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Existence Fixed Point Solutions for C-Class Functions in Bipolar Metric Spaces With Applications

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Abstract. In this study, the idea of C-class functions is introduced in the process of building a bi-polar metric space, along with often coupled fixed point theorems for these mappings in complete bi-polar metric spaces that associate altering distance function and ultra-altering distance function. Furthermore, we provide applications to integral equations as well as homotopy and we give an interpretation that demonstrates the relevance of the results obtained.

1. Introduction

Fixed point theory is a crucial topic of non-linear analysis. Numerous types of equations that exist in natural, biological, social, engineering, and other branches of science and technology are studied in order to understand their underlying relevance. Examining the situations in which single or multi-valued mappings have solutions is a common application of this technique.

Coupled fixed points was originally understood by Guo and Lakshmikantham [1] in 1987. Bhaskar and Lakshmikantham [2] developed a novel fixed point theorem for mixed monotone mapping in a metric space with partial ordering after using a weak contractivity condition. For further information

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on coupled fixed point outcomes, see the study results ([3], [4], [5], [6], [7], [8], [9]) and relevant references.

In 2014, Ansari [10] proposed the idea of C-class functions and the proofs of unique fixed point theorems for specific contractive mappings. This marked the beginning of a significant amount of work in this area (See.([11], [12], [13], [14], [15], [16], [17]).

In addition to providing variant-related (coupled) fixed point solutions for co-variant and contravariant contractive mappings, Muttu and Gurdal [18] recently developed the concept of bi-polar metric spaces. Later, we proved some fixed point theorems in our earlier papers (see. [19], [20], [21], [22], [23], [24]).

The purpose of this article is to propose a coupled common fixed point theorem for a covariant mappings of C-class functions in relation to bi-polar metric spaces. Examples that are appropriate and relevant applications to integral equations along with homotopy are also provided.

What follows is In our subsequent conversations, we compile a few suitable definitions.

2. Preliminaries

Definition 2.1. *([18]) The mapping $d : \mathcal{S} \times \mathcal{T} \to [0, \infty)$ is said to be a Bipolar-metric on pair of non empty sets $(\mathcal{S}, \mathcal{T})$.If*

- (B_1) *$d(\mu, \nu) = 0$ implies that $\mu = \nu$;*
- (B_2) *$\mu = \nu$ implies that $d(\mu, \nu) = 0$;*
- (B_3) *if $(\mu, \nu) \in (\mathcal{S}, \mathcal{T})$, then $d(\mu, \nu) = d(\nu, \mu)$;*
- (B_4) *$d(\mu_1, \nu_2) \leq d(\mu_1, \nu_1) + d(\mu_2, \nu_1) + d(\mu_2, \nu_2)$,*

for all $\mu, \mu_1, \mu_2 \in \mathcal{S}$ and $\nu, \nu_1, \nu_2 \in \mathcal{T}$, and the triple $(\mathcal{S}, \mathcal{T}, d)$ is called a Bipolar-metric space.

Example 2.1. *([18]) Let $d : \mathcal{S} \times \mathcal{T} \to [0, +\infty)$ be defined as $d(\psi, a) = \psi(a)$, for all $(\psi, a) \in (\mathcal{S}, \mathcal{T})$ where $\mathcal{S} = \{\psi/\psi : \mathbb{R} \to [1, 3]\}$ be the set of all functions and $\mathcal{T} = \mathbb{R}$. Then the triple $(\mathcal{S}, \mathcal{T}, d)$ is a disjoint Bipolar-metric space.*

Definition 2.2. *([18]) Let $\Omega : \mathcal{S}_1 \cup \mathcal{T}_1 \to \mathcal{S}_2 \cup \mathcal{T}_2$ be a function defined on two pairs of sets $(\mathcal{S}_1, \mathcal{T}_1)$ and $(\mathcal{S}_2, \mathcal{T}_2)$ is said to be*

- (i) *covariant if $\Omega(\mathcal{S}_1) \subseteq \mathcal{S}_2$ and $\Omega(\mathcal{T}_1) \subseteq \mathcal{T}_2$. This is denoted as*
  $\Omega : (\mathcal{S}_1, \mathcal{T}_1) \rightrightarrows (\mathcal{S}_2, \mathcal{T}_2)$;
- (ii) *contravariant if $\Omega(\mathcal{S}_1) \subseteq \mathcal{T}_2$ and $\Omega(\mathcal{T}_1) \subseteq \mathcal{S}_2$. It is denoted as*
  $\Omega : (\mathcal{S}_1, \mathcal{T}_1) \leftrightarrows (\mathcal{S}_2, \mathcal{T}_2)$.

Particularly, if d_1 is bipolar metrics on $(\mathcal{S}_1, \mathcal{T}_1)$ and d_2 is bipolar metrics on $(\mathcal{S}_2, \mathcal{T}_2)$, we often write
$\Omega : (\mathcal{S}_1, \mathcal{T}_1, d_1) \rightrightarrows (\mathcal{S}_2, \mathcal{T}_2, d_2)$ *and*
$\Omega : (\mathcal{S}_1, \mathcal{T}_1, d_1) \leftrightarrows (\mathcal{S}_2, \mathcal{T}_2, d_2)$ *respectively.*

Definition 2.3. *([18]) In a bipolar metric space $(\mathcal{S}, \mathcal{T}, d)$ for any $\xi \in \mathcal{S} \cup \mathcal{T}$ is left point if $\xi \in \mathcal{S}$, is right point if $\xi \in \mathcal{T}$ and is central point if $\xi \in \mathcal{S} \cap \mathcal{T}$.*

Also, $\{\alpha_i\}$ in $\mathcal{S}$ and $\{\beta_i\}$ in $\mathcal{T}$ are left and right sequence respectively. In a bipolar metric space, we call a sequence, a left or a right one. A sequence $\{\xi_i\}$ is said to be convergent to ξ iff either $\{\xi_i\}$ is a left sequence, ξ is a right point and $\lim_{i\to\infty} d(\xi_i, \xi) = 0$, or $\{\xi_i\}$ is a right sequence, ξ is a left point and $\lim_{i\to\infty} d(\xi, \xi_i) = 0$. The bisequence $(\{\alpha_i\}, \{\beta_i\})$ on $(\mathcal{S}, \mathcal{T}, d)$ is a sequence on $\mathcal{S} \times \mathcal{T}$. In the case where $\{\alpha_i\}$ and $\{\beta_i\}$ are both convergent, then $(\{\alpha_i\}, \{\beta_i\})$ is convergent.
The bi-sequence $(\{\alpha_i\}, \{\beta_i\})$ is a Cauchy bisequence if $\lim_{i,j\to\infty} d(\alpha_i, \beta_j) = 0$.

Note that every convergent Cauchy bisequence is biconvergent. The bipolar metric space is complete, if each Cauchy bisequence is convergent (and so it is biconvergent).

Definition 2.4. *([22]) Let $(\mathcal{S}, \mathcal{T}, d)$ be a bipolar metric space and a pair $(\wp, \varpi)$ is called*

- (a) *coupled fixed point of covariant mapping $\Omega : (\mathcal{S}^2, \mathcal{T}^2) \rightrightarrows (\mathcal{S}, \mathcal{T})$ if $\Omega(\wp, \varpi) = \wp$, $\Omega(\varpi, \wp) = \varpi$ for $(\wp, \varpi) \in \mathcal{S}^2 \cup \mathcal{T}^2$;*
- (b) *coupled coincident point of $\Omega : (\mathcal{S}^2, \mathcal{T}^2) \rightrightarrows (\mathcal{S}, \mathcal{T})$ and $\Lambda : (\mathcal{S}, \mathcal{T}) \rightrightarrows (\mathcal{S}, \mathcal{T})$ if $F(\wp, \varpi) = \Lambda\wp$, $\Omega(\varpi, \wp) = \Lambda\varpi$;*
- (c) *coupled common point of $\Omega : (\mathcal{S}^2, \mathcal{T}^2) \rightrightarrows (\mathcal{S}, \mathcal{T})$ and $\Lambda : (\mathcal{S}, \mathcal{T}) \rightrightarrows (\mathcal{S}, \mathcal{T})$ if $\Omega(\wp, \varpi) = \Lambda\wp = \wp$, $\Omega(\varpi, \wp) = \Lambda\varpi = \varpi$;*
- (d) *the pair (Ω, Λ) is weakly compatible if $\Lambda(\Omega(\wp, \varpi)) = \Omega(\Lambda\wp, \Lambda\varpi)$ and $\Lambda(\Omega(\varpi, \wp)) = \Omega(\Lambda\varpi, \Lambda\wp)$ whenever $\Omega(\wp, \varpi) = \Lambda\wp$, $\Omega(\varpi, \wp) = \Lambda\varpi$*

Definition 2.5. *([10]) Let $C = \{\Delta/\Delta : [0, +\infty) \times [0. +\infty) \to R\}$ be a family of continuous functions is called a C-class function if for all $s^*, t^* \in [0, \infty)$,*

- (a) $\Delta(s^*, t^*) \leq s^\star$;
- (b) $\Delta(s^*, t^*) = s^* \Rightarrow s^* = 0$ *or* $t^* = 0$.

