \lim_{j\to\infty} Jr_j \tag{2.9}$$

Now $\lim_{j\to\infty} GJ(r_j) = GJ(\pi - \frac{3}{j^3}) = \lim_{j\to\infty} G(\pi(\pi - \frac{3}{j^3})) = \pi^3$

and

$\lim_{j\to\infty} JG(r_j) = JG(\pi - \frac{3}{j^3}) = \lim_{j\to\infty} J(\pi(\pi - \frac{3}{j^3})) = \pi^4$.

Therefore $\lim_{j\to\infty} d(G(\pi^2), GJr_j) = 1$ and $\lim_{j\to\infty} d(J(\pi^2), JGr_j) = 1$.

Hence the maps G, J are reciprocally continuous.

In [1], The following Theorem was established.

Theorem 2.7 Assume that (X, d) is an MMS which is complete and the mappings B, S, A, and T are defined on X such that

(B1) $B(X) \subseteq S(X)$ and $A(X) \subseteq T(X)$

(B2) $d(Au, Bv) \leq (max\{d(Au, Su), d(Bv, Tv), \sqrt{[d(Au, Tv).d(Bv, Su)]}, d(Su, Tv)\})^p.$ $(max\{d(Au, Su), d(Bv, Tv)\})^q.(max\{d(Au, Tv), d(Bv, Su)\})^r$ for all $u, v \in X$, where $0 < h = p + q + 2r < 1$ (p,q and r are non-ve real numbers).

(B3) Among the subspaces AX or BX or SX or TX is complete

(B4) both the pairs (A,S) and (B,T) are weakly compatible.

Then the four maps A,B,S and T above share a common single fixed point.

Now we generalize the above Theorem 2.7 as below.

3. Main Result

Theorem 3.1 Assume that (X, d) is an MMS which is complete and the mappings A, B, S, and T are defined on X such that

(D1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(D2) $d(Au, Bv) \leq (max\{d(Au, Su), d(Bv, Tv), \sqrt{[d(Au, Tv).d(Bv, Su)]}, d(Su, Tv)\})^p.$ $(max\{d(Au, Su), d(Bv, Tv)\})^q.(max\{d(Au, Tv), d(Bv, Su)\})^r$ for all $u, v \in X$, where $h = p + q + 2r$ and $o < h < 1$ $(p, qandrarenon - verealnumbers)$.

(D3) the pairs (A, S) reciprocally continuous and conditionally sequentially absorbing and (B, T) is occasionally weakly compatible.

Then the four mappings share a single fixed point which is common in X.

Proof: By $(D1)$, there is a point here $u_0 \in X$ such that $Au_0 = Tu_1 = y_1$. For this point $u_1 \in X$ there exists a point u_2 in X such that $Bu_1 = Su_2 = y_2$ and so on. Similarly, we can inductively define $Bu_{2j-1} = Su_{2j} = y_{2j}; Au_{2j} = Tu_{2j+1} = y_{2j+1}$ for $n = 0, 1, 2, ...$

We can now show that the sequence $\{v_j\}$ is a Cauchy in X. Put $u = u_{2j}$ and $v = u_{2j+1}$ in $(D2)$ then

$d(v_{2j+1}, v_{2n+2}) = d(Au_{2j}, Bu_{2j+1}) \leq$
$(max\{d(Au_{2j}, Su_{2j}), d(Bu_{2j+1}, Tu_{2j+1}), \sqrt{[d(Au_{2j}, Tu_{2j+1}).d(Bu_{2j+1}, Su_{2j})]}, d(Su_{2j}, Tu_{2j+1})\})^p.$
$(max\{d(Au_{2j}, Su_{2j}), d(Bu_{2j+1}, Tu_{2j+1})\})^q.(max\{d(Au_{2j}, Tu_{2j+1}), d(Bu_{2j+1}, Su_{2j})\})^r$

