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ON SOME ISOMORPHISMS BETWEEN BOUNDED LINEAR MAPS AND NON-COMMUTATIVE L_p-SPACES

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Abstract. We define a particular space of bounded linear maps using a Von Neumann algebra and some operator spaces. By this, we prove some isomorphisms, and using interpolation in some particular cases, we get analogue of non-commutative L_p spaces.

1. Introduction

In the fifties, many authors have studied on non-commutative L_p-spaces like Segal [13], Kunze [9], Dixmier [4], Stinespring [14]. But the recent emergency of the theory of operator spaces from the late 80's to the early 90's in the works of Effros and Ruan [5],[6], [7],[12], Blecher and Paulsen [2],[3] allowed Gilles Pisier since the mid 90's to expose the general theory of non-commutative L_p-spaces [11], using for instance a Von Neumann algebra M equipped with a particular type of trace φ. As the complex interpolation method contribute to define a new Banach space using a compatible pair of Banach spaces, this method was also used to define non-commutative L_p-spaces.

Our aim in this paper is to define for each $1 \leq p \leq \infty$ a particular space of bounded linear maps denoted $\mathscr{L}_p(M, \varphi, E; F)$, using some operator spaces E, F and the non-commutative spaces $L_1(M, \varphi)$, $L_\infty(M, \varphi)$, such that those particular spaces have some properties with the non-commutative L_p spaces like the isomorphism. We firstly view those types of spaces as Banach spaces and secondly give them an operator space structure.

Before stating our results, we shall recall the concept of operator space and the complex interpolation method to make the paper more comprehensive.

2. Preliminary Notes

2.1. **Operator spaces.** H being an Hilbert space, we denote by $B(H)$ the Banach space of all bounded operators from H into H, endowed with the operator norm

$$\|T\|_\infty = \sup\{\|T\xi\|_H, \xi \in H, \|\xi\| \leq 1\}.$$

A closed subset $E \subset B(H)$ is called an operator space. But there exist an abstract characterization of an operator space given by Ruan (see [7] and [12] for more details):

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Theorem 2.1 (Ruan theorem). *A complex vector space E is an operator space if and only if for each $n \geq 1$, there is a complete norm $\|.\|_n$ on $M_n(E)$, the space of $n \times n$ matrices with entries in E, such that the following properties are satisfied:*

(1) $\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|$
(2) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$

for $x \in M_n(E), y \in M_m(E), \alpha, \beta \in M_n$.

Let M be a Von Neumann algebra on the Hilbert space H, that is, $M \subset B(H)$ is a C^*-algebra closed in the weak operator topology and contains the identity operator. We denote by M^+ the set of all positive elements of M. Then, we recall the following definitions concerning the notion of trace:

Definition 2.2. *A trace φ on M^+ is a function $\varphi : M^+ \to]0, +\infty]$, such that*

(1) *$\varphi(x + y) = \varphi(x) + \varphi(y)$ for any $x, y \in M^+$.*
(2) *$\varphi(\lambda x) = \lambda \varphi(x)$ for any $0 \leq \lambda \leq \infty$ and $x \in M^+$ with the usual convention $0.\infty = 0$.*
(3) *$\varphi(xy) = \varphi(yx)$ for any $x, y \in M^+$.*

Definition 2.3. *A trace φ is called:*

i) *faithful if $x \in M^+, \varphi(x) = 0 \Longrightarrow x = 0$.*
ii) *finite if $\varphi(x) < \infty$ for any $x \in M^+$.*
iii) *normal if for any $x \in M^+$ and any increasing net (x_α) converging to x in the strong operator topology, $\varphi(x_\alpha) \to \varphi(x)$.*
iv) *semi-finite if for any $x \in M^+$, there exist $y \in M^+$such that $\varphi(y) < \infty$ and $y \leq x$.*

In the following, the Von Neumann algebra M is assumed to be equipped with a faithful, normal and semi-finite trace φ.

2.2. **The complex interpolation method and the non-commutative L_p spaces.** A couple of (complex) Banach spaces $(X_0; X_1)$ is said to be *compatible* if they are both embedded by continuous injective linear maps into a Hausdorff topological vector space X. In this case, X_0 and X_1 are viewed as vector subspaces of X. Then their intersection $X_0 \cap X_1$ is equipped with the norm

$$\|x\|_{X_0 \cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$$

and their sum is defined by

$$X_0 + X_1 = \{x_0 + x_1 : x_k \in X_k,\, k = 0, 1\}$$

with the norm

$$\|x\|_{X_0 + X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_k \in X_k,\, k = 0, 1\}.$$

It is easy to check that $X_0 \cap X_1$ and $X_0 + X_1$ are again Banach spaces. Then we have

$$X_0 \cap X_1 \subset X_0,\, X_1 \subset X_0 + X_1,\, contractive\, injections.$$

Let $B = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\}$. Let $\mathcal{F}(X_0, X_1)$ be the family of all functions $f : B \to X_0 + X_1$ satisfying the following conditions:

(1) f is continuous on B and analytic in the interior of B;

(2) $f(k+it) \in X_k$ for $t \in \mathbb{R}$ and the function $t \mapsto f(k+it)$ is continuous from $\mathbb{R}$ into X_k, $k = 0, 1$;

(3) $\lim\limits_{|t|\to\infty} \|f(k+it)\|_{X_k} = 0$, $k = 0, 1$.

