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Solution for a System of First-Order Linear Fuzzy Boundary Value Problems

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Abstract. In this paper, we consider homogeneous and non-homogeneous system of first order linear fuzzy boundary value problems (SFOLBVPs) under granular differentiability. Using the concept of horizontal membership function, we introduced the notion of first order granular differentiability for n-dimensional fuzzy functions. We present granular integral and its properties. Theorems on the existence and uniqueness of solutions for homogeneous and non-homogeneous SFOLFBVPs are proved. We develop an algorithm for solution of non-homogeneous SFOLBVPs under granular differentiability. We provide some examples to illustrate the validity of the proposed algorithm.

1. Introduction

Mathematical models to deal with uncertainty are frequently used in fuzzy differential equations. The rich work on fuzzy differential equations (FDE) is applied to population, bioinformatics, growth and decay, economics, quantum optics, and friction models. First-order linear fuzzy systems (FOLFS) are modeled by behaviors of many dynamical systems (DS) with uncertainty. Buckley et al. [4] presented two types of solutions using the extension principle and standard interval arithmetic (SIA) to the first-order system of equations with fuzzy initial conditions. Gasilov et al. [6] proposed a geometric approach to solve the fuzzy system of differential equations (FSDEs) with crisp real coefficients and fuzzy initial conditions. Fard et al. [5] introduced an iterative technique to solve FSDEs with fuzzy constant coefficients using the H-differentiability concept. Hashemi et al. [7] developed the series solution to SFDEs under H-differentiability. Mondal et al [10] analyzed adaptive schemes to study

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the FSDEs in two types, fuzzy and in the crip sense. Bara et al. [1] analyzed numerical solutions for FSDEs using the variational iteration technique. Keshavarz et al [8] proposed to get an analytical solution for FSDEs under gH-differentiability. Boukezzoula et al. [3] proposed a new technique to solve the FSDEs with variables as fuzzy intervals. Suhhiem and Khwayyit [16] proposed to get a semi-analytical solution for autonomous FDEs using the Adomian decomposition method. But, these derivatives possess some drawbacks such as derivatives may not always exist, doubling property, a multiplicity of solutions, unnatural behavior in modeling (UBM) phenomenon, and monotonicity of the uncertainty.

Piegat et al. [14] presented a horizontal membership function (HMF) for fuzzy function (FF) and solved distinct granular problems. Recently Piegat et al. [15], provide a detailed comparison of HMFs and inverse HMFs, highlighting the key distinctions between the two kinds of functions. Mazandarani et al. [9] established granular differentiability (gr-differentiability), a novel idea of FF differentiability based on RDM-IA and horizontal membership functions (HMF). Najariyan et al. [12] investigated the solution of singular FDEs with the concept of gr-differentiability. Under the concept of gr-differentiability, Najariyan et al. [11] effectively tune the fuzzy granular PID controller using a particle swarm optimization algorithm. These findings were made by studying FDEs under the gr-differentiability to overcome all the shortcomings as discussed.

In this present work, we consider SFOLBVPs under gr-differentiability. Section 2, presents basic definitions and results related to HMFs, gr-metric, gr-differentiability, and gr-integration of n-dimensional FF. The existence and uniqueness of theorems for SFOLFIVPs under gr-differentiability are established in Section 3. Section 4 presents a working method to solve SFOFBVPs under gr-differentiability and highlighted proposed results with suitable examples. Concluding remarks and future works are discussed in Section 5.

2. Preliminaries

This section presents some useful definitions, notations, and results that are useful to establish the main results.

Definition 2.1. *A non-empty fuzzy subset p of $\mathbf{R}$, with membership function, $p : \mathbf{R} \to [0, 1]$, is said to be a fuzzy number, if it is semi continuous, fuzzy convex, normal and compactly supported on $\mathbf{R}$. Here $p(y)$ is the membership degree of y, for every $y \in \mathbf{R}$. Let $\mathbf{R}_F$ denotes the space of fuzzy numbers(FNs) in $\mathbf{R}$. The β-level sets of p are defined by $[p]^\beta = \{y \in \mathbf{R} : p(y) \geq \beta\} = [p_l^\beta, p_r^\beta]$, for $0 < \beta \leq 1$ and $[p]^0 = cl\{y \in \mathbf{R} : p(y) > 0\}$.*

For notations, definitions and basic results related to HMF, gr-derivative and gr-integrations of fuzzy numbers refer to [9].

Definition 2.2. *Let* $\mathbf{R}_F^n = \underbrace{\mathbf{R}_F \times \mathbf{R}_F \times \mathbf{R}_F \times \cdots \times \mathbf{R}_F}_{n\,times}$, *be the space of n-tuple fuzzy numbers. Then the addition and scalar multiplication defined component wise as follows:*
If $u = (u_1, u_2, \cdots, u_n), v = (v_1, v_2, \cdots, v_n) \in \mathbf{R}_F^n$, *then*

(i) $u + v = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$,
(ii) $ku = (ku_1, ku_2, \cdots, ku_n)$,

where $u_i, v_i \in \mathbf{R}_F,\ i = 1, 2, \cdots n$ *and* $k \in \mathbf{R}$.