$d(v_{2j+1}, v_{2n+2}) \leq$
$(max\{d(v_{2j+1}, v_{2j}), d(v_{2j+2}, v_{2j+1}), \sqrt{[d(v_{2j+1}, v_{2j+1}).d(v_{2j+2}, v_{2j})]}, d(v_{2j}, v_{2j+1})\})^p.$
$(max\{d(v_{2j+1}, v_{2j}), d(v_{2j+2}, v_{2j+1})\})^q.(max\{d(v_{2j+1}, v_{2j+1}), d(v_{2j+2}, v_{2j})\})^r$

$d(v_{2j+1}, v_{2n+2}) \leq$
$(max\{d(v_{2j+1}, v_{2j}), d(v_{2j+2}, v_{2j+1}), \sqrt{[d(v_{2j+1}, v_{2j+1}).d(v_{2j+1}, v_{2j}).d(v_{2j+1}, v_{2j+2})]}, d(v_{2j}, v_{2j+1})\})^p.$
$(max\{d(v_{2j+1}, v_{2j}), d(v_{2j+2}, v_{2j+1})\})^q.(max\{d(v_{2j+1}, v_{2j+1}), d(v_{2j+1}, v_{2j}).d(v_{2j+1}, v_{2j+2})\})^r$

In the above equation, if $d(v_{2j+2}, v_{2j+1}) > d(v_{2j+1}, v2j)$ for some +ve integer j, then we have $d(v_{2j+1}, v_{2j+2}) \leq d(v_{2j+1}, v_{2j+2})^h$, where $o < h = p + q + 2r < 1$, a contradiction.

Therefore we have

$d(v_{2j+2}, v_{2j+1}) \leq d(v_{2j}, v_{2j+1})^h.$

Likewise, we have

$d(v_j, v_{2j+1}) \leq (d(v_{j-1}, v_j)^h) \leq (d(v_{j-2}, vj - 1)^{h^2} \leq .... \leq (d(v_0, v_1))^{h^n}.$

Let l, j In N such that $l > j$, we get

$d(v_l, v_j) \leq d(v_l, v_{l-1})....d(v_{j+1}, v_j)$
$\leq (d(v_1, v_0))^{h^{l-1}+....h^j}$
$\leq (d(v_1, v_0))^{\frac{h^j}{1-h}} \to 1$ as $l, j \to \infty$.

As a result, the sequence $\{v_j\}$ is a Cauchy.

By the completeness of X $\exists$ $w \in X$ such that $v_j \to w$ as $j \to \infty$.
Accordingly, the sequences

$$Au_{2j}, Su_{2j} \to z, Tu_{2j+1}, Bu_{2j+1} \to z \tag{3.1}$$

as $j \to \infty$.

Use the notion
$L\{A, S\} = \{(u_j) : \lim_{j\to\infty} Au_j = \lim_{j\to\infty} Su_j\}$.

By $(D3)$ the pair of mapping (A, S) is conditionally sequentially absorbing from (3.1) $L\{A, S\} \neq \emptyset \Rightarrow \exists(v_j)$ such that

$$\lim_{j\to\infty} Av_j = \lim_{j\to\infty} Sv_j = \psi \tag{3.2}$$

$$\Longrightarrow d(Av_j, ASv_j) = 1 and d(Sv_j, SAv_j) = 1 \tag{3.3}$$

By the reciprocally continuous of the pair (A, S) implies whenever

$$\lim_{j\to\infty} Av_j = \lim_{j\to\infty} Sv_j = \psi \tag{3.4}$$

$$\Longrightarrow d(A\psi, ASv_j) = 1 and d(S\psi, SAv_j) = 1. \tag{3.5}$$

Using (3.2) and (3.5) in (3.3), we get
$A\psi = S\psi = \psi$.
Since $A\psi$ is an element in $A(X)$ by $(D1)$ there exists φ such that

$$\psi = S\psi = A\psi = T\varphi. \tag{3.6}$$

Claim $B\varphi = T\varphi$.
Putting $u = \psi$, $v = \varphi$ in (D2)