We equip $\mathcal{F}(X_0, X_1)$ with the norm:

$$\|f\|_{\mathcal{F}(X_0,X_1)} = \{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\}.$$

Then it is easy to check that $\mathcal{F}(X_0, X_1)$ becomes a Banach space. Let $0 < \theta < 1$, the complex interpolation space $(X_0, X_1)_\theta$ is defined as the space of all those $x \in X_0 + X_1$ for which there exists $f \in \mathcal{F}(X_0, X_1)$ such that $f(\theta) = x$. Equipped with

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} : f(\theta) = x,\ f \in \mathcal{F}(X_0, X_1)\},$$

$(X_0, X_1)_\theta$ becomes a Banach space. Indeed, by the maximum principle, the map $f \mapsto f(\theta)$ is a contraction from $\mathcal{F}(X_0, X_1)$ to $X_0 + X_1$. Then $(X_0, X_1)_\theta$ can be isometrically identified with the quotient of $\mathcal{F}(X_0, X_1)$ by the kernel of this map.

Remark 2.1. *The following properties are easy:*

i) $(X_0, X_1)_\theta = (X_1, X_0)_{1-\theta}$ *isometrically.*

ii) $X_0 \cap X_1$ *is dense in* $(X_0, X_1)_\theta$.

iii) *Let* (X_0, X_1) *and* (Y_0, Y_1) *be two compatible couples.*
Let $T : X_0 + X_1 \to Y_0 + Y_1$ *be a linear map which is bounded from* X_k *to* Y_k *for* $k = 0$ *and* $k = 1$. *Then* T *is bounded from* $(X_0, X_1)_\theta$ *to* $(Y_0, Y_1)_\theta$ *for any* $0 < \theta < 1$; *moreover*

$$\|T : (X_0, X_1)_\theta \to (Y_0, Y_1)_\theta\| \le \|T : X_0 \to Y_0\|^{1-\theta} \|T : X_1 \to Y_1\|^\theta.$$

This statement is usually called interpolation theorem.

Note that by tradition in interpolation theory, the assumption on T in the statement (iii) above means that T maps X_k into Y_k and its restriction to X_k belongs to $B(X_k, Y_k)$ $(k = 0, 1)$.

Set $I_p = \{x \in M : \varphi(|x|^p) < \infty\}$, $(1 \le p < \infty)$, equipped with the norm $\|x\|_p = (\varphi(|x|^p))^{1/p}$. The completion of I_p under this norm is a Banach space and is denoted $L_p(M, \varphi)$ by G. Pisier in [11] where it is called non-commutative L_p space.

Since all $L_p(M, \varphi)$, $1 \le p \le \infty$, are injected into $L_1(M, \varphi)$, $(L_{p_0}(M, \varphi), L_{p_1}(M, \varphi))$ is a compatible couple for any $p_0, p_1 \in [1; \infty]$. The following is the complex interpolation theorem on non-commutative L_p-spaces.

Theorem 2.4. *Let* $1 \le p_0 < p_1 \le \infty, 0 < \theta < 1$ *and let* p *be determined by* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *Then*

$$(L_{p_0}(M, \varphi), L_{p_1}(M, \varphi))_\theta = L_p(M, \varphi)$$

with equal norms.

For more details, see [1] and [8].

Particularly, for $p_0 = 1$ and $p_1 = \infty$ we get

$$(L_1(M, \varphi), L_\infty(M, \varphi))_\theta = L_p(M, \varphi)$$

whith $\theta = \frac{1}{p}$

Remark 2.2. (i) *$E \subset B(H)$ being an operator space, the non-commutative vector valued L_p spaces for $1 \leq p \leq \infty$ are defined as follow:*

$$L_1(M, \varphi, E) = L_1(M, \varphi)\hat{\otimes} E$$

$$L^0_\infty(M, \varphi, E) = M \otimes_{min} E$$

$$L_p(M, \varphi, E) = (L_1(M, \varphi, E), L^0_\infty(M, \varphi, E))_\theta \quad 1 < p < \infty$$

whith $\theta = \frac{1}{p}$.

(ii) *The dual space of $L_p(M, \varphi)$ is $L_q(M, \varphi)$ for $1 \leq p < \infty$ $(\frac{1}{p} + \frac{1}{q} = 1)$ with respect to the following duality*

$$\langle x, y\rangle = \varphi(xy),\ x \in L_p(M, \varphi),\ y \in L_q(M, \varphi).$$

In over words

$$(L_p(M, \varphi))' = L_q(M, \varphi) \quad \textit{isometrically.}$$

(iii) *In the Lebesgue-Bochner theory, if E is a Banach space, it is well known that the dual of $L_p(\Omega, \mu; E)$ is not in general $L_q(\Omega, \mu; E')$ unless E' possesses the Radon Nikodym property (in short the RNP). In* [11]*, it was introduced an operator space analogue of the RNP which is called the ORNP. Thus, if E is an operator space such that is dual E' has the ORNP, then the dual theorem is confirmed. Namely the dual of $L_p(M, \varphi, E)$ is completely isometric to $L_q(M, \varphi, E')$. One must also note that the ORNP of an operator space implies the RNP of the underlying Banach space, but the converse is not true.*

The following theorem has been proved by G. Pisier in [11] pages 49,50.