Definition 2.3. *If* $g : [b, c] \to \mathbf{R}_F^n$, *is a FF, then it is called an n-dimensional vector of FN valued function on* $[b, c]$.

Definition 2.4. *If* $g : [b, c] \to \mathbf{R}_F^n$ *is a n-dimensional FF, include* $mn \in N$, *distinct FNs such that* $u_i = (u_{i1}, u_{i2}, \ldots, u_{in})$, $i = 1, 2, \cdots m$, *then the HMF of g is indicated by* $H(g(x)) \equiv g_{gr}(x, \beta, \alpha_g)$, *and interpreted as* $g_{gr} : [b, c] \times [0, 1] \times \underbrace{[0, 1] \times [0, 1] \times \cdots \times [0, 1]}_{mn\,times} \to \mathbf{R}^n$, *in which* $\alpha_g \equiv (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$, $i = 1, 2, \cdots m$, *where* $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}$, *are the RDM variables related to* $u_{i1}, u_{i2}, \ldots, u_{in}$.

Definition 2.5. *Let p and q be two n-dimensional FNs. Then* $H(p) = H(q)$, *for all* $\alpha_p = \alpha_q \in [0, 1]$ *if and only if p and q are said to be equal.*

Definition 2.6. *Let* $p, q \in \mathbf{R}_F^n$. *The function* $\mathcal{D}_{gr}^n : \mathbf{R}_F^n \times \mathbf{R}_F^n \to \mathbf{R}^+ \cup \{0\}$, *defined by*

$$\mathcal{D}_{gr}^n(p, q) = \sup_{\beta} \max_{\alpha_p, \alpha_q} \|p_{gr}(\beta, \alpha_p) - q_{gr}(\beta, \alpha_q)\|,$$

which is called a n-dimensional granular distance between two n-dimentional FNs p and q, where $\|.\|$ *represents Euclidean norm in* $\mathbf{R}^n$.

Definition 2.7. *If* $\mathbf{g}, \mathbf{h} : [b, c] \to \mathbf{R}_F^n$ *are n-dimensional FFs, then the granular distance is*

$$\mathcal{D}_{gr}(\mathbf{g}(y), \mathbf{h}(y)) = \sup_{\beta} \max_{\alpha_g, \alpha_h} \|\mathbf{g}_{gr}(y, \beta, \alpha_g) - \mathbf{h}_{gr}(y, \beta, \alpha_h)\|,$$

where $y \in [b, c] \subset \mathbf{R}$ *and* $\beta, \alpha_g, \alpha_h \in [0, 1]$.

Now, we define first order gr-differentiability for n-dimensional FF.

Definition 2.8. *Let* $\mathbf{g} : [b, c] \to \mathbf{R}_F^n$, *be the n-dimensional FF. If there exists* $\frac{d_{gr}\mathbf{g}(y_0)}{dy} \in \mathbf{R}_F^n$, *such that*

$$\lim_{h \to 0} \frac{\mathbf{g}(y_0 + h) - \mathbf{g}(y_0)}{h} = \frac{d_{gr}\mathbf{g}(y_0)}{dy} = \mathbf{g}'_{gr}(y_0),$$

this limit is taken in the metric space $(\mathbf{R}_F^n, \mathcal{D}_{gr}^n)$. *Then* $\mathbf{g}$ *is said to be first order gr- differentiable at a point* $y_0 \in [b, c]$.

Theorem 2.1. *Let* $\mathbf{g} : [b, c] \to \mathbf{R}_F$ *be a n-dimensional FF, then* $\mathbf{g}$ *is gr-differentiable if and only if its HMF is differentiable with respect to* $y \in [b, c]$*. Moreover,*

$$H\left(\frac{d_{gr}\mathbf{g}(y)}{dy}\right) = \frac{\partial \mathbf{g}_{gr}(y, \beta, \alpha_f)}{\partial y}.$$

Proof. Suppose $\mathbf{g}$ is gr-differentiable and $y \in (b, c)$. Based on the Definition 2.8, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $|h| < \delta_1 \implies \mathcal{D}^n{}_{gr}(\frac{g(y+h)-g(y)}{h}, \frac{d_{gr}g(y)}{dy}) < \epsilon_1$

$$\implies \sup_{\beta} \max_{\alpha_g} \| \frac{\mathbf{g}_{gr}(y + h, \beta, \alpha_g) - \mathbf{g}_{gr}(y, \beta, \alpha_g)}{h} - \frac{d_{gr}\mathbf{g}_{gr}(y, \beta, \alpha_g)}{dy} \| < \epsilon_1$$

$$\implies \| \frac{\mathbf{g}_{gr}(y + h, \beta, \alpha_g) - \mathbf{g}_{gr}(y, \beta, \alpha_g)}{h} - \frac{d_{gr}\mathbf{g}_{gr}(y, \beta, \alpha_g)}{dy} \| < \epsilon_1$$

$$\implies \lim_{h\to 0} \frac{\mathbf{g}_{gr}(y + h, \beta, \alpha_g) - \mathbf{g}_{gr}(y, \beta, \alpha_g)}{h} = \frac{d_{gr}\mathbf{g}_{gr}(y, \beta, \alpha_g)}{dy}$$

$$\implies \frac{\partial \mathbf{g}_{gr}(y, \beta, \alpha_g)}{\partial y} = H\left(\frac{d_{gr}\mathbf{g}(y)}{dy}\right).$$