Example 2.2. *([10]) Each of the functions $\Delta : [0, +\infty) \times [0. +\infty) \to R$ defined below are elements of C.*

- (a) $\Delta(s^*, t^*) = s^\star - t^\star$;
- (b) $\Delta(s^*, t^*) = ms^*$ *where* $m \in (0, 1)$.
- (c) $\Delta(s^*, t^*) = \frac{s^*}{(1+t^\star)^r}$ *where* $r \in (0, \infty)$.
- (d) $\Delta(s^*, t^*) = s^\star \eta(s^\star)$ *where* $\eta : [0, \infty) \to [0, \infty)$ *is continuous function.*
- (e) $\Delta(s^*, t^*) = s^\star - \varphi(s^\star)$ *for all* $s^*, t^* \in [0, +\infty)$ *where, the continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *such that* $\varphi(s^\star) = 0 \Leftrightarrow s^\star = 0$.
- (f) $\Delta(s^*, t^*) = s\Omega(s^\star, t^\star)$ *for all* $s^*, t^* \in [0, +\infty)$ *where, the continuous function* $\Omega : [0, \infty)^2 \to [0, \infty)$ *such that* $\Omega(s^\star, t^\star) < 1$.

Khan et al. [25] and A. H. Ansari et al. [11] both addressed a new category of contractive fixed point outcomes. The idea of an altering distance function and ultra altering distance functions, which are control functions that vary the distance between two locations in a metric space, were introduced in their work.

We say $\mathfrak{F} = \{\psi_\star/\psi_\star : [0,\infty) \to [0,\infty)\}$ and $\mathfrak{G} = \{\phi_\star/\phi_\star : [0,\infty) \to [0,\infty)\}$ be the class of all altering distance and ultra altering distance functions satisfying the following condition:

(ψ_0) $\psi_\star$ is nondecreasing and continuous;
(ψ_1) $\psi_\star(t) = 0$ if and only if $t = 0$.
(ψ_2) $\psi_\star(t)$ is subadditivity, $\psi_\star(a+b) \leq \psi_\star(a) + \psi_\star(b)$;
(ϕ_0) $\phi_\star$ is continuous;
(ϕ_1) $\phi_\star(t) > 0,\ t > 0$ and $\phi_\star(0) \geq 0$.

3. Main Results

In this section, two covariant mappings that meet new type contractive criteria in bipolar metric spaces are given some common coupled fixed point theorems via C-class functions.

Theorem 3.1. *Let $(\mathcal{S}, \mathcal{T}, d)$ be a complete bipolar metric space. Suppose that $\Gamma : \left(\mathcal{S}^2, \mathcal{T}^2\right) \rightrightarrows (\mathcal{S}, \mathcal{T})$ and $\Lambda : (\mathcal{S}, \mathcal{T}) \rightrightarrows (\mathcal{S}, \mathcal{T})$ be two covariant mappings satiesfies*

$$\psi_\star\left(d(\Gamma(u,v), \Gamma(p,q))\right) \leq \Delta\left(\psi_\star\left(M(u,v,p,q)\right), \phi_\star\left(M(u,v,p,q)\right)\right) \tag{3.1}$$

where, $M(u,v,p,q) = \ell \max\left\{\ d\left(\Lambda u, \Lambda p\right), d\left(\Lambda v, \Lambda q\right)\ \right\}$ for all $u, v \in \mathcal{S}$ and $p, q \in \mathcal{T}$ and $\Delta \in C$, $\psi_\star \in \mathfrak{F}$, $\phi_\star \in \mathfrak{G}$ with $\ell \in (0,1)$

(ξ_0) $\Gamma(\mathcal{S}^2 \cup \mathcal{T}^2) \subseteq \Lambda(\mathcal{S} \cup \mathcal{T})$ and $\Lambda(\mathcal{S} \cup \mathcal{T})$ is a complete subspace of $\mathcal{S} \cup \mathcal{T}$,
(ξ_1) pair (Γ, Λ) is ω-compatible.

Then there is a unique common coupled fixed point of Γ and Λ in $\mathcal{S} \cup \mathcal{T}$.

Proof. Let $x_0, y_0 \in \mathcal{S}$ and $p_0, q_0 \in \mathcal{T}$ be arbitrary, and from (ξ_0), we construct the bisequences $(\{\alpha_\kappa\}, \{\zeta_\kappa\}), (\{\beta_\kappa\}, \{\eta_\kappa\})$ in $(\mathcal{S}, \mathcal{T})$ as

$$\begin{aligned} \Gamma\left(x_\kappa, y_\kappa\right) &= \Lambda x_{\kappa+1} = \alpha_\kappa, \quad \Gamma\left(p_\kappa, q_\kappa\right) = \Lambda p_{\kappa+1} = \zeta_\kappa \\ \Gamma\left(y_\kappa, y_\kappa\right) &= \Lambda y_{\kappa+1} = \beta_\kappa, \quad \Gamma\left(q_\kappa, p_\kappa\right) = \Lambda q_{\kappa+1} = \eta_\kappa \end{aligned}$$

where $\kappa = 0, 1, 2, \ldots$.

Then from (3.1), we can get

$$\begin{aligned} \psi_\star\left(d(\alpha_\kappa, \zeta_{\kappa+1})\right) &= \psi_\star\left(d(\Gamma\left(x_\kappa, y_\kappa\right), \Gamma\left(p_{\kappa+1}, q_{\kappa+1}\right))\right) \\ &\leq \Delta\left(\psi_\star\left(M(x_\kappa, y_\kappa, p_{\kappa+1}, q_{\kappa+1})\right), \phi_\star\left(M(x_\kappa, y_\kappa, p_{\kappa+1}, q_{\kappa+1})\right)\right) \end{aligned} \tag{3.2}$$

where,

$$\begin{aligned} M(x_\kappa, y_\kappa, p_{\kappa+1}, q_{\kappa+1}) &= \ell \max\left\{ d(\Lambda x_\kappa, \Lambda p_{\kappa+1}), d(\Lambda y_\kappa, \Lambda q_{\kappa+1}) \right\} \\ &= \ell \max\left\{ d(\alpha_{\kappa-1}, \zeta_\kappa), d(\beta_{\kappa-1}, \eta_\kappa) \right\} \end{aligned}$$

From (3.2), deduce that

$$\begin{aligned} \psi_\star(d(\alpha_\kappa, \zeta_{\kappa+1})) &\leq \Delta\left(\psi_\star\left(\ell \max\left\{ \begin{array}{c} d(\alpha_{\kappa-1}, \zeta_\kappa), \\ d(\beta_{\kappa-1}, \eta_\kappa) \end{array} \right\}\right), \phi_\star\left(\ell \max\left\{ \begin{array}{c} d(\alpha_{\kappa-1}, \zeta_\kappa), \\ d(\beta_{\kappa-1}, \eta_\kappa) \end{array} \right\}\right)\right) \\ &\leq \psi_\star\left(\ell \max\left\{ d(\alpha_{\kappa-1}, \zeta_\kappa), d(\beta_{\kappa-1}, \eta_\kappa) \right\}\right) \end{aligned}$$

By using (ψ_0), we have

$$d(\alpha_\kappa, \zeta_{\kappa+1}) \leq \ell \max\left\{ d(\alpha_{\kappa-1}, \zeta_\kappa), d(\beta_{\kappa-1}, \eta_\kappa) \right\} \tag{3.3}$$

Similarly, we can prove

$$d(\beta_\kappa, \eta_{\kappa+1}) \leq \ell \max\left\{ d(\alpha_{\kappa-1}, \zeta_\kappa), d(\beta_{\kappa-1}, \eta_\kappa) \right\} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\begin{aligned} \max\left\{ d(\alpha_\kappa, \zeta_{\kappa+1}), d(\beta_\kappa, \eta_{\kappa+1}) \right\} &\leq \ell \max\left\{ d(\alpha_{\kappa-1}, \zeta_\kappa), d(\beta_{\kappa-1}, \eta_\kappa) \right\} \\ &\leq \ell^2 \max\left\{ d(\alpha_{\kappa-2}, \zeta_{\kappa-1}), d(\beta_{\kappa-2}, \eta_{\kappa-1}) \right\} \\ &\vdots \\ &\leq \ell^\kappa \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1) \right\} \to 0 \text{ as } \kappa \to \infty. \end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned} \psi_\star(d(\alpha_{\kappa+1}, \zeta_\kappa)) &= \psi_\star(d(\Gamma(x_{\kappa+1}, y_{\kappa+1}), \Gamma(p_\kappa, q_\kappa))) \\ &\leq \Delta(\psi_\star(M(x_{\kappa+1}, y_{\kappa+1}, p_\kappa, q_\kappa)), \phi_\star(M(x_{\kappa+1}, y_{\kappa+1}, p_\kappa, q_\kappa))) \\ &\leq \psi_\star\left(\ell \max\left\{ d(\alpha_\kappa, \zeta_{\kappa-1}), d(\beta_\kappa, \eta_{\kappa-1}) \right\}\right) \end{aligned}$$