Theorem 2.5 (Pisier)**.** *Let (M, φ) be any non-commutative probability space (in short n.c.p. space). Let E be an operator space. If E' has the $ORNP_{p'}$ with $1 < p < \infty$ and $q = p/(p-1)$, then we have a completely isometric identity*

$$L_p(M, \varphi, E)' = L_q(M, \varphi, E').$$

3. Main Results

3.1. **The spaces $\mathscr{L}_1(M, \varphi, E; F)$ and $\mathscr{L}_\infty(M, \varphi, E; F)$..** Let $E, F \subset B(H)$ two operator spaces. We denote by $\|.\|_E$ and $\|.\|_F$ the operator norm on E and F inherited from $B(H)$ and $\mathcal{L}(E, F)$ the space of all bounded linear maps from E into F equipped with the norm $\|f\|_{\mathcal{L}(E,F)} = \sup\{\|f(x)\|_F : \|x\|_E \leq 1\}$.

Set $\mathscr{L}_1(M, \varphi, E; F)$ the space of all bounded linear maps from $L_1(M, \varphi)$ into $\mathcal{L}(E, F)$.

Theorem 3.1. *The mapping*

$$u \longmapsto \|u\|_{\varphi,1} = \sup\{\|u(x)\|_{\mathcal{L}(E,F)} : \varphi(|x|) \leq 1\}$$

is a norm on $\mathscr{L}_1(M, \varphi, E; F)$. $\mathscr{L}_1(M, \varphi, E; F)$ equipped with this norm is a Banach space.

Proof. It is obvious that $\|.\|_{\varphi,1}$ is a norm. Since $L_1(M, \varphi)$, $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are Banach spaces, then $(\mathscr{L}_1(M, \varphi, E; F), \|.\|_{\varphi,1})$ is a Banach space. $\square$

Moreover, $\mathscr{L}_1(M,\varphi,E;F)$ is a Banach algebra if endowed with the inner product denoted $\cdot$ as follow: for all $u,v\in\mathscr{L}_1(M,\varphi,E;F)$, $u\cdot v=w$ such that $w(x)(y)=u(x)(y)\circ v(x)(y)$, with $x\in L_1(M,\varphi)$ and $y\in E$. Here, the product between the two elements $u(x)(y),v(x)(y)$ of F is the natural product of operator inherited from the Banach algebra $B(H)$. More precisely,

$$\forall T_1,T_2\in B(H), T_1\circ T_2(\xi)=T_1(T_2(\xi))$$

with $\xi\in H$. And we obtain

$$\begin{aligned}
\|u\cdot v\|_{\varphi,1} &= \sup\{\|(u\cdot v)(x)\|_{\mathcal{L}(E,F)} : \varphi(|x|)\le 1\}\\
&= \sup\{\sup\{\|(u\cdot v)(x)(y)\|_F : \|y\|_E\le 1\} : \varphi(|x|)\le 1\}\\
&= \sup\{\sup\{\|u(x)(y)\circ v(x)(y)\|_F : \|y\|_E\le 1\} : \varphi(|x|)\le 1\}\\
&\le \sup\{\sup\{\|u(x)(y)\|_F\|v(x)(y)\|_F : \|y\|_E\le 1\} : \varphi(|x|)\le 1\}\\
&\le \sup\{\sup_{\|y\|_E\le 1}\{\|u(x)(y)\|_F\|v(x)(y)\|_F\} : \varphi(|x|)\le 1\}\\
&\le \sup\{(\sup_{\|y\|_E\le 1}\|u(x)(y)\|_F)(\sup_{\|y\|_E\le 1}\|v(x)(y)\|_F) : \varphi(|x|)\le 1\}\\
&\le \sup\{\left(\|u(x)\|_{\mathcal{L}(E,F)}\right)\left(\|v(x)\|_{\mathcal{L}(E,F)}\right) : \varphi(|x|)\le 1\}\\
&\le \sup_{\varphi(|x|)\le 1}\{\left(\|u(x)\|_{\mathcal{L}(E,F)}\right)\left(\|v(x)\|_{\mathcal{L}(E,F)}\right)\}\\
&\le (\sup_{\varphi(|x|)\le 1}\|u(x)\|_{\mathcal{L}(E,F)})(\sup_{\varphi(|x|)\le 1}\|v(x)\|_{\mathcal{L}(E,F)})\\
&\le \|u\|_{\varphi,1}\|v\|_{\varphi,1}
\end{aligned}$$

Definition 3.2. *Set $\mathscr{L}_\infty(M,\varphi,E;F)$ the space of all bounded linear maps from M into $\mathcal{L}(E,F)$ equipped with the norm*

$$\|u\|_{\varphi,\infty}=\sup\{\|u(x)\|_{\mathcal{L}(E,F)} : \|x\|_\infty\le 1\},$$

where $\|.\|_\infty$ is the operator norm on $M\subset B(H)$.