$d(A\psi, B\varphi) \leq (max\{d(A\psi, S\psi), d(B\varphi, T\varphi), \sqrt{[d(A\psi, T\varphi).d(B\varphi, S\psi)]}, d(S\psi, T\varphi)\})^p.$
$(max\{d(A\psi, S\psi), d(B\varphi, T\varphi)\})^q.(max\{d(A\psi, T\varphi), d(B\varphi, S\psi)\})^r$.
Letting $n \to \infty$ we get,
$d(T\varphi, B\varphi) \leq (max\{d(\psi, \psi), d(B\varphi, T\varphi), \sqrt{[d(\psi, \psi).d(B\varphi, T\varphi)]}, d(\psi, \psi)\})^p.$
$(max\{d(\psi, \psi), d(B\varphi, T\varphi)\})^q.(max\{d(\psi, \psi), d(B\varphi, T\varphi)\})^r$

$d(T\varphi, B\varphi) \leq d(T\varphi, B\varphi)^{p+q+r}$,
which is a contradiction.

Hence $T\varphi = B\varphi$.

Which gives

$$\psi = S\psi = A\psi = T\varphi = B\varphi. \tag{3.7}$$

From (D3) we have the pair (B, T) is occasionally weakly compatible which gives $BT\varphi = TB\varphi$ implies that $B\psi = T\psi$ from (3.7).

Claim $\psi = B\psi$.

Putting $u = v = \psi$ in (D2)

$d(\psi, B\psi) \leq (max\{d(\psi, \psi), d(T\psi, T\psi), \sqrt{[d(\psi, B\psi).d(B\psi, \psi)]}, d(\psi, B\psi)\})^p.$

$(max\{d(\psi, \psi), d(T\psi, T\psi)\})^q.(max\{d(\psi, B\psi), d(B\psi, \psi)\})^r$

$d(\psi, B\psi) \leq d(\psi, B\psi)^{p+r}$, a contradiction

which impies $\psi = B\psi$.

Therefore $\psi = S\psi = A\psi = T\psi = B\psi$.

Which implies that ψ is the required common fixed point.

For Uniqueness:

Assume that ρ be the another fixed point then $\rho = S\rho = A\rho = T\rho = B\rho$.

Putting $u = \psi$ and $v = \rho$ in (D2), we get

$d(A\psi, B\rho) \leq (max\{d(A\psi, S\psi), d(B\rho, T\rho), \sqrt{[d(A\psi, T\rho).d(B\rho, S\psi)]}, d(S\psi, T\rho)\})^p.$

$(max\{d(A\psi, S\psi), d(B\rho, T\rho)\})^q.(max\{d(A\psi, T\rho), d(B\rho, S\psi)\})^r$

$d(\psi, \rho) \leq (max\{d(\psi\psi), d(\rho, \rho), \sqrt{[d(\psi, \rho).d(\rho, \psi)]}, d(\psi, \rho)\})^p.$

$(max\{d(\psi, \psi), d(\rho, \rho)\})^q.(max\{d(\psi, \rho), d(\rho, \psi)\})^r$

$d(\psi, \rho) \leq d(\psi, \rho)^{p+q+r}$, a contradiction

which implies $\psi = \rho$.

This proves the uniqueness.

Now we discuss an example.

Example 3.2 Assume that (X, d) is an MMS space with $d(u, v) = e^{|u-v|}$ for all $u, v \in X$.