By using (ψ_0), we have

$$d(\alpha_{\kappa+1}, \zeta_\kappa) \leq \ell \max\left\{ d(\alpha_\kappa, \zeta_{\kappa-1}), d(\beta_\kappa, \eta_{\kappa-1}) \right\} \tag{3.6}$$

Because of

$$\begin{aligned} M(x_{\kappa+1}, y_{\kappa+1}, p_\kappa, q_\kappa) &= \ell \max\left\{ d(\Lambda x_{\kappa+1}, \Lambda p_\kappa), d(\Lambda y_{\kappa+1}, \Lambda q_\kappa) \right\} \\ &= \ell \max\left\{ d(\alpha_\kappa, \zeta_{\kappa-1}), d(\beta_\kappa, \eta_{\kappa-1}) \right\} \end{aligned}$$

Similarly, we can prove

$$d\left(\beta_{\kappa+1},\eta_{\kappa}\right) \leq \ell \max\left\{ d\left(\alpha_{\kappa},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa},\eta_{\kappa-1}\right) \right\} \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned} \max\left\{ d\left(\alpha_{\kappa+1},\zeta_{\kappa}\right), d\left(\beta_{\kappa+1},\eta_{\kappa}\right) \right\} &\leq \ell \max\left\{ d\left(\alpha_{\kappa},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa},\eta_{\kappa-1}\right) \right\} \\ &\leq \ell^{2} \max\left\{ d\left(\alpha_{\kappa-1},\zeta_{\kappa-2}\right), d\left(\beta_{\kappa-1},\eta_{\kappa-2}\right) \right\} \\ &\vdots \\ &\leq \ell^{\kappa} \max\left\{ d\left(\alpha_{1},\zeta_{0}\right), d\left(\beta_{1},\eta_{0}\right) \right\} \to 0 \text{ as } \kappa \to \infty. \end{aligned} \tag{3.8}$$

Moreover,

$$\begin{aligned} \psi_{\star}\left(d(\alpha_{\kappa},\zeta_{\kappa})\right) &= \psi_{\star}\left(d(\Gamma\left(x_{\kappa},y_{\kappa}\right),\Gamma\left(p_{\kappa},q_{\kappa}\right))\right) \\ &\leq \Delta\left(\psi_{\star}\left(M(x_{\kappa},y_{\kappa},p_{\kappa},q_{\kappa})\right),\phi_{\star}\left(M(x_{\kappa},y_{\kappa},p_{\kappa},q_{\kappa})\right)\right) \\ &\leq \psi_{\star}\left(\ell \max\left\{ d\left(\alpha_{\kappa-1},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa-1},\eta_{\kappa-1}\right) \right\}\right) \end{aligned}$$

By using (ψ_0), we have

$$d(\alpha_{\kappa},\zeta_{\kappa}) \leq \ell \max\left\{ d\left(\alpha_{\kappa-1},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa-1},\eta_{\kappa-1}\right) \right\} \tag{3.9}$$

Because of

$$\begin{aligned} M(x_{\kappa},y_{\kappa},p_{\kappa},q_{\kappa}) &= \ell \max\left\{ d\left(\Lambda x_{\kappa},\Lambda p_{\kappa}\right), d\left(\Lambda y_{\kappa},\Lambda q_{\kappa}\right) \right\} \\ &= \ell \max\left\{ d\left(\alpha_{\kappa-1},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa-1},\eta_{\kappa-1}\right) \right\} \end{aligned}$$

Similarly, we can prove

$$d\left(\beta_{\kappa},\eta_{\kappa}\right) \leq \ell \max\left\{ d\left(\alpha_{\kappa-1},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa-1},\eta_{\kappa-1}\right) \right\} \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\begin{aligned} \max\left\{ d\left(\alpha_{\kappa},\zeta_{\kappa}\right), d\left(\beta_{\kappa},\eta_{\kappa}\right) \right\} &\leq \ell \max\left\{ d\left(\alpha_{\kappa-1},\zeta_{\kappa-1}\right), d\left(\beta_{\kappa-1},\eta_{\kappa-1}\right) \right\} \\ &\leq \ell^{2} \max\left\{ d\left(\alpha_{\kappa-2},\zeta_{\kappa-2}\right), d\left(\beta_{\kappa-2},\eta_{\kappa-2}\right) \right\} \\ &\vdots \\ &\leq \ell^{\kappa} \max\left\{ d\left(\alpha_{0},\zeta_{0}\right), d\left(\beta_{0},\eta_{0}\right) \right\} \to 0 \text{ as } \kappa \to \infty. \end{aligned} \tag{3.11}$$

For each $\kappa, \delta \in \mathbf{N}$ with $\kappa < \delta$. Then, from (3.5), (3.8), (3.11) and using property (B_4), we have

$$\begin{aligned} d(\alpha_\kappa, \zeta_\delta) + d(\beta_\kappa, \eta_\delta) &\leq (d(\alpha_\kappa, \zeta_{\kappa+1}) + d(\beta_\kappa, \eta_{\kappa+1})) \\ &\qquad + (d(\alpha_{\kappa+1}, \zeta_{\kappa+1}) + d(\beta_{\kappa+1}, \eta_{\kappa+1})) \\ &\quad + \cdots + (d(\alpha_{\delta-1}, \zeta_{\delta-1}) + d(\beta_{\delta-1}, \eta_{\delta-1})) \\ &\quad + (d(\alpha_{\delta-1}, \zeta_\delta) + d(\beta_{\delta-1}, \eta_\delta)) \\ &\leq 2\left(\ell^\kappa + \ell^{\kappa+1} + \cdots + \ell^{\delta-1}\right) \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1) \right\} \\ &\qquad + 2\left(\ell^{\kappa+1} + \ell^{\kappa+2} + \cdots + \ell^{\delta-1}\right) \max\left\{ d(\alpha_0, \zeta_0), d(\beta_0, \eta_0) \right\} \\ &\leq \frac{2\ell^\kappa}{1-\ell} \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1) \right\} \\ &\quad + \frac{2\ell^{\kappa+1}}{1-\ell} \max\left\{ d(\alpha_0, \zeta_1), d(\beta_0, \eta_1) \right\} \to 0 \text{ as } \kappa \to \infty. \end{aligned}$$

Similarly, we can prove that $(d(\alpha_\delta, \zeta_\kappa) + d(\beta_\delta, \eta_\kappa)) \to 0$ as $\kappa, \delta \to \infty$. Then the bisequence $(\alpha_\kappa, \zeta_\delta)$ and $(\beta_\kappa, \eta_\delta)$ are Cauchy bisequences in $(\mathcal{S}, \mathcal{T})$. Suppose $\Lambda(\mathcal{S} \cup \mathcal{T})$ is complete subspace of $(\mathcal{S}, \mathcal{T}, d)$, then the sequences $\{\alpha_\kappa\}, \{\beta_\kappa\}$ and $\{\zeta_\kappa\}, \{\eta_\kappa\} \subseteq f(\mathcal{S} \cup \mathcal{T})$ are convergence in complete bipolar metric spaces $(\Lambda(\mathcal{S}), \Lambda(\mathcal{T}), d)$. Therefore, there exist $a, b \in \Lambda(\mathcal{S})$ and $l, m \in \Lambda(\mathcal{T})$ such that

$$\lim_{\kappa\to\infty} \alpha_\kappa = l \quad \lim_{\kappa\to\infty} \beta_\kappa = m \quad \lim_{\kappa\to\infty} \zeta_\kappa = a \quad \lim_{\kappa\to\infty} \eta_\kappa = b. \tag{3.12}$$

Since $\Lambda : \mathcal{S} \cup \mathcal{T} \to \mathcal{S} \cup \mathcal{T}$ and $a, b \in \Lambda(\mathcal{S})$ and $l, m \in \Lambda(\mathcal{T})$, there exist $x, y \in \mathcal{S}$ and $p, q \in \mathcal{T}$ such that $\Lambda x = a, \Lambda y = b$ and $\Lambda p = l, \Lambda q = m$. Hence

$$\lim_{\kappa\to\infty} \alpha_\kappa = l = \Lambda p \quad \lim_{\kappa\to\infty} \beta_\kappa = m = \Lambda q \quad \lim_{\kappa\to\infty} \zeta_\kappa = a = \Lambda x \quad \lim_{\kappa\to\infty} \eta_\kappa = b = \Lambda y.$$