□

Definition 2.9. *Suppose that* $\mathbf{g} : [b, c] \to \mathbf{R}^n_F$*, is continuous and the HMF* $H(\mathbf{g}(y)) = \mathbf{g}_{gr}(y, \beta, \alpha_g)$ *is integrable on* $[b, c]$*. If there exists a* $\mathbf{m}$ *such that* $H(\mathbf{m}) = \int_b^c H(\mathbf{g}(y))\mathrm{d}y$*, then* $\mathbf{m}$ *is called the gr-integral of* $\mathbf{g}$ *on* $[b, c]$ *and* $\mathbf{m} = \int_b^c \mathbf{g}(y)\mathrm{d}y$*.*

Proposition 2.1. *Assume that* $F : [b, c] \to \mathbf{R}^n_F$ *is gr-differentiable and* $\mathbf{g}(y) = \frac{d_{gr}F(y)}{dy}$ *is continuous on* $[b, c]$*. Then,* $\int_b^c \mathbf{g}(y)\mathrm{d}y = F(c) - F(b)$*.*

Theorem 2.2. *Assume that* $\mathbf{g}, \mathbf{h} : [b, c] \to \mathbf{R}^n_F$ *are gr-integrable n-dimensional FFs and* $l, m \in \mathbf{R}$*. Then the following properties hold:*

(i) $\int_b^c [l\mathbf{g}(y) + m\mathbf{h}(y)]\mathrm{d}y = l \int_b^c \mathbf{g}(y)\mathrm{d}y + m \int_b^c \mathbf{h}(y)\mathrm{d}y$*;*

(ii) $\mathcal{D}^n_{gr}(\mathbf{g}, \mathbf{h})$ *is integrable;*

(iii) $\mathcal{D}^n_{gr}\left(\int_b^c \mathbf{g}(y)\mathrm{d}y, \int_b^c \mathbf{h}(y)\mathrm{d}y\right) \le \int_b^c \mathcal{D}^n_{gr}(\mathbf{g}(y), \mathbf{h}(y))\mathrm{d}y$*;*

(iv) $\int_b^c \mathbf{g}(y)\mathrm{d}y = \int_b^a \mathbf{g}(y)\mathrm{d}y + \int_a^c \mathbf{g}(y)\mathrm{d}y$*, for each* $a \in (b, c)$*.*

Proof. (i) Consider

$$\begin{aligned}
H\left(\int_b^c [l\mathbf{g}(y) + m\mathbf{h}(y)]\,\mathrm{d}y\right) &= \int_b^c H\left(l\mathbf{g}(y) + m\mathbf{h}(y)\right)\mathrm{d}y \\
&= H(l)\int_b^c H\left(\mathbf{g}(y)\right)\mathrm{d}y + H(m)\int_b^c H\left(\mathbf{h}(y)\right)\mathrm{d}y \\
&= H(l)H\left(\int_b^c \mathbf{g}(y)\mathrm{d}y\right) + H(m)H\left(\int_b^c \mathbf{h}(y)\mathrm{d}y\right) \\
&= H\left(l\int_b^c \mathbf{g}(y)\mathrm{d}y\right) + H\left(m\int_b^c \mathbf{h}(y)\mathrm{d}y\right) \\
&= H\left(l\int_b^c \mathbf{g}(y)\mathrm{d}y + m\int_b^c \mathbf{h}(y)\mathrm{d}y\right).
\end{aligned}$$

By the Definition 2.5

$$\int_b^c [l\mathbf{g}(y) + m\mathbf{h}(y)]\,\mathrm{d}y = l\int_b^c \mathbf{g}(y)\mathrm{d}y + m\int_b^c \mathbf{h}(y)\mathrm{d}y.$$

(ii) Consider

$$\mathcal{D}_{gr}^n([g(y)]^\beta, [h(y)]^\beta) = \max_{\alpha_g,\alpha_h} \|g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)\|,\ \beta \in [0,1].$$

Since $\mathbf{f}, \mathbf{g}$ are integrable n-dimensional FFs on $[b,c]$, so that $g_{gr}(y,\beta,\alpha_g)$, $h_{gr}(y,\beta,\alpha_h)$ are also integrable on $[b,c]$ or all $\beta, \alpha_g, \alpha_h \in [0,1]$. Let $\beta \in [0,1]$ be fixed. Then, $\mathcal{D}_{gr}^n([g(y)]^\beta, [h(y)]^\beta)$ is measurable on $[b,c]$. By the definition of gr-distance, we have $\mathcal{D}_{gr}^n(f(y), g(y)) = \sup_\beta \mathcal{D}_{gr}^n([g(y)]^\beta, [h(y)]^\beta)$, $\beta \in [0,1]$. Further more, we have

$$\begin{aligned}\mathcal{D}_{gr}^n(g(y), h(y)) &\leq \mathcal{D}_{gr}^n(g(y), 0) + \mathcal{D}_{gr}^n(0, h(y))\\ &\leq g_1(y) + h_1(y),\end{aligned}$$

where g_1, h_1 are integrable bounded functions for g, h respectively.