A, B, S, and T are the self maps that are defined on $X = [0, 1]$ as follows:

$$A(\eta) = \begin{cases} \frac{\eta^2+1}{2} & \text{if } 0 \leq \eta < \frac{1}{5} \\ \eta & \text{if } \frac{1}{5} \leq \eta \leq 1; \end{cases}$$

$$S(\eta) = \begin{cases} \frac{\eta^2+\eta+1}{2} & \text{if } 0 \leq \eta < \frac{1}{5} \\ \eta^2 & \text{if } \frac{1}{5} \leq \eta \leq 1; \end{cases}$$

$$B(\eta) = \begin{cases} \frac{\eta^2+4\eta+1}{2} & \text{if } 0 \leq \eta < \frac{1}{5} \\ \frac{1}{5} & \text{if } \frac{1}{5} \leq \eta \leq 1; \end{cases}$$

$$T(\eta) = \begin{cases} \frac{\eta^2+3\eta+1}{2} & \text{if } 0 \leq \eta < \frac{1}{5} \\ \eta & \text{if } \frac{1}{5} \leq \eta \leq 1; \end{cases}$$

Now $A(X) = [\frac{1}{2}, 0.52] \cup (\frac{1}{5}, 1]$, $S(X) = [\frac{1}{2}, 0.9) \cup \{\frac{1}{5}\}$, $B(X) = [\frac{1}{2}, 0.62] \cup \{\frac{1}{5}\}$ and $T(X) = [\frac{1}{2}, 0.52] \cup (\frac{1}{5}, 1]$.

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ so that $(D1)$ is satisfied.

For the pair of mappings (A, S) and (B, T), it is evident that 0 and 1 are coincidence points.

At $\eta = 0 \Rightarrow A(0) = S(0) = \frac{1}{2}$.

But $AS(0) = A(\frac{1}{2}) = \frac{1}{2}$ and
$SA(0) = S(\frac{1}{2}) = \frac{1}{5}$.
Therefore $AS(0) \neq SA(0)$.
Also at $\eta = 0 \Rightarrow B(0) = T(0) = \frac{1}{2}$.
But $BT(0) = B(\frac{1}{2}) = \frac{1}{8}$ and
$TB(0) = T(\frac{1}{2}) = \frac{1}{2}$.
Therefore $BT(0) \neq TB(0)$.

As a result, the mappings (A, S) and (B, T) are not weakly compatible.

Take a sequence $\eta_k = \frac{3}{2k}$ for all $k \geq 0$.
Then $\lim_{k\to\infty} A\eta_k = A(\frac{3}{2k}) = \frac{(\frac{3}{2k})^2+1}{2} = \frac{1}{2}$ and $\lim_{n\to\infty} S\eta_k = S(\frac{3}{2k}) = \frac{1}{2}$.
Implies $\lim_{k\to\infty} A\eta_k = \lim_{k\to\infty} S\eta_k = \frac{1}{2}$.
Now $\lim_{k\to\infty} AS\eta_k = \lim_{k\to\infty} AS(\frac{3}{2k}) = \lim_{k\to\infty} A(\frac{1}{2} + \frac{9+6k}{8k^2}) = \frac{1}{2}$
and $\lim_{k\to\infty} SA\eta_k = \lim_{k\to\infty} SA(\frac{3}{2k^2} + \frac{1}{2}) = (\frac{1}{2} + \frac{9}{8k^2})^2 = \frac{1}{4}$.
Therefore the pair (A, S) is non-compatible so that $\exists$ another sequence $\beta_k = \frac{1}{5} + \frac{2}{3k}$ for all $k \geq 1$.
$\lim_{k\to\infty} A\beta_k = \lim_{k\to\infty} S\beta_k = \frac{1}{5}$.
Also we have $\lim_{k\to\infty} AS\beta_k = \lim_{k\to\infty} AS(\frac{1}{5} + \frac{2}{3m}) = \lim_{k\to\infty} A(\frac{1}{5}) = \frac{1}{5}$
and
$\lim_{k\to\infty} SA\beta_k = \lim_{k\to\infty} SA(\frac{1}{5} + \frac{2}{3k}) = \lim_{k\to\infty} S(\frac{1}{5} + \frac{2}{3k}) = \frac{1}{5}$.
Thus from above $\lim_{k\to\infty} d(A\beta_k, AS\beta_k) = d(\frac{1}{5}, \frac{1}{5}) = e^{|\frac{1}{5} - \frac{1}{5}|} = 1$ and
$\lim_{k\to\infty} d(S\beta_k, SA\beta_k) = d(\frac{1}{5}, \frac{1}{5}) = e^{|\frac{1}{5} - \frac{1}{5}|} = 1$.
Further $\lim_{k\to\infty} d(AS\beta_k, A(\frac{1}{5})) = d(\frac{1}{5}, \frac{1}{5}) = e^{|\frac{1}{5} - \frac{1}{5}|} = 1$.
From the above we can conclude that the pairs (A,S) and (B,T) are non-compatible reciprocally continuous and conditionally sequential absorbing mappings.
Also it is observed that $A(\frac{1}{5}) = S(\frac{1}{5}) = B(\frac{1}{5}) = T(\frac{1}{5}) = \frac{1}{5}$.