Claim that $\Gamma(x, y) = l, \Gamma(y, x) = m$ and $\Gamma(p, q) = a, \Gamma(q, p) = b$.
By using (3.1), (B_4), (ψ_1) and (ψ_2), we have

$$\begin{aligned} \psi_\star(d(\Gamma(x, y), l)) &\leq \psi_\star(d(\Gamma(x, y), \zeta_{\kappa+1})) + \psi_\star(d(\alpha_{\kappa+1}, \zeta_{\kappa+1})) + \psi_\star(d(\alpha_{\kappa+1}, l)) \\ &\leq \psi_\star(d(\Gamma(x, y), \Gamma(p_{\kappa+1}, q_{\kappa+1}))) + \psi_\star(d(\alpha_{\kappa+1}, \zeta_{\kappa+1})) + \psi_\star(d(\alpha_{\kappa+1}, l)) \\ &\leq \Delta(\psi_\star(M(x, y, p_{\kappa+1}, q_{\kappa+1})), \phi_\star(M(x, y, p_{\kappa+1}, q_{\kappa+1}))) \\ &\qquad + \psi_\star(d(\alpha_{\kappa+1}, \zeta_{\kappa+1})) + \psi_\star(d(\alpha_{\kappa+1}, l)) \\ &\leq \psi_\star\left(\ell \max\left\{ d(\Lambda x, \zeta_\kappa), d(\Lambda y, \eta_\kappa) \right\}\right) \\ &\qquad + \psi_\star(d(\alpha_{\kappa+1}, \zeta_{\kappa+1})) + \psi_\star(d(\alpha_{\kappa+1}, l)) \to 0 \text{ as } \kappa \to \infty. \end{aligned}$$

It follows that $\psi_\star(d(\Gamma(x, y), l)) = 0$ implies that $d(\Gamma(x, y), l) = 0$, which deduce that $\Gamma(x, y) = l$. Similarly, we can prove that $\Gamma(y, x) = m$ and $\Gamma(p, q) = a$, $\Gamma(q, p) = b$. Therefore, it follows that $\Gamma(x, y) = l = \Lambda p, \Gamma(y, x) = m = \Lambda q$ and $\Gamma(p, q) = a = \Lambda x, \Gamma(q, p) = b = \Lambda y$.

Since $\{\Gamma, \Lambda\}$ is ω-compatible pair, we have $\Gamma(l, m) = \Lambda l$, $\Gamma(m, l) = \Lambda m$ and $\Gamma(a, b) = \Lambda a$, $\Gamma(b, a) = \Lambda b$. Now we prove that $\Lambda l = l$, $\Lambda m = m$ and $\Lambda a = a$, $\Lambda b = b$. Now we have

$$\begin{aligned}\psi_\star\left(d(\Lambda a, \zeta_\kappa)\right) &\leq \psi_\star\left(d(\Gamma(a, b), \Gamma(p_\kappa, q_\kappa))\right)\\ &\leq \Delta\left(\psi_\star\left(M(a, b, p_\kappa, q_\kappa)\right), \phi_\star\left(M(a, b, p_\kappa, q_\kappa)\right)\right)\\ &\leq \psi_\star\left(\ell \max\left\{ d\left(\Lambda a, \zeta_{\kappa-1}\right), d\left(\Lambda b, \eta_{\kappa-1}\right) \right\}\right)\end{aligned}$$

By using (ψ_0), we have

$$d(\Lambda a, \zeta_\kappa) \leq \ell \max\left\{ d\left(\Lambda a, \zeta_{\kappa-1}\right), d\left(\Lambda b, \eta_{\kappa-1}\right) \right\}$$

Letting $\kappa \to \infty$, we have

$$d(\Lambda a, a) \leq \ell \max\left\{ d\left(\Lambda a, a\right), d\left(\Lambda b, b\right) \right\}$$

and

$$\begin{aligned}\psi_\star\left(d(\Lambda b, \eta_\kappa)\right) &\leq \psi_\star\left(d(\Gamma(b, a), \Gamma(q_\kappa, p_\kappa))\right)\\ &\leq \Delta\left(\psi_\star\left(M(b, a, q_\kappa, p_\kappa)\right), \phi_\star\left(M(b, a, q_\kappa, p_\kappa)\right)\right)\\ &\leq \psi_\star\left(\ell \max\left\{ d\left(\Lambda b, \eta_{\kappa-1}\right), d\left(\Lambda a, \zeta_{\kappa-1}\right) \right\}\right)\end{aligned}$$

By using (ψ_0), we have

$$d(\Lambda b, \eta_\kappa) \leq \ell \max\left\{ d\left(\Lambda b, \eta_{\kappa-1}, d\left(\Lambda a, \zeta_{\kappa-1}\right)\right) \right\}$$

Letting $\kappa \to \infty$, we have

$$d(\Lambda b, b) \leq \ell \max\left\{ d\left(\Lambda b, b\right), d\left(\Lambda a, a\right) \right\}$$

Therefore,

$$\max\left\{ d\left(\Lambda a, a\right), d\left(\Lambda b, b\right) \right\} \leq \ell \max\left\{ d\left(\Lambda a, a\right), d\left(\Lambda b, b\right) \right\}$$

which implies that $d\left(\Lambda a, a\right) = 0$ and $d\left(\Lambda b, b\right) = 0$ and hence $\Lambda a = a$ and $\Lambda b = b$. Therefore, $\Gamma(a, b) = \Lambda a = a$, $\Gamma(b, a) = \Lambda b = b$.
Similarly, we can prove $\Gamma(l, m) = \Lambda l = l$, $\Gamma(m, l) = \Lambda m = m$. Therefore,

$$\Gamma(p, q) = \Lambda x = a = \Lambda a = \Gamma(a, b) \quad \Gamma(x, y) = \Lambda p = l = \Lambda l = \Gamma(l, m)$$

$$\Gamma(q, p) = \Lambda y = b = \Lambda b = \Gamma(b, a) \quad \Gamma(y, x) = \Lambda q = m = \Lambda m = \Gamma(m, l)$$

On the other hand, from (3.12), we get

$$d\left(\Lambda p, \Lambda x\right) = d(l, a) = d\left(\lim_{\kappa\to\infty} \alpha_\kappa, \lim_{\kappa\to\infty} \zeta_\kappa\right) = \lim_{\kappa\to\infty} d(\alpha_\kappa, \zeta_\kappa) = 0$$

and

$$d\left(\Lambda q, \Lambda y\right) = d(m, b) = d\left(\lim_{\kappa\to\infty} \beta_\kappa, \lim_{\kappa\to\infty} \eta_\kappa\right) = \lim_{\kappa\to\infty} d(\beta_\kappa, \eta_\kappa) = 0.$$

Thus $a = l, b = m$. Therefore, $(a, b) \in \mathcal{S}^2 \cap \mathcal{T}^2$ is a common coupled fixed point of Γ and Λ. In the following we will show the uniqueness. Assume that there is another coupled fixed point (a', b') of Γ, Λ. Then from (3.1), we have

$$\begin{aligned}
\psi_\star\left(d(a, a')\right) &= \psi_\star\left(d(\Gamma(a, b), \Gamma(a', b')\right) \\
&\leq \Delta\left(\psi_\star\left(M(a, b, a', b')\right), \phi_\star\left(M(a, b, a', b')\right)\right) \\
&\leq \psi_\star\left(\ell \max\left\{ d\left(\Lambda a, \Lambda a'\right), d\left(\Lambda b, \Lambda b'\right) \right\}\right) \\
&\leq \psi_\star\left(\ell \max\left\{ d\left(a, a'\right), d\left(b, b'\right) \right\}\right)
\end{aligned}$$

by the property of (ψ_0), we have

$$d(a, a') \leq \ell \max\left\{ d\left(a, a'\right), d\left(b, b'\right) \right\}$$

Therefore, we have

$$\max\left\{ d\left(a, a'\right), d\left(b, b'\right) \right\} \leq \ell \max\left\{ d\left(a, a'\right), d\left(b, b'\right) \right\}$$

hence, we get $a = a', b = b'$. Therefore, (a, b) is a unique common coupled fixed point of Γ and Λ. Finally we will show $a = b$.

$$\begin{aligned}
\psi_\star\left(d(a, b)\right) &= \psi_\star\left(d(\Gamma(a, b), \Gamma(b, a)\right) \\
&\leq \Delta\left(\psi_\star\left(M(a, b, b, a)\right), \phi_\star\left(M(a, b, b, a)\right)\right) \\
&\leq \psi_\star\left(\ell \max\left\{ d\left(\Lambda a, \Lambda b\right), d\left(\Lambda b, \Lambda a\right) \right\}\right) \\
&\leq \psi_\star\left(\ell \max\left\{ d\left(a, b\right), d\left(b, a\right) \right\}\right)
\end{aligned}$$

by the property of (ψ_0), we have

$$d(a, b) \leq \ell \max\left\{ d\left(a, b\right), d\left(b, a\right) \right\}$$