Theorem 3.3. *$\mathscr{L}_\infty(M,\varphi,E;F)$ is a Banach space.*

Proof. The proof of this theorem is in the same spirit of the one of the previous theorem . ☐

It is also easy to check that $\mathscr{L}_\infty(M,\varphi,E;F)$ endowed with the same inner product used for $\mathscr{L}_1(M,\varphi,E;F)$ is a Banach algebra.

Theorem 3.4 (duality)**.** *The topological dual of $\mathscr{L}_1(M,\varphi,E;F)$ is isometrically isomorph to $\mathscr{L}_\infty(M,\varphi,E;F')$ where F' is the dual of F:*

$$(\mathscr{L}_1(M,\varphi,E;F))'\simeq\mathscr{L}_\infty(M,\varphi,E;F')$$

Proof. Let us consider the mapping

$$\begin{aligned}
T:\mathscr{L}_\infty(M,\varphi,E;F') &\longrightarrow (\mathscr{L}_1(M,\varphi,E;F))'\\
u\longmapsto &\quad Tu\,,
\end{aligned}$$

such that

$$\langle Tu,v\rangle=\sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}|\langle u(x)(y),v(x)(y)\rangle|,$$

where $u\in\mathscr{L}_\infty(M,\varphi,E,F')$, $v\in(\mathscr{L}_1(M,\varphi,E;F))'$, and so:
$\forall x\in\mathscr{L}_1(M,\varphi),y\in E$ we have $u(x)(y)\in F'$, $v(x)(y)\in F$.

(1) Linearity and boundedness of T: The linearity of T is trivial. Let us prove T is bounded. We have:

$$\begin{aligned}
|\langle Tu, v\rangle| &= \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} |\langle u(x)(y), v(x)(y)\rangle| \\
&\le \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} \|u(x)(y)\|_{F'} \|v(x)(y)\|_F \\
&\le \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} \|u(x)(y)\|_{F'} \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} \|v(x)(y)\|_F \\
&\le \sup_{\|x\|_\infty\le 1, \|y\|_E\le 1} \|u(x,y)\|_{F'} \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} \|v(x)(y)\|_F \\
&\le \|u\|_{\varphi,\infty}\|v\|_{\varphi,1}
\end{aligned}$$

Thus, $\|Tu\| \le \|u\|_{\varphi,\infty}$ and T is bounded with $\|T\| \le 1$.

(2) We prove now that $\|T\| = 1$. Since $\|T\| \le 1$, all we have to do is to prove $\|T\| \ge 1$.

Let $a \in F$ such that $\|a\|_F = 1$. Since $a \neq 0$, there exist $b^* \in F^{'}$ such that $\|b^*\| = 1$ and $\langle b^*, a\rangle = \|a\|_F = 1$. For (x_0, y_0) fixed in $L_1(M, \varphi) \times E$, one define $u \in \mathscr{L}_\infty(M, \varphi, E; F')$ as follow:

$$u(x)(y) = \begin{cases} b^* \text{ if } (x, y) = (x_0, y_0) \\ \quad 0 \text{ if not} \end{cases}$$

and $v \in \mathscr{L}_1(M, \varphi, E; F)$ by

$$v(x)(y) = \begin{cases} a \text{ if } (x, y) = (x_0, y_0) \\ \quad 0 \text{ if not} \end{cases}$$

Afterwards,

$$\begin{aligned}
\langle Tu, v\rangle &= \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} |\langle u(x)(y), v(x)(y)\rangle| \\
&= \langle u(x_0)(y_0), v(x_0)(y_0)\rangle \\
&= \langle b^*, a\rangle \\
&= 1
\end{aligned}$$

Since $\langle Tu, v\rangle \le \|T\|\|u\|\|v\|$, with $\|u\| = \|v\| = 1$, then $\|T\| \ge 1$ and $\|T\| = 1$.

(3) subjectivity of T

Suppose $f \in (\mathscr{L}_1(M, \varphi, E; F))'$ and (x_0, y_0) fixed in $L_1(M, \varphi) \times E$.
Let $\phi \in \mathscr{L}_1(M, \varphi, E; F)$ such that $\phi(x)(y) = 0$ if $(x, y) \neq (x_0, y_0)$.
We set $a_{(x_0,y_0)} = \phi(x_0)(y_0) \in F$,
then there exist a linear functional $b_{(x_0,y_0)} \in F'$ such that

$$\langle f, \phi\rangle = \langle b_{(x_0,y_0)}, a_{(x_0,y_0)}\rangle > 0$$

Set $\Phi \in \mathscr{L}_\infty(M, \varphi, E; F)$ as follow:

$$\Phi(x)(y) = \begin{cases} b_{(x_0,y_0)} \text{ if } (x, y) = (x_0, y_0) \\ \quad 0 \text{ if not} \end{cases}$$

So we have:

$$\begin{aligned}
\langle f, \phi\rangle &= \langle \Phi(x_0)(y_0), \phi(x_0)(y_0)\rangle \\
&= \sup_{\varphi(|x|)\le 1, \|y\|_E\le 1} |\langle \Phi(x)(y), \phi(x)(y)\rangle| \\
&= \langle T\Phi, \phi\rangle.
\end{aligned}$$

Hence the linear functional f and $T\Phi$ coincide on $\mathscr{L}_\infty(M, \varphi, E; F')$. In over words $f = T\Phi$ and T is subjective.