It is found that the only common fixed point shared by the four self-maps is $\frac{1}{5}$.

Now we prove another generalization of Theorem 2.7, as given below.

Theorem 3.3 Assume that (X, d) is an MMS which is complete and the mappings A,B,S, and T are defined on X such that

(E1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$
(E2) $d(Au, Bv) \leq (max\{d(Au, Su), d(Bv, Tv), \sqrt{[d(Au, Tv).d(Bv, Su)]}, d(Su, Tv)\})^p.$
$(max\{d(Au, Su), d(Bv, Tv)\})^q.(max\{d(Au, Tv), d(Bv, Su)\})^r$
for all $u, v \in X$, where $h = p + q + 2r$ and $0 < h < 1$ (p,q and r are non-ve real numbers).
(E3) The mappings for the pairs (A, S) and (B, T) are non-compatible reciprocally continuous and conditionally sequential absorbing mappings.

Then the four mappings share a single fixed point which is common in X.

Proof:

By $(E3)$ we have the pair (A, S) non-compatible $\implies$ there is a sequence (u_j) with

$$\lim_{j\to\infty} Au_j = \lim_{j\to\infty} Su_j = \psi \tag{3.8}$$

for some $\psi \in X$.

$\implies lim_{j\to\infty}\ d(ASu_j, SAu_j)$ not exist or $lim_{j\to\infty}\ d(ASu_j, SAu_j) \neq 1$.

Considering that the pair (A, S) is conditionally sequentially absorbing from (3.8) we have $L\{A, S\} \neq \emptyset \implies \exists(v_j)$ such that

$\lim_{j\to\infty} Av_j = \lim_{j\to\infty} Sv_j = \psi$ (say)

$\implies lim_{j\to\infty}\ d(Av_j, ASv_j) = 1$ and $lim_{j\to\infty}\ d(Sv_j, SAv_j) = 1$.

Also from (E3) we have (A, S) is reciprocally continuous means whenever

$$\lim_{j\to\infty} Av_j = \lim_{j\to\infty} Sv_j = \psi. \tag{3.9}$$

$\implies lim_{j\to\infty}\ d(A\psi, ASv_j) = 1$ and $lim_{j\to\infty}\ d(S\psi, SAv_j) = 1$.

Using the above equations, we get

$$A\psi = S\psi = \psi. \tag{3.10}$$

Since the pair (B, T) is non compatible implies there is sequence (u_j) with

$$\lim_{j\to\infty} Bu_j = \lim_{j\to\infty} Tu_j = \varphi \tag{3.11}$$

for some $\varphi \in X$.

$\implies lim_{j\to\infty}\ d(BTu_j, TBu_j)1$ not exist or $lim_{j\to\infty}\ d(BTu_j, TBu_j) \neq 1$.