Therefore, we have

$$\max\left\{ d\left(a, b\right), d\left(b, a\right) \right\} \leq \ell \max\left\{ d\left(a, b\right), d\left(b, a\right) \right\}$$

hence, we get $a = b$. Which means that Γ and Λ have a unique common fixed point of the form (a, a). □

Corollary 3.1. *Let $(\mathcal{S}, \mathcal{T}, d)$ be a complete bipolar metric space. Suppose that $\Gamma : \left(\mathcal{S}^2, \mathcal{T}^2\right) \rightrightarrows (\mathcal{S}, \mathcal{T})$ be a covariant mapping satisfy*

$$\psi_\star\left(d(\Gamma(u, v), \Gamma(p, q))\right) \leq \Delta\left(\psi_\star\left(\ell \max\left\{ \begin{array}{c} d\left(u, p\right), \\ d\left(v, q\right) \end{array} \right\}\right), \phi_\star\left(\ell \max\left\{ \begin{array}{c} d\left(u, p\right), \\ d\left(v, q\right) \end{array} \right\}\right)\right)$$

for all $u, v \in \mathcal{S}$ and $p, q \in \mathcal{T}$ and $\Delta \in C$, $\psi_\star \in \mathfrak{F}$, $\phi_\star \in \mathfrak{G}$ with $\ell \in (0, 1)$ Then there is a unique coupled fixed point of Γ in $\mathcal{S} \cup \mathcal{T}$.

Corollary 3.2. *Let $(\mathcal{S}, \mathcal{T}, d)$ be a complete bipolar metric space. Suppose that $\Gamma : (\mathcal{S}\times\mathcal{T}, \mathcal{T}\times\mathcal{S}) \rightrightarrows (\mathcal{S}, \mathcal{T})$ be a covariant mapping satisfy*

$$\psi_\star\left(d(\Gamma(u,p),\Gamma(q,v))\right) \leq \Delta\left(\psi_\star\left(\ell\max\left\{\begin{array}{c} d(u,q), \\ d(v,p) \end{array}\right\}\right), \phi_\star\left(\ell\max\left\{\begin{array}{c} d(u,q), \\ d(v,p) \end{array}\right\}\right)\right)$$

for all $u, v \in \mathcal{S}$ and $p, q \in \mathcal{T}$ and $\Delta \in C$, $\psi_\star \in \mathfrak{F}$, $\phi_\star \in \mathfrak{G}$ with $\ell \in (0,1)$ Then there is a unique coupled fixed point of Γ in $\mathcal{S}\cup\mathcal{T}$.

Example 3.1. *Let $\mathcal{S} = \mathfrak{U}_n(\mathbb{R})$ and $\mathcal{T} = \mathfrak{L}_n(\mathbb{R})$ be the set of all $n\times n$ upper and lower triangular matrices over $\mathbb{R}$. Define $d : \mathcal{S}\times\mathcal{T} \to [0,\infty)$ as $d(X,Y) = \sum_{i,j=1}^{\kappa} |\alpha_{ij} - \beta_{ij}|$*

for all $X = (\alpha_{ij})_{n\times n} \in \mathfrak{U}_n(\mathbb{R})$ and $Y = (\beta_{ij})_{n\times n} \in \mathfrak{L}_n(\mathbb{R})$. Then obviously $(\mathcal{S}, \mathcal{T}, d)$ is a Bipolar-metric space. And define $\Gamma : \mathcal{S}^2\cup\mathcal{T}^2 \to \mathcal{S}\cup\mathcal{T}$ as $\Gamma(A,B) = (\frac{a_{ij}-b_{ij}}{10})_{n\times n}$ where $(A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}) \in \mathfrak{U}_n(\mathbb{R})^2 \cup \mathfrak{L}_n(\mathbb{R})^2$ and define $\Lambda : \mathcal{S}\cup\mathcal{T} \to \mathcal{S}\cup\mathcal{T}$ as $\ell(A) = (\frac{a_{ij}}{2})_{n\times n}$ and let $\Delta : [0,+\infty)\times[0.+\infty) \to R$ as $\Delta(s^*, t^*) = s^* - t^*$, also define $\psi_\star : [0,\infty) \to [0,\infty)$, $\phi_\star : [0,\infty) \to [0,\infty)$ as $\psi_\star(t^*) = t^*$ and $\phi_\star(t^*) = \frac{t^*}{2}$ respectively. Then obviously, $\Gamma(\mathcal{S}^2\cup\mathcal{T}^2) \subseteq \Lambda(\mathcal{S}\cup\mathcal{T})$ and the pairs (Γ, Λ) is ω-compatible.
In fact, we have

$$\begin{aligned}
\psi_\star\left(d(\Gamma(A,B),\Gamma(X,Y))\right) &= d(\Gamma(A,B),\Gamma(X,Y)) \\
&= \sum_{i,j=1}^{\kappa} \left|\frac{a_{ij}-b_{ij}}{10} - \frac{x_{ij}-y_{ij}}{10}\right| \\
&\leq \frac{1}{4}\left(\sum_{i,j=1}^{\kappa}\left|\frac{a_{ij}}{2} - \frac{x_{ij}}{2}\right| + \sum_{i,j=1}^{\kappa}\left|\frac{b_{ij}}{2} - \frac{y_{ij}}{2}\right|\right) \\
&\leq \frac{1}{4}\left(d(\Lambda A,\Lambda X) + d(\Lambda B,\Lambda Y)\right) \\
&\leq \frac{1}{2}\left(\frac{1}{2}\max\{d(\Lambda A,\Lambda X), d(\Lambda B,\Lambda Y)\}\right) \\
&\leq \Delta\left(\psi_\star\left(\ell\max\left\{\begin{array}{c} d(\Lambda A,\Lambda X), \\ d(\Lambda B,\Lambda Y) \end{array}\right\}\right), \phi_\star\left(\ell\max\left\{\begin{array}{c} d(\Lambda A,\Lambda X), \\ d(\Lambda B,\Lambda Y) \end{array}\right\}\right)\right)
\end{aligned}$$

Thus all the conditions of the theorem (3.1) are satisfied and $(O_{n\times n}, O_{n\times n})$ is unique coupled fixed point.

3.1. **Application to the existence of solutions of integral equations.**

Let $\mathcal{S} = C(L^\infty(E_1))$, $\mathcal{T} = C(L^\infty(E_2))$ be the set of essential bounded measurable continuous functions on E_1 and E_2 where E_1, E_2 are two Lebesgue measurable sets with $m(E_1\cup E_2) < \infty$. Define $d : \mathcal{S}\times\mathcal{T} \to R^+$ as $d(\ell,\sigma) = ||\ell - \sigma||$ for all $\ell \in \mathcal{S}, \sigma \in \mathcal{T}$. Therefore, $(\mathcal{S}, \mathcal{T}, d)$ is a complete bipolar metric space.

In this section, we apply our theorem (3.1) to establish the existence and uniqueness solution of nonlinear integral equation defined by:

$$x(t) = f(t) + \kappa \int\limits_{E_1 \cup E_2} \Omega(t, \ell, (x, y)) d\ell. \tag{3.13}$$

where $x, y \in C(L^\infty(E_1) \cup L^\infty(E_2))$, $\kappa \in R$ and $t, \ell \in E_1 \cup E_2$,
$\Omega : E_1^2 \cup E_2^2 \times L^\infty(E_1)^2 \cup L^\infty(E_2)^2 \to R$ and $f : E_1 \cup E_2 \to R$ are given continuous functions

Theorem 3.2. *Assume that the following conditions are fulfilled*

(i) *Define,* $\Delta : [0, +\infty) \times [0. + \infty) \to R$ *as* $\Delta(s^*, t^*) = \theta s^*$ *where* $\theta \in (0, 1)$, *let* $\psi_\star : [0, \infty) \to [0, \infty)$ *as* $\psi_\star(t^*) = t^*$. *Let* $\Lambda : \mathcal{S} \cup \mathcal{T} \to \mathcal{S} \cup \mathcal{T}$ *as* $\Lambda(x) = x$ *and* $\Gamma : \mathcal{S}^2 \cup \mathcal{T}^2 \to \mathcal{S} \cup \mathcal{T}$ *by* $\Gamma(x, y)(t) = f(t) + \kappa \int\limits_{E_1 \cup E_2} \Omega(t, \ell, (x, y)) d\ell$

(ii) *There exists a continuous function* $\chi : E_1^2 \cup E_2^2 \to R^+$ *such that for all* $x, y \in \mathcal{S}, p, q \in \mathcal{T}$, $\kappa \in R$ *and* $t, \ell \in E_1 \cup E_2$, *we get that*
$||\Omega(t, \ell, (x, y)) - \Omega(t, \ell, (p, q))|| \leq \chi(t, \ell) M(x, y, p, q)$ *where,*
$M(x, y, p, q) = \lambda \max\{d(\Lambda x, \Lambda p), d(\Lambda y, \Lambda q)\}$ *where* $\lambda \in (0, 1)$