In conclusion, T is isometric.

$\square$

3.2. The spaces $\mathscr{L}_p^0(M,\varphi,E;F)$ with $1<p<\infty$.

Definition 3.5. *Let $\mathscr{L}_p^0(M,\varphi,E;F)$ be the space of all bounded linear maps from $L_p(M,\varphi)$ into $\mathcal{L}(E,F)$ for $1<p<\infty$.*

Theorem 3.6. *For each $1<p<\infty$,the space $\mathscr{L}_p^0(M,\varphi,E;F)$ endowed with the norm*

$$\|u\|_{\varphi,p}=\sup\{\|u(x)\|_{\mathcal{L}(E,F)}:\varphi(|x|^p)\leq 1\}$$

is a Banach space.

Proof. In the same spirit as in theorem 3.1 $\square$

Theorem 3.7 (Duality). *The topological dual of $\mathscr{L}_p^0(M,\varphi,E;F)$ is isometrically isomorph to $\mathscr{L}_q^0(M,\varphi,E;F')$, $(\dfrac{1}{p}+\dfrac{1}{q}=1)$ where F' is the dual of F:*

$$(\mathscr{L}_p^0(M,\varphi,E;F))'\simeq\mathscr{L}_q^0(M,\varphi,E,F')$$

Proof. Let us consider the mapping

$$\begin{array}{rcl} T:\mathscr{L}_q(M,\varphi,E;F') & \longrightarrow & (\mathscr{L}_p(M,\varphi,E;F))' \\ u\longmapsto & & Tu\,, \end{array}$$

such that

$$\langle Tu,v\rangle=\sup_{\varphi(|x|)\leq 1,\|y\|_E\leq 1}|\langle u(x)(y),v(x)(y)\rangle|,$$

where $u\in\mathscr{L}_q(M,\varphi,E,F')$, $v\in(\mathscr{L}_p(M,\varphi,E;F))'$, and so:
$\forall x\in\mathscr{L}_1(M,\varphi),y\in E$ we have $u(x)(y)\in F'$, $v(x)(y)\in F$.

(1) Linearity and boundedness of T: The linearity of T is trivial. Let us prove T is bounded. We have:

$$\begin{array}{rcl} |\langle Tu,v\rangle| & = & \displaystyle\sup_{\varphi(|x|)\leq 1,\|y\|_E\leq 1}|\langle u(x)(y),v(x)(y)\rangle| \\ & \leq & \displaystyle\sup_{\varphi(|x|)\leq 1,\|y\|_E\leq 1}\|u(x)(y)\|_{F'}\|v(x)(y)\|_F \\ & \leq & \displaystyle\sup_{\varphi(|x|)\leq 1,\|y\|_E\leq 1}\|u(x)(y)\|_{F'}\sup_{\varphi(|x|)\leq 1,\|y\|_E\leq 1}\|v(x)(y)\|_F \\ & \leq & \displaystyle\sup_{\varphi(|x|^q)\leq 1,\|y\|_E\leq 1}\|u(x)(y)\|_{F'}\sup_{\varphi(|x|^p)\leq 1,\|y\|_E\leq 1}\|v(x)(y)\|_F \\ & \leq & \displaystyle\left(\sup_{\varphi(|x|^q)\leq 1}\|u(x)\|_{\mathcal{L}(E,F')}\right)\left(\sup_{\varphi(|x|^p)\leq 1}\|v(x)\|_{\mathcal{L}(E,F}\right) \\ & \leq & \|u\|_{\varphi,q}\|v\|_{\varphi,p} \end{array}$$

Thus, $\|Tu\|\leq\|u\|_{\varphi,q}$ and T is bounded with $\|T\|\leq 1$.

(2) We prove now that $\|T\|=1$. Since $\|T\|\leq 1$, all we have to do is to prove $\|T\|\geq 1$.
Let $a\in F$ such that $\|a\|_F=1$. Since $a\neq 0$, there exist $b^*\in F^{'}$ such that $\|b^*\|=1$ and $\langle b^*,a\rangle=\|a\|_F=1$. For (x_0,y_0) fixed in $L_p(M,\varphi)\times E$, one define $u\in\mathscr{L}_q(M,\varphi,E;F')$ as follow:

$$u(x)(y)=\begin{cases} b^* \text{ if } (x,y)=(x_0,y_0) \\ \quad 0 \text{ if not}\end{cases}$$

and $v \in \mathscr{L}_p(M, \varphi, E; F)$ by

$$v(x)(y) = \begin{cases} a \text{ if } (x, y) = (x_0, y_0) \\ 0 \text{ if not} \end{cases}$$

Afterwards,

$$\begin{aligned} \langle Tu, v\rangle &= \sup_{\varphi(|x|)\leq 1, \|y\|_E\leq 1} |\langle u(x)(y), v(x)(y)\rangle| \\ &= \langle u(x_0)(y_0), v(x_0)(y_0)\rangle \\ &= \langle b^*, a\rangle \\ &= 1 \end{aligned}$$

Since $\langle Tu, v\rangle \leq \|T\|\|u\|\|v\|$, with $\|u\| = \|v\| = 1$, then $\|T\| \geq 1$ and $\|T\| = 1$.