From (E3) the pair (B, T) is conditionally sequential absorbing from (3.11)

$L\{B, T\} \neq \emptyset \implies \exists(v_j)$ such that

$\lim_{j\to\infty} Bv_j = \lim_{j\to\infty} Tv_j = \beta$ (say)

$\implies lim_{j\to\infty}\ d(Bv_j, BTv_j) = 1$ and $lim_{j\to\infty}\ d(Tv_j, TBv_j) = 1$.

Also the pair (B, T) is reciprocally continuous implies whenever $\lim_{j\to\infty} Bv_j = \lim_{j\to\infty} Tv_j = \beta$ (say)

$\implies lim_{j\to\infty}\ d(B\beta, BTv_j) = 1$ and $lim_{j\to\infty}\ d(T\beta, TBv_j) = 1$.

Using the above equation, we get

$$B\beta = T\beta = \beta. \tag{3.12}$$

Claim $\beta = \psi$.

Assume that $\beta \neq \psi$.

Putting $u = \psi$ and $v = \beta$ in $(E2)$

$d(A\psi, B\beta) \leq (max\{d(A\psi, S\psi), d(B\beta, T\beta), \sqrt{[d(A\psi, T\beta).d(B\beta, S\psi)]}, d(S\psi, T\beta)\})^p.$
$(max\{d(A\psi, S\psi), d(B\beta, T\beta)\})^q.(max\{d(A\psi, T\beta), d(B\beta, S\psi)\})^r.$

Letting as $n \to \infty$

$d(\psi,\beta) \leq (max\{d(\psi,\psi), d(\beta,\beta), \sqrt{[d(\psi,\beta).d(\beta,\psi)]}, d(\psi,\beta)\})^p.$

$(max\{d(\psi,\psi), d(\beta,\beta)\})^q.(max\{d(\psi,\beta), d(\beta,\psi)\})^r.$

$d(\psi,\beta) \leq d(\psi,\beta)^{p+r}.$

A contradiction hence $\psi = \beta$.

Therefore $A\psi = S\psi = B\psi = T\psi = \psi$.

Which implies that ψ is the required unique common fixed point.

From (E2) uniqueness follows easily.

Now we give an example to support Theorem 3.3.

Example 3.4 Assume that (X, d) is a MMS space with $d(u, v) = e^{|u-v|}$ for all $u, v \in X$.

A, B, S, and T are the self maps that are defined on $X = [0, 12]$ as follows:

$$A(\eta) = \begin{cases} \eta^4 & \text{if } 0 \leq \eta \leq 1 \\ 2 & \text{if } 1 < \eta \leq 12; \end{cases}$$

$$S(\eta) = \begin{cases} \eta^2 & \text{if } 0 \leq \eta \leq 1 \\ 2log\eta & \text{if } 1 < \eta \leq 12; \end{cases}$$

$$B(\eta) = \begin{cases} \eta^5 & \text{if } 0 \leq \eta \leq 1 \\ 4 & \text{if } 1 < \eta \leq 12; \end{cases}$$

$$T(\eta) = \begin{cases} \eta^3 & \text{if } 0 \leq \eta \leq 1 \\ 4log\eta & \text{if } 1 < \eta \leq 12. \end{cases}$$

Now $A(X) = [0, 1] \cup \{2\}$, $S(X) = [0, 4.96]$, $B(X) = [0, 1] \cup 4$ and $T(X) = [0, 5.45]$.

We have from above maps $\eta = e$ and 1 are coincidence points for (A,S) and (B,T).

At $\eta = e$, $A(e) = S(e) = 2$ and $B(e) = T(e) = 4$.

$AS(e) = A(2) = 2, SA(e) = S(2) = 2log2$ and $BT(e) = B(1) = 1, TB(e) = T(4) = 4log4$.

Clearly $AS(e) \neq SA(e)$ and $BT(e) \neq TB(e)$.

As a result, the mappings (A,S) and (B,T) are not weakly comparable.