(iii) $||\kappa|| \int\limits_{E_1 \cup E_2} \chi(t, \ell) d\ell \leq \theta$

(iv) $\Gamma\left(\mathcal{S}^2 \cup \mathcal{T}^2\right) \subseteq \Lambda(\mathcal{S} \cup \mathcal{T})$, $\Lambda(\mathcal{S} \cup \mathcal{T})$ *is closed and the pair* (Γ, Λ) *is weakly compatible.*

Then there exists unique solution in $C(L^\infty(E_1) \cup L^\infty(E_2))$ *for the initial value problem 3.13.*

Proof. The existence of a solution of (3.13) is equivalent to the existence of a common fixed point of Γ and Λ. Obviously, $\Gamma\left(\mathcal{S}^2 \cup \mathcal{T}^2\right) \subseteq \Lambda(\mathcal{S} \cup \mathcal{T})$, $\Lambda(\mathcal{S} \cup \mathcal{T})$ is closed and the pair (Γ, Λ) is weakly compatible. Using the inequalities, (i), (ii) and (iii), we have

$$\begin{aligned}
\psi_\star(d(\Gamma(x, y), \Gamma(p, q))) &= d(\Gamma(x, y), \Gamma(p, q)) \\
&= ||\kappa \int\limits_{E_1 \cup E_2} (\Omega(t, \ell, (x, y))) d\ell - \kappa \int\limits_{E_1 \cup E_2} (\Omega(t, \ell, (p, q))) d\ell|| \\
&\leq ||\kappa|| \int\limits_{E_1 \cup E_2} ||\Omega(t, \ell, (x, y)) - \Omega(t, \ell, (p, q))|| d\ell \\
&\leq ||\kappa|| \int\limits_{E_1 \cup E_2} \chi(t, \ell) M(x, y, p, q) d\ell \\
&\leq ||\kappa|| \left(\int\limits_{E_1 \cup E_2} \chi(t, \ell) d\ell \right) M(x, y, p, q) \\
&\leq \theta M(x, y, p, q) \\
&\leq \Delta\left(\psi_\star\left(\lambda \max\left\{\begin{array}{c} d(\Lambda x, \Lambda p), \\ d(\Lambda y, \Lambda q) \end{array}\right\}\right), \phi_\star\left(\lambda \max\left\{\begin{array}{c} d(\Lambda x, \Lambda p), \\ d(\Lambda y, \Lambda q) \end{array}\right\}\right)\right)
\end{aligned}$$

Hence, all the conditions of Theorem (3.1) hold, we conclude that Γ and Λ have a unique solution in $\mathcal{S} \cup \mathcal{T}$ to the integral equation (3.13). □

3.2. **Application to the existence of solutions of Homotopy.**

In this part, we examine the possibility that homotopy theory has a unique solution.

Theorem 3.3. *Let $(\mathcal{S}, \mathcal{T}, d)$ be complete bipolar metric space, $(\mathcal{P}, \mathcal{Q})$ and $(\overline{\mathcal{P}}, \overline{\mathcal{Q}})$ be an open and closed subset of $(\mathcal{S}, \mathcal{T})$ such that $(\mathcal{P}, \mathcal{Q}) \subseteq (\overline{\mathcal{P}}, \overline{\mathcal{Q}})$. Suppose*
$\mathcal{H} : (\overline{\mathcal{P}} \times \overline{\mathcal{Q}}) \cup (\overline{\mathcal{Q}} \times \overline{\mathcal{P}}) \times [0,1] \to \mathcal{S} \cup \mathcal{T}$ be an operator with following conditions are satisfying,
ℓ_0) $\wp \neq \mathcal{H}(\wp, \varpi, s)$, $\varpi \neq \mathcal{H}(\varpi, \wp, s)$, for each $\wp \in \partial\mathcal{P}, \varpi \in \partial\mathcal{Q}$ and $s \in [0,1]$ (Here $\partial\mathcal{P} \cup \partial\mathcal{Q}$ is boundary of $\mathcal{P} \cup \mathcal{Q}$ in $\mathcal{S} \cup \mathcal{T}$);
ℓ_1) for all $\wp, \varpi \in \overline{\mathcal{P}}$, $\imath, \jmath \in \overline{\mathcal{Q}}$, $s \in [0,1]$ and $\psi_\star \in \mathfrak{F}, \phi_\star \in \mathfrak{G}$ $\Delta \in C$ and $\ell \in (0,1)$ such that

$$\psi_\star\left(d\left(\mathcal{H}(\wp, \imath, s), \mathcal{H}(\jmath, \varpi, s)\right)\right) \leq \Delta\left(\psi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp, \jmath), \\ d(\varpi, \imath) \end{array}\right\}\right), \phi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp, \jmath), \\ d(\varpi, \imath) \end{array}\right\}\right)\right)$$

ℓ_2) $\exists\, M \geq 0 \ni d(\mathcal{H}(\wp, \imath, s), \mathcal{H}(\jmath, \varpi, t)) \preceq M|s - t|$
for every $\wp, \varpi \in \overline{\mathcal{P}}$, $\imath, \jmath \in \overline{\mathcal{Q}}$ and $s, t \in [0,1]$.
Then $\mathcal{H}(., 0)$ has a coupled fixed point $\iff$ $\mathcal{H}(., 1)$ has a coupled fixed point.

Proof. Let the set

$$\Theta = \left\{ s \in [0,1] : \mathcal{H}(\wp, \imath, s) = \wp, \mathcal{H}(\imath, \wp, s) = \imath \text{ for some } \wp \in \mathcal{P}, \imath \in \mathcal{Q} \right\}.$$

$$\Upsilon = \left\{ t \in [0,1] : \mathcal{H}(\jmath, \varpi, t) = \jmath, \mathcal{H}(\varpi, \jmath, t) = \varpi \text{ for some } \varpi \in \mathcal{P}, \jmath \in \mathcal{Q} \right\}.$$

Suppose that $\mathcal{H}(., 0)$ has a coupled fixed point in $(\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$, we have that $(0,0) \in (\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$. Now we show that $(\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$ is both closed and open in $[0,1]$ and hence by the connectedness $\Theta = \Upsilon = [0,1]$. As a result, $\mathcal{H}(., 1)$ has a coupled fixed point in $(\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$. First we show that $(\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$ closed in $[0,1]$. To see this, Let $(\{a_p\}_{p=1}^{\infty}, \{x_p\}_{p=1}^{\infty}) \subseteq (\Theta, \Upsilon)$ and $(\{y_p\}_{p=1}^{\infty}, \{b_p\}_{p=1}^{\infty}) \subseteq (\Upsilon, \Theta)$ with $(a_p, x_p) \to (\alpha, \beta), (y_p, b_p) \to (\beta, \alpha) \in [0,1]$ as $p \to \infty$. We must show that $(\alpha, \beta) \in (\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$.
Since $(a_p, x_p) \in (\Theta, \Upsilon), (y_p, b_p) \in (\Upsilon, \Theta)$ for $p = 0, 1, 2, 3, \cdots$, there exists sequences $(\{\wp_p\}, \{\varpi_p\})$ and $(\{\imath_p\}, \{\jmath_p\})$ with $\wp_{p+1} = \mathcal{H}(\wp_p, \varpi_p, a_p)$, $\varpi_{p+1} = \mathcal{H}(\varpi_p, \wp_p, x_p)$ and $\imath_{p+1} = \mathcal{H}(\imath_p, \jmath_p, y_p)$, $\jmath_{p+1} = \mathcal{H}(\jmath_p, \imath_p, b_p)$
Consider

$$\begin{aligned} \psi_\star\left(d(\wp_p, \jmath_{p+1})\right) &= \psi_\star\left(d\left(\mathcal{H}(\wp_{p-1}, \varpi_{p-1}, a_{p-1}), \mathcal{H}(\jmath_p, \imath_p, b_p)\right)\right) \\ &\leq \Delta\left(\psi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp_{p-1}, \jmath_p), \\ d(\imath_p, \varpi_{p-1}) \end{array}\right\}\right), \phi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp_{p-1}, \jmath_p), \\ d(\imath_p, \varpi_{p-1}) \end{array}\right\}\right)\right) \\ &\leq \psi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp_{p-1}, \jmath_p), \\ d(\imath_p, \varpi_{p-1}) \end{array}\right\}\right) \end{aligned}$$

By using (ψ_0), we have

$$d(\wp_p, \jmath_{p+1}) \leq \ell \max \left\{ \begin{array}{c} d(\wp_{p-1}, \jmath_p), \\ d(\imath_p, \varpi_{p-1}) \end{array} \right\}$$