(3) subjectivity of T
Suppose $f \in (\mathscr{L}_p(M, \varphi, E; F))'$ and (x_0, y_0) fixed in $L_p(M, \varphi) \times E$.
Let $\phi \in \mathscr{L}_p(M, \varphi, E; F)$ such that $\phi(x)(y) = 0$ if $(x, y) \neq (x_0, y_0)$.
We set $a_{(x_0,y_0)} = \phi(x_0)(y_0) \in F$,
then there exist a linear functional $b_{(x_0,y_0)} \in F'$ such that

$$\langle f, \phi\rangle = \langle b_{(x_0,y_0)}, a_{(x_0,y_0)}\rangle > 0$$

Set $\Phi \in \mathscr{L}_q(M, \varphi, E; F)$ as follow:

$$\Phi(x)(y) = \begin{cases} b_{(x_0,y_0)} \text{ if } (x, y) = (x_0, y_0) \\ 0 \text{ if not} \end{cases}$$

So we have:

$$\begin{aligned} \langle f, \phi\rangle &= \langle \Phi(x_0)(y_0), \phi(x_0)(y_0)\rangle \\ &= \sup_{\varphi(|x|)\leq 1, \|y\|_E\leq 1} |\langle \Phi(x)(y), \phi(x)(y)\rangle| \\ &= \langle T\Phi, \phi\rangle. \end{aligned}$$

Hence the linear functional f and $T\Phi$ coincide on $\mathscr{L}_q(M, \varphi, E; F')$. In over words $f = T\Phi$ and T is subjective.

In conclusion, T is isometric.

□

Corollary 3.8. *for* $1 \leq p < \infty$ $\mathscr{L}_p^0(M, \varphi, \mathbb{C}; \mathbb{C})$ *is isomorphic to* $L_q(M, \varphi)$*:*

$$\mathscr{L}_p^0(M, \varphi, \mathbb{C}; \mathbb{C}) \simeq L_q(M, \varphi)$$

where q is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ *(called the conjugate of p).*

Proof.

$$\begin{aligned} \mathscr{L}_p^0(M, \varphi, \mathbb{C}; \mathbb{C}) &= \mathcal{L}(L_p(M, \varphi), \mathcal{L}(\mathbb{C}, \mathbb{C})) \\ &\simeq \mathcal{L}(L_p(M, \varphi), \mathbb{C}) \\ &\simeq (L_p(M, \varphi))' \\ &\simeq L_q(M, \varphi) \end{aligned}$$

□

3.3. The spaces $\mathscr{L}_p(M,\varphi,E;F)$ by using the interpolation method.

Theorem 3.9. *There is a contractive injection from $\mathscr{L}_\infty(M,\varphi,E;F)$ into $\mathscr{L}_1(M,\varphi,E;F)$*

Proof. Since $\mathscr{L}_1(M,\varphi,E;F)$ denotes the space of all bounded linear maps from $L_1(M,\varphi)$ into $\mathcal{L}(E,F)$ and $\mathscr{L}_\infty(M,\varphi,E;F)$ the space of all bounded linear maps from M into $\mathcal{L}(E,F)$, with $M \subset L_1(M,\varphi)$, it is obvious to claim that $\mathscr{L}_\infty(M,\varphi,E;F) \subset \mathscr{L}_1(M,\varphi,E;F)$. Let z_0 be an element of F.

Now, we consider the injection:

$$\begin{array}{rcl} f : \mathscr{L}_\infty(M,\varphi,E;F) & \longrightarrow & \mathscr{L}_1(M,\varphi,E;F) \\ u & \longmapsto & f(u) \end{array}$$

such that $\forall(x,y) \in L_\infty(M,\varphi) \times E$,

$$f(u)(x)(y) = \left\{ \begin{array}{c} u(x)(y) \text{ if } x \in L_1(M,\varphi) \\ z_0 \text{ if not} \end{array} \right.$$

Then we have:

$$\begin{array}{rcl} \|f(u)\|_{\varphi,1} & = & \sup\{\|f(u)(x)(y)\|_F : \varphi(|x|) \leq 1, \|y\|_E \leq 1\} \\ & \geq & \sup\{\|f(u)(x)(y)\|_F : \|x\|_\infty \leq 1, \|y\|_E \leq 1\} \\ & \geq & \sup\{\|u(x)(y)\|_F : \|x\|_\infty \leq 1, \|y\|_E \leq 1\} \\ & \geq & \|u\|_{\varphi,\infty} \end{array}$$

□

This theorem allows as to view the pair $(\mathscr{L}_\infty(M,\varphi,E;F), \mathscr{L}_1(M,\varphi,E;F))$ as a compatible couple of Banach spaces and so we can apply the complex interpolation method to define a new Banach space.