Thus $\mathcal{D}_{gr}^n(g(y), h(y))$ is integrable.

(iii) Consider,

$$\mathcal{D}_{gr}^n\left(\int_b^c \mathbf{g}(y)\mathrm{d}y, \int_b^c \mathbf{h}(y)\mathrm{d}y\right) = \sup_\beta \max_{\alpha_g,\alpha_h} \|\int_b^c g_{gr}(y,\beta,\alpha_g)\mathrm{d}y - \int_b^c h_{gr}(y,\beta,\alpha_h)\mathrm{d}y\|,$$

for all $\beta, \alpha_g, \alpha_h \in [0,1]$. Since the fact that

$$\begin{aligned}&\|\int_b^c g_{gr}(y,\beta,\alpha_g)\mathrm{d}y - \int_b^c h_{gr}(y,\beta,\alpha_h)\mathrm{d}y\|\\ &= \|\int_b^c [g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)]\,\mathrm{d}y\|\\ &\leq \int_b^c \|[g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)]\|\mathrm{d}y.\end{aligned}$$

It follows that

$$\max_{\alpha_g,\alpha_h} \|\int_b^c [g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)]\,\mathrm{d}y\| \leq \max_{\alpha_g,\alpha_h} \int_b^c \|[g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)]\|\mathrm{d}y.$$

Thus

$$\sup_\beta \max_{\alpha_g,\alpha_h} \|\int_b^c [g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)]\,\mathrm{d}y\| \leq \sup_\beta \max_{\alpha_g,\alpha_h} \int_b^c \|[g_{gr}(y,\beta,\alpha_g) - h_{gr}(y,\beta,\alpha_h)]\|\mathrm{d}y,$$

we get inequality (iii).

The proof of (iv) is deduced directly by the Definition 2.6 □

Definition 2.10. *[13] Let A be a square matrix of order n with real numbers. The exponential of A is represented by the notation* $\exp(Ay)$ *and defined as* $\exp(Ay) = I + Ay + \frac{A^2y^2}{2!} + \frac{A^3y^3}{3!} + \cdots$, *for all $y \in \mathbf{R}$ and the following results hold:*

(i) $\exp(Ay)|_{y=0} = I$.

(ii) $\frac{d_{gr}}{dy}(\exp(Ay)) = A(\exp(Ay)) = (\exp(Ay))A$.

(iii) $(\exp(Ay))^{-1} = (\exp(-Ay))$.

3. Main Results

3.1. The fundamental theorem for SFOLFBVPs.

Theorem 3.1. *Let A be a square matrix of order n with real numbers. Then for a given $Z(x_0), Z(x_1) \in \mathbf{R}^n_F$, the FBVP $Z'_{gr}(x) = AZ(x)$, with $MZ(x_0) + NZ(x_1) = 0$, has a unique trivial solution if $[Me^{Ax_0} + Ne^{Ax_1}]$ is non-singular.*

Proof. Consider the FBVP

$$Z'_{gr}(x) = AZ(x), \tag{3.1}$$

$$\text{with, } MZ(x_0) + NZ(x_1) = 0. \tag{3.2}$$

$$\begin{aligned} & e^{-Ax}Z'_{gr}(x) - e^{-Ax}AZ(x) = 0 \\ \implies & \frac{d_{gr}}{dx}(e^{-Ax}Z(x)) = 0 \\ \implies & e^{-Ax}Z(x) = K,\ K \in \mathbf{R}^n_F \\ \implies & Z(x) = e^{Ax}K. \end{aligned}$$

Now

$$\begin{aligned} & MZ(x_0) + NZ(x_1) = 0 \\ \implies & Me^{Ax_0}K + Ne^{Ax_1}K = 0 \\ \implies & [Me^{Ax_0} + Ne^{Ax_1}]K = 0. \end{aligned}$$

Thus (3.1) and (3.2), has a unique trivial solution if $[Me^{Ax_0}k + Ne^{Ax_1}]$ is non-singular. □

3.2. Non-homogeneous SFOLFBVPs.

Consider the following SFOLFBVP

$$Z'_{gr}(x) = AZ(x) + F(x), \tag{3.3}$$

$$\text{with, } MZ(x_0) + NZ(x_1) = R. \tag{3.4}$$

Theorem 3.2. *The non-homogeneous SFOLFBVPs (3.3) and (3.4), has a unique solution if the corresponding homogeneous system (3.1) and (3.2) has only the trivial solution. If this condition holds then the solution of system (3.3) and (3.4) given by $Z(x) = Z_0(x) + \int_{x_0}^{x_1} G(x,s)F(s)ds$, where $Z_0(x)$ is a solution of the homogeneous system (3.1) and (3.4) and G is the Green's matrix of non homogeneous system (3.3) and (3.4).*