Similar lines we can prove that

$$d(\imath_{p+1}, \varpi_p) \leq \ell \max \left\{ \begin{array}{c} d(\wp_{p-1}, \jmath_p), \\ d(\imath_p, \varpi_{p-1}) \end{array} \right\}$$

Therefore, we get

$$\begin{aligned} \max \left\{ \begin{array}{c} d(\wp_p, \jmath_{p+1}), \\ d(\imath_{p+1}, \varpi_p) \end{array} \right\} &\leq \ell \max \left\{ \begin{array}{c} d(\wp_{p-1}, \jmath_p), \\ d(\imath_p, \varpi_{p-1}) \end{array} \right\} \\ &\leq \ell^2 \max \left\{ \begin{array}{c} d(\wp_{p-2}, \jmath_{p-1}), \\ d(\imath_{p-1}, \varpi_{p-2}) \end{array} \right\} \\ &\vdots \\ &\leq \ell^p \max \left\{ \begin{array}{c} d(\wp_0, \jmath_1), \\ d(\imath_1, \varpi_0) \end{array} \right\} \end{aligned} \tag{3.14}$$

Similarly, we can prove

$$\max \left\{ \begin{array}{c} d(\wp_{p+1}, \jmath_p), \\ d(\imath_p, \varpi_{p+1}) \end{array} \right\} \leq \ell^p \max \left\{ \begin{array}{c} d(\wp_1, \jmath_0), \\ d(\imath_0, \varpi_1) \end{array} \right\} \tag{3.15}$$

and

$$\max \left\{ \begin{array}{c} d(\wp_p, \jmath_p), \\ d(\imath_p, \varpi_p) \end{array} \right\} \leq \ell^p \max \left\{ \begin{array}{c} d(\wp_0, \jmath_0), \\ d(\imath_0, \varpi_0) \end{array} \right\} \tag{3.16}$$

For each $p, q \in \mathbf{N}$ with $p < q$. Then, from (3.14), (3.15), (3.16) and using property (B_4), we have

$$\begin{aligned} & d(\wp_p, \jmath_q) + d(\imath_p, \varpi_q) \\ \leq\; & (d(\wp_p, \jmath_{p+1}) + d(\imath_p, \varpi_{p+1})) + (d(\wp_{p+1}, \jmath_{p+1}) + d(\imath_{p+1}, \varpi_{p+1})) \\ & + \cdots + (d(\wp_{q-1}, \jmath_{q-1}) + d(\imath_{q-1}, \varpi_{q-1})) + (d(\wp_{q-1}, \jmath_q) + d(\imath_{q-1}, \varpi_q)) \\ \leq\; & (M|a_{p-1} - b_p| + M|x_p - y_{p-1}|) + \cdots + (M|a_{q-2} - b_{q-1}| + M|x_{q-1} - y_{q-2}|) \\ & \quad +2\left(\ell^{p+1} + \ell^{p+2} + \cdots + \ell^{q-1}\right) \max \left\{ \begin{array}{c} d(\wp_0, \jmath_0), \\ d(\imath_0, \varpi_0) \end{array} \right\} \\ \leq\; & (M|a_{p-1} - b_p| + M|x_p - y_{p-1}|) + \cdots + (M|a_{q-2} - b_{q-1}| + M|x_{q-1} - y_{q-2}|) \\ & \quad + \frac{2\ell^{p+1}}{1-\ell} \max \left\{ \begin{array}{c} d(\wp_0, \jmath_0), \\ d(\imath_0, \varpi_0) \end{array} \right\} \to 0 \text{ as } p, q \to \infty. \end{aligned}$$

It follows that $\lim_{p,q\to\infty}(d(\wp_p, \jmath_q) + d(\imath_p, \varpi_q)) = 0$. Similarly, we can prove that $\lim_{p,q\to\infty}(d(\wp_q, \jmath_p) + d(\imath_q, \varpi_p)) = 0$. Therefore, $(\{\wp_p\}, \{\varpi_p\})$ and $(\{\imath_p\}, \{\jmath_p\})$ are Cauchy bi-sequences in $(\mathcal{P}, \mathcal{Q})$. By completeness, there exist $(a, x) \in \mathcal{P} \times \mathcal{Q}$ and $(y, b) \in \mathcal{Q} \times \mathcal{P}$ with

$$\lim_{p\to\infty} \wp_{p+1} = x \quad \lim_{p\to\infty} \imath_{p+1} = y \quad \lim_{p\to\infty} \varpi_{p+1} = a \quad \lim_{p\to\infty} \jmath_{p+1} = b \tag{3.17}$$

we have

$$\begin{aligned} d(\mathcal{H}(b, y, \alpha), x) &\le d(\mathcal{H}(b, y, \alpha), \jmath_{p+1}) + d(\wp_{p+1}, \jmath_{p+1}) + d(\wp_{p+1}, x) \\ &\le d(\mathcal{H}(b, y, \alpha), \mathcal{H}(\jmath_p, \imath_p, b_p)) + M|a_p - b_p| + d(\wp_{p+1}, x) \end{aligned}$$

Letting $p \to \infty$ in the above inequality and $\psi_\star$ is continuous and non-decreasing, we have

$$\begin{aligned} \psi_\star(d(\mathcal{H}(b, y, \alpha), x)) &\le \psi_\star(d(\mathcal{H}(b, y, \alpha), \mathcal{H}(\jmath_p, \imath_p, b_p))) \\ &\le \Delta\left(\psi_\star\left(\ell \max\left\{\begin{array}{c} d(b, \jmath_p), \\ d(\imath_p, y) \end{array}\right\}\right), \phi_\star\left(\ell \max\left\{\begin{array}{c} d(b, \jmath_p), \\ d(\imath_p, y) \end{array}\right\}\right)\right) \\ &\le \psi_\star\left(\ell \max\left\{\begin{array}{c} d(b, \jmath_p), \\ d(\imath_p, y) \end{array}\right\}\right) \end{aligned}$$

By using (ψ_0) and letting as $p \to \infty$, we get that $d(\mathcal{H}(b, y, \alpha), x) = 0$ implies that $\mathcal{H}(b, y, \alpha) = x$. Similarly, we can prove that $\mathcal{H}(y, b, \beta) = a$ and $\mathcal{H}(x, a, \alpha) = y$, $\mathcal{H}(a, x, \beta) = b$. On the other hand, from (3.17), we get

$$d(a, y) = d\left(\lim_{p\to\infty} \varpi_p, \lim_{p\to\infty} \imath_p\right) = \lim_{p\to\infty} d(\imath_p, \varpi_p) = 0$$

and

$$d(b, x) = d\left(\lim_{p\to\infty} \jmath_p, \lim_{p\to\infty} \wp_p\right) = \lim_{p\to\infty} d(\wp_p, \jmath_p) = 0.$$

Therefore, $a = y$ and $b = x$ and hence $(\alpha, \beta) \in (\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$.

Clearly $(\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$ is closed in $[0, 1]$. Let $(\alpha_0, \beta_0) \in \Theta \times \Upsilon$, there exists bisequences $(\wp_0, \varpi_0)$ and $(\imath_0, \jmath_0)$ with $\wp_0 = \mathcal{H}(\wp_0, \varpi_0, \alpha_0)$, $\varpi_0 = \mathcal{H}(\varpi_0, \wp_0, \beta_0)$ and $\imath_0 = \mathcal{H}(\imath_0, \jmath_0, \beta_0)$, $\jmath_0 = \mathcal{H}(\jmath_0, \imath_0, \alpha_0)$. Since $(\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$ is open, then there exist $\delta > 0$ such that $B_d(\wp_0, \delta) \subseteq (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$, $B_d(\varpi_0, \delta) \subseteq (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$, $B_d(\imath_0, \delta) \subseteq (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$ and $B_d(\jmath_0, \delta) \subseteq (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$. Choose $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$, $\beta \in (\beta_0 - \epsilon, \beta_0 + \epsilon)$ such that $|\alpha - \alpha_0| \le \frac{1}{M^p} < \frac{\epsilon}{2}$, $|\beta - \beta_0| \le \frac{1}{M^p} < \frac{\epsilon}{2}$ and $|\alpha_0 - \beta_0| \le \frac{1}{M^p} < \frac{\epsilon}{2}$.