Definition 3.10. *Let φ be a semi-finite normal faithful trace on an injective Von Neumann algebra $M \subset B(H)$ and let $E, F \subset B(H)$ two operators spaces. If $1 < p < \infty$, we define*

$$\mathscr{L}_p(M,\varphi,E;F) = (\mathscr{L}_\infty(M,\varphi,E;F), \mathscr{L}_1(M,\varphi,E;F))_\theta$$

where $\theta = \dfrac{1}{p}$.

Theorem 3.11. *$\mathscr{L}_1(M,\varphi,E;F)$ is isomorphic to $\mathcal{L}(L_1(M,\varphi)\otimes E, F)$ and $\mathscr{L}_\infty(M,\varphi,E;F)$ is isomorphic to $\mathcal{L}(M \otimes E, F)$:*

$$\mathscr{L}_1(M,\varphi,E;F) \simeq \mathcal{L}(L_1(M,\varphi) \otimes E, F),$$

$$\mathscr{L}_\infty(M,\varphi,E;F) \simeq \mathcal{L}(M \otimes E, F).$$

Proof. We want to prove firstly that

$$\mathscr{L}_1(M,\varphi,E;F) \simeq \mathcal{L}(L_1(M,\varphi) \otimes E, F).$$

Let us consider the map:

$$\begin{array}{rll} \mathcal{F} : & \mathscr{L}_1(M,\varphi,E;F) & \longrightarrow \mathcal{L}(L_1(M,\varphi) \otimes E, F) \\ & u & \longmapsto \mathcal{F}(u) \end{array}$$

such that

$$\mathcal{F}(u)(x \otimes y) = u(x)(y)$$

i) Linearity:
We have

$$\begin{aligned}\mathcal{F}(\lambda u)(x\otimes y) &= (\lambda u)(x)(y)\\ &= \lambda u(x)(y)\\ &= \lambda\mathcal{F}u(x\otimes y)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(u+v)(x\otimes y) &= (u+v)(x)(y)\\ &= u(x)(y)+v(x)(y)\\ &= \mathcal{F}(u)(x\otimes y)+\mathcal{F}(v)(x\otimes y)\\ &= [\mathcal{F}(u)+\mathcal{F}(v)](x\otimes y)\end{aligned}$$

ii) $\mathcal{F}$ is bijective
For all $v\in\mathcal{L}(L_1(M,\varphi)\otimes E,F)$ and $x\in L_1(M,\varphi)$, set:
$\mathcal{G}(v)(x)$ the mapping $y\mapsto v(x\otimes y)$, where $y\in E$. This mapping is an element of $\mathcal{L}(E,F)$. In fact for x fixed, $y\mapsto v(x\otimes y)$ is linear, and is bounded since v is a bounded linear map. Thus, we get a map:

$$\mathcal{G}(v):L_1(M,\varphi)\longrightarrow\mathcal{L}(E,F).$$

Now we want to prove that this map is linear. For all $x,x'\in L_1(M,\varphi)$, $y\in E$,

$$\begin{aligned}\mathcal{G}(v)(x+\lambda x')(y) &= v((x+\lambda x')\otimes y)\\ &= v(x\otimes y+\lambda(x'\otimes y))\\ &= v(x\otimes y)+\lambda v(x'\otimes y)\\ &= \mathcal{G}(v)(x)(y)+\lambda\mathcal{G}(v)(x')(y)\\ &= [\mathcal{G}(v)(x)+\lambda\mathcal{G}(v)(x')](y)\end{aligned}$$

Since $\forall y,\mathcal{G}(v)(x+\lambda x')(y)=[\mathcal{G}(v)(x)+\lambda\mathcal{G}(v)(x')](y)$, then

$$\mathcal{G}(v)(x+\lambda x')=\mathcal{G}(v)(x)+\lambda\mathcal{G}(v)(x'),$$

and $\mathcal{G}(v)$ is linear. By this, we claim that $\mathcal{G}$ is a map from $\mathcal{L}(L_1(M,\varphi)\otimes E,F)$ into $\mathscr{L}_1(M,\varphi,E;F)$.
Finally we prove that $\mathcal{G}$ is the inverse of $\mathcal{F}$:

$$\begin{aligned}(\mathcal{G}\circ\mathcal{F})(u)(x\otimes y) &= \mathcal{G}[\mathcal{F}(u)(x\otimes y)]\\ &= \mathcal{G}(u(x)(y))\\ &= u(x\otimes y)\end{aligned}$$

Since $(\mathcal{G}\circ\mathcal{F})(u)(x\otimes y)=u(x\otimes y)$, then

$$(\mathcal{G}\circ\mathcal{F})(u)=u$$

$$\begin{aligned}(\mathcal{F}\circ\mathcal{G})(v)(x)(y) &= \mathcal{F}[\mathcal{G}(v)(x)(y)]\\ &= \mathcal{F}(v(x\otimes y))\\ &= v(x)(y)\end{aligned}$$

Since $(\mathcal{F}\circ\mathcal{G})(v)(x)(y)=v(x)(y)$, then

$$(\mathcal{G}\circ\mathcal{F})(v)=v.$$

So $\mathscr{L}_1(M,\varphi,E;F)\simeq\mathcal{L}(L_1(M,\varphi)\otimes E,F)$.