Proof. Let e^{Ax} be the fundamental matrix of the system (3.1) and (3.2). Then the general solution of non homogeneous system (3.3) is,

$$Z(x) = e^{Ax}K + e^{Ax}\int_{x_0}^{x} e^{-As}F(s)ds. \tag{3.5}$$

Using boundary condition (3.4), we have

$$Me^{Ax_0}K + N[e^{Ax_1}K + e^{Ax_1}\int_{x_0}^{x_1} e^{-As}F(s)ds] = R$$

$$\implies [Me^{Ax_0} + Ne^{Ax_1}]K = R - Ne^{Ax_1}\int_{x_0}^{x_1} e^{-As}F(s)ds$$

$$\implies K = D^{-1}R - D^{-1}Ne^{Ax_1}\int_{x_0}^{x_1} e^{-As}F(s)ds, \text{ where } D = [Me^{Ax_0} + Ne^{Ax_1}].$$

$$\begin{aligned} Z(x) &= e^{Ax}[D^{-1}R - D^{-1}Ne^{Ax_1}\int_{x_0}^{x_1} e^{-As}F(s)ds] + e^{Ax}\int_{x_0}^{x} e^{-As}F(s)ds \\ &= e^{Ax}D^{-1}R + \int_{x_0}^{x_1} G(x,s)F(s)ds, \end{aligned}$$

where

$$G(x,s) = \begin{cases} e^{Ax}D^{-1}Me^{Ax_0}e^{-As} & \text{if } x_0 \le s \le x \le x_1 \\ -e^{Ax}D^{-1}Ne^{Ax_1}e^{-As} & \text{if } x_0 \le x \le s \le x_1, \end{cases}$$

which is the Greens matrix of SFOLFBVP (3.1) and (3.2). □

4. An Algorithm for solving system of first-order linear fuzzy boundary value problems under gr-differentiability

Consider a SFOLFBVP,

$$Z'_{gr}(x) = AZ(x) + F(x), \; with \; MZ(x_0) + NZ(x_1) = R. \tag{4.1}$$

The matrix form of (4.1) is,

$$\begin{bmatrix} y'_{gr}(x) \\ z'_{gr}(x) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y(x) \\ z(x) \end{bmatrix} + \begin{bmatrix} \mathbf{f}(x) \\ \mathbf{g}(x) \end{bmatrix}, \tag{4.2}$$

$$\text{subject to, } \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} y(x_0) \\ z(x_0) \end{bmatrix} + \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} y(x_1) \\ z(x_1) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{4.3}$$

The following algorithm describes the procedure to compute β-cut solution of SFOLFBVP (4.1) if it exists.

Step 1 : Applying HMF on both sides of (4.2) and (4.3), we get

$$\begin{bmatrix} \frac{\partial y_{gr}(x,\beta,\alpha_y)}{\partial x} \\ \frac{\partial z_{gr}(x,\beta,\alpha_z)}{\partial x} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_{gr}(x,\beta,\alpha_y) \\ z_{gr}(x,\beta,\alpha_z) \end{bmatrix} + \begin{bmatrix} f_{gr}(x,\beta,\alpha_f) \\ g_{gr}(x,\beta,\alpha_g) \end{bmatrix}, \tag{4.4}$$

$$\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} y_{gr}(x_0,\beta,\alpha_{y_0}) \\ z_{gr}(x_0,\beta,\alpha_{z_0}) \end{bmatrix} + \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} y_{gr}(x_1,\beta,\alpha_{y_1}) \\ z_{gr}(x_1,\beta,\alpha_{z_1}) \end{bmatrix} = \begin{bmatrix} r_1(\beta,\alpha_{r_1}) \\ r_2(\beta,\alpha_{r_2}) \end{bmatrix}, \tag{4.5}$$

where β,α_y, α_z,α_f, α_g, α_{r_1}, α_{r_2},α_{y_0}, α_{z_0}, α_{y_1}, $\alpha_{z_1} \in [0,1]$. Here, (4.4) is a system of partial differential equations with single independent variable x. Therefore, (4.4) and (4.5) taken as a ordinary second order system of differential equations.

Step 2 : Solving (4.4) and (4.5), we get

$$H(y(x)) = y_{gr}(x,\beta,\alpha_y), and \tag{4.6}$$

$$H(z(x)) = z_{gr}(x,\beta,\alpha_z). \tag{4.7}$$

Step 3 : Applying inverse HMF on both sides of (4.6) and (4.7), we get

$$[y(x)]^\beta = [\inf_{\beta\le\alpha\le1} \min_{\alpha_y} y_{gr}(x,\alpha,\alpha_y), \sup_{\beta\le\alpha\le1} \max_{\alpha_y} z_{gr}(x,\alpha,\alpha_y)], \tag{4.8}$$

$$[z(x)]^\beta = [\inf_{\beta\le\alpha\le1} \min_{\alpha_z} z_{gr}(x,\alpha,\alpha_z), \sup_{\beta\le\alpha\le1} \max_{\alpha_z} z_{gr}(x,\alpha,\alpha_z)], \tag{4.9}$$

which is the required β-cut solution of SFOLFBVP (4.1).