Then for, $\jmath \in \overline{B}_{\mathcal{P}\cup\mathcal{Q}}(\wp_0, \delta) = \{\jmath, \jmath_0 \in \mathcal{Q} / d(\wp_0, \jmath) \le d(\wp_0, \jmath_0) + \delta\}$,

$\imath \in \overline{B}_{\mathcal{P}\cup\mathcal{Q}}(\delta, \varpi_0) = \{\imath, \imath_0 \in \mathcal{P} / d(\imath, \varpi_0) \le d(\imath_0, \varpi_0) + \delta\}$

$\wp \in \overline{B}_{\mathcal{P}\cup\mathcal{Q}}(\delta, \jmath_0) = \{\wp, \wp_0 \in \mathcal{P} / d(\wp, \jmath_0) \le d(\wp_0, \jmath_0) + \delta\}$

$\varpi \in \overline{B}_{\mathcal{P}\cup\mathcal{Q}}(\imath_0, \delta) = \{\varpi, \varpi_0 \in \mathcal{Q} / d(\imath_0, \varpi) \le d(\imath_0, \varpi_0) + \delta\}$

$$\begin{aligned}
d\left(\mathcal{H}(\wp,\varpi,\alpha),\jmath_0)\right) &= d\left(\mathcal{H}(\wp,\varpi,\alpha),\mathcal{H}(\jmath_0,\imath_0,\alpha_0)\right) \\
&\leq d\left(\mathcal{H}(\wp,\varpi,\alpha),\mathcal{H}(\jmath,\imath,\alpha_0)\right) + d\left(\mathcal{H}(\wp_0,\varpi_0,\alpha),\mathcal{H}(\jmath,\imath,\alpha_0)\right) \\
&\qquad + d\left(\mathcal{H}(\wp_0,\varpi_0,\alpha),\mathcal{H}(\jmath_0,\imath_0,\alpha_0)\right) \\
&\leq 2M|\alpha-\alpha_0| + d\left(\mathcal{H}(\wp_0,\varpi_0,\alpha),\mathcal{H}(\jmath,\imath,\alpha_0)\right) \\
&\leq \frac{2}{M^{p-1}} + d\left(\mathcal{H}(\wp_0,\varpi_0,\alpha),\mathcal{H}(\jmath,\imath,\alpha_0)\right)
\end{aligned}$$

Letting $p \to \infty$ and using (ψ_0), then we have

$$\begin{aligned}
\psi_\star\left(d\left(\mathcal{H}(\wp,\varpi,\alpha),\jmath_0)\right)\right) &\leq \psi_\star\left(d\left(\mathcal{H}(\wp_0,\varpi_0,\alpha),\mathcal{H}(\jmath,\imath,\alpha_0)\right)\right) \\
&\leq \Delta\left(\psi_\star\left(\ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath\right), \\ d\left(\imath,\varpi_0\right)\end{array}\right\}\right), \phi_\star\left(\ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath\right), \\ d\left(\imath,\varpi_0\right)\end{array}\right\}\right)\right) \\
&\leq \psi_\star\left(\ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath\right), \\ d\left(\imath,\varpi_0\right)\end{array}\right\}\right)
\end{aligned}$$

Using the property of $\psi_\star$, we get

$$d\left(\mathcal{H}(\wp,\varpi,\alpha),\jmath_0)\right) \leq \ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath\right), \\ d\left(\imath,\varpi_0\right)\end{array}\right\}$$

Similarly we can prove

$$d\left(\imath_0,\mathcal{H}(\varpi,\wp,\beta)\right)) \leq \ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath\right), \\ d\left(\imath,\varpi_0\right)\end{array}\right\}$$

Therefore,

$$\begin{aligned}
\max\left\{\begin{array}{c} d\left(\mathcal{H}(\wp,\varpi,\alpha),\jmath_0)\right), \\ d\left(\imath_0,\mathcal{H}(\varpi,\wp,\beta)\right))\end{array}\right\} &\leq \ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath\right), \\ d\left(\imath,\varpi_0\right)\end{array}\right\} \\
&\leq \ell\max\left\{\begin{array}{c} d\left(\wp_0,\jmath_0\right)+\delta, \\ d\left(\imath_0,\varpi_0\right)+\delta\end{array}\right\}
\end{aligned}$$

Thus, $d\left(\mathcal{H}(\wp,\varpi,\alpha),\jmath_0)\right) \leq d\left(\wp_0,\jmath_0\right)+\delta$ and $d\left(\imath_0,\mathcal{H}(\varpi,\wp,\beta)\right)) \leq d\left(\imath_0,\varpi_0\right)+\delta$.
Similarly, we can prove
$d\left(\mathcal{H}(\imath,\jmath,\beta),\varpi_0)\right) \leq d\left(\imath_0,\varpi_0\right)+\delta$ and $d\left(\wp_0,\mathcal{H}(\jmath,\imath,\alpha)\right)) \leq d\left(\wp_0,\jmath_0\right)+\delta$.

On the other hand,

$$d(\wp_0,\varpi_0) = d\left(\mathcal{H}(\wp_0,\varpi_0,\alpha_0),\mathcal{H}(\varpi_0,\wp_0,\beta_0)\right) \leq M|\alpha_0-\beta_0| < \frac{1}{M^{p-1}} \to 0 \text{ as } p \to \infty.$$

and

$$d(\imath_0,\jmath_0) = d\left(\mathcal{H}(\imath_0,\jmath_0,\beta_0),\mathcal{H}(\jmath_0,\imath_0,\alpha_0)\right) \leq M|\alpha_0-\beta_0| < \frac{1}{M^{p-1}} \to 0 \text{ as } p \to \infty.$$

So $\wp_0 = \varpi_0$ and $\imath_0 = \jmath_0$ and hence $\alpha = \beta$. Thus for each fixed $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$, $\mathcal{H}(., \alpha) : \overline{B}_{\Theta \cup \Upsilon}(\wp_0, \delta) \to \overline{B}_{\Theta \cup \Upsilon}(\wp_0, \delta)$ and $\mathcal{H}(., \alpha) : \overline{B}_{\Theta \cup \Upsilon}(\imath_0, \delta) \to \overline{B}_{\Theta \cup \Upsilon}(\imath_0, \delta)$. Thus, we conclude that $\mathcal{H}(., \alpha)$ has a coupled fixed point in $(\overline{\mathcal{P}} \times \overline{\mathcal{Q}}) \cap (\overline{\mathcal{Q}} \times \overline{\mathcal{P}})$. But this must be in $(\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$. Therefore, $(\alpha, \alpha) \in (\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$ for $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$.Hence, $(\alpha_0 - \epsilon, \alpha_0 + \epsilon) \subseteq (\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$. Clearly, $(\Theta \times \Upsilon) \cap (\Upsilon \times \Theta)$ is open in $[0, 1]$. For the reverse implication, we use the same strategy. □

Theorem 3.4. *Let $(\mathcal{S}, \mathcal{T}, d)$ be complete bipolar metric space, $(\mathcal{P}, \mathcal{Q})$ and $(\overline{\mathcal{P}}, \overline{\mathcal{Q}})$ be an open and closed subset of $(\mathcal{S}, \mathcal{T})$ such that $(\mathcal{P}, \mathcal{Q}) \subseteq (\overline{\mathcal{P}}, \overline{\mathcal{Q}})$. Suppose*
$\mathcal{H} : \left(\overline{\mathcal{P}}^2 \cup \overline{\mathcal{Q}}^2\right) \times [0, 1] \to \mathcal{S} \cup \mathcal{T}$ be an operator with following conditions are satisfying,
$\ell_0)$ $\wp \neq \mathcal{H}(\wp, \varpi, s)$, $\varpi \neq \mathcal{H}(\varpi, \wp, s)$, for each $\wp, \varpi \in \partial\mathcal{P} \cup \partial\mathcal{Q}$ and $s \in [0, 1]$ (Here $\partial\mathcal{P} \cup \partial\mathcal{Q}$ is boundary of $\mathcal{P} \cup \mathcal{Q}$ in $\mathcal{S} \cup \mathcal{T}$);
$\ell_1)$ for all $\wp, \varpi \in \overline{\mathcal{P}}$, $\imath, \jmath \in \overline{\mathcal{Q}}$, $s \in [0, 1]$ and $\psi_\star \in \mathfrak{F}$,$\phi_\star \in \mathfrak{G}$ $\Delta \in C$ and $\ell \in (0, 1)$ such that

$$\psi_\star\left(d\left(\mathcal{H}(\wp, \varpi, s), \mathcal{H}(\imath, \jmath, s)\right)\right) \leq \Delta\left(\psi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp, \imath), \\ d(\varpi, \jmath) \end{array}\right\}\right), \phi_\star\left(\ell \max\left\{\begin{array}{c} d(\wp, \imath), \\ d(\varpi, \jmath) \end{array}\right\}\right)\right)$$

$\ell_2)$ $\exists\ M \geq 0 \ni d(\mathcal{H}(\wp, \varpi, s), \mathcal{H}(\imath, \jmath, t)) \preceq M|s - t|$
for every $\wp, \varpi \in \overline{\mathcal{P}}$, $\imath, \jmath \in \overline{\mathcal{Q}}$ and $s, t \in [0, 1]$.
Then $\mathcal{H}(., 0)$ has a coupled fixed point $\iff$ $\mathcal{H}(., 1)$ has a coupled fixed point.

CONCLUSION

We ensured the existence and uniqueness of a common coupled fixed point for two covariant mappings in the class of complete bipolar metric spaces with examples via C-class functions. Two illustrated application has been provided.

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