Now take a sequence $\eta_j = e + \frac{3}{4j}$, for all $j \geq 1$.

Then

$$\lim_{j\to\infty} A\eta_j = \lim_{j\to\infty} A(e + \frac{3}{4j}) = \lim_{j\to\infty} 2 = 2 \tag{3.13}$$

and

$$\lim_{j\to\infty} S\eta_j = \lim_{j\to\infty} S(e + \frac{3}{4j}) = \lim_{j\to\infty} 2log(e + \frac{3}{4j}) = 2. \tag{3.14}$$

Implies $\lim_{j\to\infty} A\eta_j = \lim_{j\to\infty} S\eta_j = 2$.

Now $\lim_{j\to\infty} AS(e + \frac{3}{4j})$=$\lim_{j\to\infty} A(2log(e + \frac{3}{4m}))$= $\lim_{j\to\infty} 2 = 2$

and $\lim_{j\to\infty} SA(e + \frac{3}{4j})$=$\lim_{j\to\infty} S(2)$= $2log2$.

Therefore the pair (A, S) is non-compatible.

Further from (3.13) and (3.14) we get

$\{(\eta_j) : \lim_{j\to\infty} A\eta_j = \lim_{j\to\infty} S\eta_j\} \neq \emptyset$.

Further there exists another sequence $\beta_j = 1 - \frac{5}{4j}$ for all $j \geq 1$.

$$\lim_{j\to\infty} A\beta_j = \lim_{j\to\infty} A(1 - \frac{5}{4j}) = \lim_{j\to\infty} (1 - \frac{5}{4j})^4 = 1 \quad (3.15)$$

and

$$\lim_{j\to\infty} S\beta_j = \lim_{j\to\infty} S(1 - \frac{5}{4j}) = \lim_{j\to\infty} (1 - \frac{5}{4j})^2 = 1. \quad (3.16)$$

Now $\lim_{j\to\infty} AS(\beta_j) = \lim_{j\to\infty} AS(1 - \frac{5}{4j}) = \lim_{j\to\infty} A(1 - \frac{5}{4m})^2 = \lim_{j\to\infty} A(1 - \frac{15}{16j}) = \lim_{j\to\infty} (1 - \frac{5}{4j})^4 = 1$
and $\lim_{j\to\infty} SA(\beta_j) = \lim_{j\to\infty} SA(1 - \frac{5}{4j}) = \lim_{j\to\infty} S(1 - \frac{5}{4j})^4 = \lim_{j\to\infty} (1 - \frac{5}{4j})^8 = 1$.
Therefore

$\lim_{j\to\infty} d(AS\beta_j, A\beta_j)$ =d(1,1)=$e^{|1-1|} = 1$

and

$\lim_{j\to\infty} d(SA\beta_j, S\beta_j)$ =d(1,1)= $e^{|1-1|} = 1$.

Further

$\lim_{j\to\infty} d(AS\beta_j, A(1))$ =d(1,1)=$e^{|1-1|} = 1$

and

$\lim_{j\to\infty} d(SA\beta_j, S(1))$ = d(1,1) $e^{|1-1|} = 1$.

It follows that the pairs (A,S) and (B,T) have unique fixed point $\eta = 1$ and are non-compatible reciprocally continuous and conditionally sequentially absorbing mappings. Further the maps A,S,T and B are discontinuous at $\eta = 1$.Moreover the pairs (A,S) and (B,T) are not weakly compatible and hence all the conditions of Theorem 3.3 are satisfied.

4. Conclusion

In this paper we generalized Theorem 2.7 using

(i) conditionally sequential absorbing, reciprocally continuous and OWC by removing weakly compatible mappings in Theorem 3.1.

(ii) Further the weakly compatible mappings are replaced by non-compatible reciprocally continuous and conditionally sequential absorbing mappings in Theorem 3.3.

Moreover the above two results are substantiated by two suitable examples.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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