Example 4.1. *Consider a homogeneous SFOLFBVP with fuzzy boundary conditions,*

$$y'_{gr}(x) = 3y(x) + 2z(x),$$
$$z'_{gr}(x) = y(x) + 4z(x),$$
with fuzzy boundary values, $y(0) + 2y(1) = r_1$, $z(0) + 3z(1) = r_2$.

Suppose that the β*-level sets of fuzzy boundary values are* $[r_1]^\beta = [1+\beta, 3-\beta]$, $[r_2]^\beta = [2+\beta, 4-\beta]$. *Then the matrix equation is,*

$$\begin{bmatrix} y'_{gr}(x) \\ z'_{gr}(x) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}, \tag{4.10}$$

$$\textit{subject to, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(x_0) \\ z(x_0) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y(x_1) \\ z(x_1) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{4.11}$$

Taking HMF on both sides of (4.10) *and* (4.11), *we have*

$$\begin{bmatrix} \frac{\partial y_{gr}(x,\beta,\alpha_y)}{\partial x} \\ \frac{\partial z_{gr}(x,\beta,\alpha_z)}{\partial x} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_{gr}(x,\beta,\alpha_y) \\ z_{gr}(x,\beta,\alpha_z) \end{bmatrix}, \tag{4.12}$$

$$\text{subject to, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{gr}(x_0) \\ z_{gr}(x_0) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_{gr}(x_1) \\ z_{gr}(x_1) \end{bmatrix} = \begin{bmatrix} r_1(\beta,\alpha_1) \\ r_2(\beta,\alpha_2) \end{bmatrix}, \tag{4.13}$$

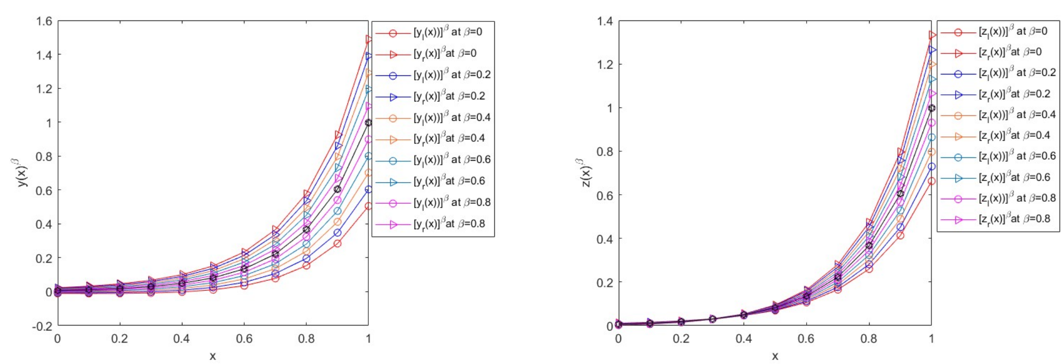
where the granule of boundary values are $r_{1_{gr}}(\beta,\alpha_1) = [1+\beta+2(1-\beta)\alpha_1]$, $r_{2_{gr}}(\beta,\alpha_2) = [2+\beta+2(1-\beta)\alpha_2]$, *where* β, α_1, $\alpha_2 \in [0,1]$.
The solution for system of equations (4.12) *and* (4.13) *is*

$$y_{gr}(x,\beta,\alpha_1,\alpha_2) \text{ and } z_{gr}(x,\beta,\alpha_1,\alpha_2). \tag{4.14}$$

Applying inverse HMF on (4.14), *we get*

$$[y(x)]^\beta = [\inf_{\beta\le\alpha\le 1} \min_{\alpha_1,\alpha_2} y_{gr}(x,\alpha,\alpha_1,\alpha_2), \sup_{\beta\le\alpha\le 1} \max_{\alpha_1,\alpha_2} y_{gr}(x,\alpha,\alpha_1,\alpha_2)],$$
$$[z(x)]^\beta = [\inf_{\beta\le\alpha\le 1} \min_{\alpha_1,\alpha_2} z_{gr}(x,\alpha,\alpha_1,\alpha_2), \sup_{\beta\le\alpha\le 1} \max_{\alpha_1,\alpha_2} z_{gr}(x,\alpha,\alpha_1,\alpha_2)].$$

The β*-cut solution is computed using MATLAB and is depicted in Fig.* 1.



(a) The black curve represents $y(x)$ at $\beta = 1$.

(b) The black curve represents $z(x)$ at $\beta = 1$.

Figure 1. The span of the information granule (β-level sets) of $y(x)$ and $z(x)$.

Example 4.2. *Consider a non-homogeneous SFOLFBVP with fuzzy force functions,*

$$y'_{gr}(x) = 3y(x) + 2z(x) + f(x),$$
$$z'_{gr}(x) = y(x) + 4z(x) + g(x),$$
$$\textit{with boundary values, } y(0) - 2y(1) = r_1 = 2,\ z(0) - 3z(1) = r_2 = 3.$$

Suppose that the β-level sets of fuzzy boundary values are $[f]^\beta = [1+\beta, 3-\beta]$, $[g]^\beta = [2+\beta, 4-\beta]$. *Then the matrix equation is,*

$$\begin{bmatrix} y'_{gr}(x) \\ z'_{gr}(x) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y(x) \\ z(x) \end{bmatrix} + \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \tag{4.15}$$

$$\text{subject to, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(x_0) \\ z(x_0) \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y(x_1) \\ z(x_1) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{4.16}$$

Taking HMF on both sides of (4.15) *and* (4.16), *we have*

$$\begin{bmatrix} \frac{\partial y_{gr}(x,\beta,\alpha_y)}{\partial x} \\ \frac{\partial z_{gr}(x,\beta,\alpha_z)}{\partial x} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_{gr}(x,\beta,\alpha_y) \\ z_{gr}(x,\beta,\alpha_z) \end{bmatrix} + \begin{bmatrix} f_{gr}(x,\beta,\alpha_f) \\ g_{gr}(x,\beta,\alpha_g) \end{bmatrix} \tag{4.17}$$

$$\text{subject to } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{gr}(x_0) \\ z_{gr}(x_0) \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_{gr}(x_1) \\ z_{gr}(x_1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \tag{4.18}$$

where the granule of fuzzy force functions are $f_{gr}(\beta,\alpha_1) = [1+\beta+2(1-\beta)\alpha_1]$, $g_{gr}(\beta,\alpha_2) = [2+\beta+2(1-\beta)\alpha_2]$, *where* β, α_1, $\alpha_2 \in [0,1]$.

The solution for system of equations (4.17) *and* (4.18) *is*

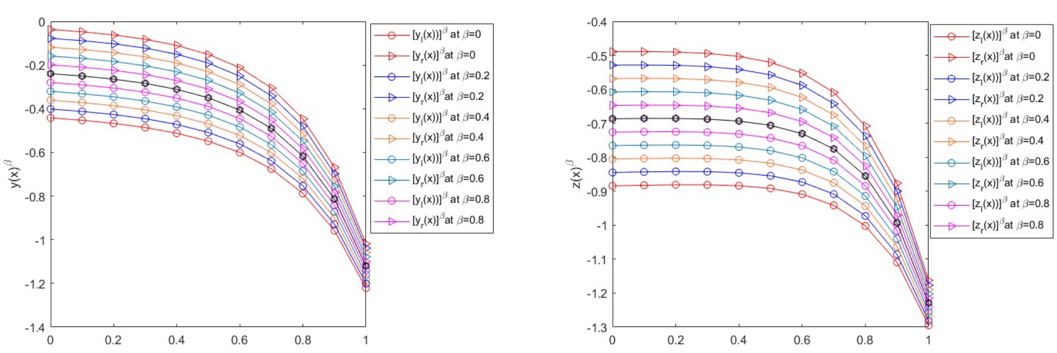
$$y_{gr}(x,\beta,\alpha_1,\alpha_2) \text{ and } z_{gr}(x,\beta,\alpha_1,\alpha_2). \tag{4.19}$$

Applying inverse HMF on (4.19), *we get*

$$[y(x)]^\beta = [\inf_{\beta\le\alpha\le 1} \min_{\alpha_1,\alpha_2} y_{gr}(x,\alpha,\alpha_1,\alpha_2), \sup_{\beta\le\alpha\le 1} \max_{\alpha_1,\alpha_2} y_{gr}(x,\alpha,\alpha_1,\alpha_2)],$$

$$[z(x)]^\beta = [\inf_{\beta\le\alpha\le 1} \min_{\alpha_1,\alpha_2} z_{gr}(x,\alpha,\alpha_1,\alpha_2), \sup_{\beta\le\alpha\le 1} \max_{\alpha_1,\alpha_2} z_{gr}(x,\alpha,\alpha_1,\alpha_2)].$$

The β-cut solution is computed using MATLAB and is depicted in Fig. 2.



(a) The black curve represents $y(x)$ at $\beta = 1$.

(b) The black curve represents $z(x)$ at $\beta = 1$.

Figure 2. The span of the information granule (β-level sets) of $y(x)$ and $z(x)$.

Example 4.3. *Consider a non-homogeneous SFOLFBVP with fuzzy boundary conditions and fuzzy force functions are,*

$$y'_{gr}(x) = 3y(x) + 2z(x) + f(x),$$
$$z'_{gr}(x) = y(x) + 4z(x) + g(x),$$
$$\text{with fuzzy boundary values, } y(0) + 2y(1) = r_1,\ z(0) + 3z(1) = r_2.$$

Suppose that the β-level sets of fuzzy force functions are $[f]^\beta = [1+\beta, 3-\beta]$, $[g]^\beta = [2+\beta, 4-\beta]$, $[r_1]^\beta = [\beta, 2-\beta]$, $[r_2]^\beta = [1+\beta, 3-\beta]$. Then the matrix equation is,

$$\begin{bmatrix} y'_{gr}(x) \\ z'_{gr}(x) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y(x) \\ z(x) \end{bmatrix} + \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \tag{4.20}$$

$$\text{subject to, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(x_0) \\ z(x_0) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y(x_1) \\ z(x_1) \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{4.21}$$

Taking HMF on both sides of (4.20) and (4.21), we have

$$\begin{bmatrix} \frac{\partial y_{gr}(x,\beta,\alpha_y)}{\partial x} \\ \frac{\partial z_{gr}(x,\beta,\alpha_z)}{\partial x} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_{gr}(x,\beta,\alpha_y) \\ z_{gr}(x,\beta,\alpha_z) \end{bmatrix} + \begin{bmatrix} f_{gr}(x,\beta,\alpha_1) \\ g_{gr}(x,\beta,\alpha_2) \end{bmatrix}, \tag{4.22}$$

$$\text{subject to, } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{gr}(x_0) \\ z_{gr}(x_0) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_{gr}(x_1) \\ z_{gr}(x_1) \end{bmatrix} = \begin{bmatrix} r_1(\beta,\alpha_3) \\ r_2(\beta,\alpha_1) \end{bmatrix}, \tag{4.23}$$

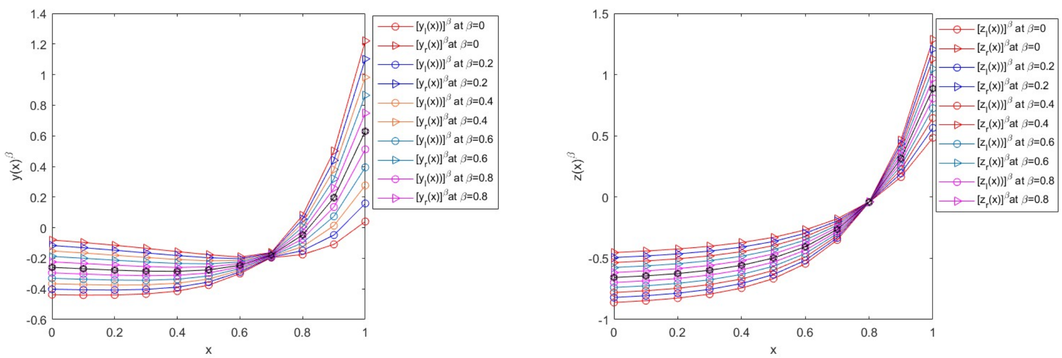
where the granule of fuzzy boundary values and force functions are $f_{gr}(\beta,\alpha_1) = [1+\beta+2(1-\beta)\alpha_1]$, $g_{gr}(\beta,\alpha_2) = [2+\beta+2(1-\beta)\alpha_2]$ $r_{1_{gr}}(\beta,\alpha_3) = [\beta+2(1-\beta)\alpha_3]$, $r_{2_{gr}}(\beta,\alpha_1) = [1+\beta+2(1-\beta)\alpha_1]$, where β, α_1, α_2, $\alpha_3 \in [0,1]$.
The solution for system of equations (4.17) and (4.18) is

$$y_{gr}(x,\beta,\alpha_1,\alpha_2,\alpha_3) \text{ and } z_{gr}(x,\beta,\alpha_1,\alpha_2,\alpha_3). \tag{4.24}$$

Applying inverse HMF on (4.24), we get

$$[y(x)]^\beta = [\inf_{\beta\le\alpha\le1} \min_{\alpha_1,\alpha_2,\alpha_3} y_{gr}(x,\alpha,\alpha_1,\alpha_2,\alpha_3), \sup_{\beta\le\alpha\le1} \max_{\alpha_1,\alpha_2,\alpha_3} y_{gr}(x,\alpha,\alpha_1,\alpha_2,\alpha_3)],$$
$$[z(x)]^\beta = [\inf_{\beta\le\alpha\le1} \min_{\alpha_1,\alpha_2,\alpha_3} z_{gr}(x,\alpha,\alpha_1,\alpha_2,\alpha_3), \sup_{\beta\le\alpha\le1} \max_{\alpha_1,\alpha_2,\alpha_3} z_{gr}(x,\alpha,\alpha_1,\alpha_2,\alpha_3)].$$

The β-cut solution is computed using MATLAB and is depicted in Fig. 3.



(a) The black curve represents $y(x)$ at $\beta = 1$.

(b) The black curve represents $z(x)$ at $\beta = 1$.

Figure 3. The span of the information granule (β-level sets) of $y(x)$ and $z(x)$.

5. Conclusions

The results proposed in this paper are useful for examining and determining solutions for SFOLFBVPs. The granular differentiability and integrability are extended to an n-dimensional fuzzy function. The SFOLFBVPs with fuzzy boundary conditions are researched under granular differentiability. We have established the existence and uniqueness of solutions for homogeneous and non-homogeneous SFOLFBVPs. The proposed algorithm is useful to determine the solution of the first-order FSDEs with fuzzy boundary conditions. We provide various examples to demonstrate the effectiveness and applicability of our method. In the future, this work will be extended for higher-order FSDEs with fuzzy boundary conditions and investigating applications in real-life.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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