The proof of the second isomorphism use the same method by replacing for instance $L_1(M,\varphi)$ by M and $\mathscr{L}_1(M,\varphi,E;F)$ by $\mathscr{L}_\infty(M,\varphi,E;F)$. □

Corollary 3.12.

$$\mathscr{L}_1(M,\varphi,E;F) \simeq \mathcal{L}(L_1(M,\varphi),E),F)$$

and

$$\mathscr{L}_\infty(M,\varphi,E;F) \simeq \mathcal{L}(L_\infty(M,\varphi,E),F)$$

Proof.

$$\begin{array}{rcl}\mathscr{L}_1(M,\varphi,E;F) & \simeq & \mathcal{L}(L_1(M,\varphi)\otimes E,F) \\ & \simeq & \mathcal{L}(L_1(M,\varphi)\widehat{\otimes}E,F) \\ & \simeq & \mathcal{L}(L_1(M,\varphi,E),F)\end{array}$$

and

$$\begin{array}{rcl}\mathscr{L}_\infty(M,\varphi,E;F) & \simeq & \mathcal{L}(M\otimes E,F) \\ & \simeq & \mathcal{L}(M\otimes_{\min} E,F) \\ & \simeq & \mathcal{L}(L_\infty(M,\varphi,E),F)\end{array}$$

□

Corollary 3.13. *For all $1 < p < \infty$, $\mathscr{L}_p(M,\varphi,E;F)$ is isomorphic to $(\mathcal{L}(L_\infty(M,\varphi,E),F),\mathcal{L}(L_1(M,\varphi,E),F))_\theta$:*

$$\mathscr{L}_p(M,\varphi,E;F) \simeq (\mathcal{L}(L_\infty(M,\varphi,E),F),\mathcal{L}(L_1(M,\varphi,E),F))_\theta$$

with $\theta = \dfrac{1}{p}$

Proof. This is a direct consequence of Definition 3.10 and Theorem 3.11. □

Corollary 3.14. *for $1 \leq p < \infty$ $\mathscr{L}_p(M,\varphi,\mathbb{C};\mathbb{C})$ is isomorphic to $L_q(M,\varphi)$:*

$$\mathscr{L}_p(M,\varphi,\mathbb{C};\mathbb{C}) \simeq L_q(M,\varphi)$$

where q is such that $\dfrac{1}{p}+\dfrac{1}{q}=1$ (called the conjugate of p).

Proof.

$$\begin{array}{rcl}\mathscr{L}_p(M,\varphi,\mathbb{C};\mathbb{C}) & = & (\mathscr{L}_\infty(M,\varphi,\mathbb{C};\mathbb{C}),\mathscr{L}_1(M,\varphi,\mathbb{C};\mathbb{C}))_\theta \\ & = & (\mathcal{L}(L_\infty(M,\varphi),\mathcal{L}(\mathbb{C};\mathbb{C})),\mathcal{L}(L_1(M,\varphi),\mathcal{L}(\mathbb{C};\mathbb{C})))_\theta \\ & \simeq & (\mathcal{L}(L_\infty(M,\varphi),\mathbb{C})),\mathcal{L}(L_1(M,\varphi),\mathbb{C})))_\theta \\ & \simeq & ((L_\infty(M,\varphi))',(L_1(M,\varphi))')_\theta \\ & \simeq & ((L_1(M,\varphi)),(L_\infty(M,\varphi)))_\theta \\ & \simeq & ((L_\infty(M,\varphi)),(L_1(M,\varphi)))_{1-\theta} \\ & \simeq & ((L_\infty(M,\varphi)),(L_1(M,\varphi)))_{1/q} \\ & \simeq & L_q(M,\varphi)\end{array}$$

□

3.4. **Operator Space structure.** For any integer $n \in \mathbb{N}^*$, we identify $M_n(\mathscr{L}_1(M,\varphi,E;F))$ with $\mathscr{L}_1(M,\varphi,E;M_n(F))$:

$$M_n(\mathscr{L}_1(M,\varphi,E;F)) \approx \mathscr{L}_1(M,\varphi,E;M_n(F)),$$

via the correspondence

$$\begin{array}{rcl}M_n(\mathscr{L}_1(M,\varphi,E;F)) & \to & \mathscr{L}_1(M,\varphi,E;M_n(F)) \\ (\varphi_{kl})_{1\leq k,l\leq n} & \mapsto & \varphi_n\end{array}$$

where φ_n is an element of $\mathscr{L}_1(M,\varphi,E;M_n(F))$ defined by
$\forall (x,y) \in L_\infty(M,\varphi)\times E, \quad \varphi_n(x,y) = (\varphi_{kl}(x,y))_{1\leq k,l\leq n}.$

Now, by identifying $M_n(\mathscr{L}_1(M,\varphi,E;F))$ with $\mathscr{L}_1(M,\varphi,E;M_n(F))$, we can set an operator space structure on $\mathscr{L}_1(M,\varphi,E;F)$. Thus, the norm $\|\cdot\|_n$ in $M_n(\mathscr{L}_1(M,\varphi,E;F))$ is the one defined on $\mathscr{L}_1(M,\varphi,E;M_n(F))$. In fact, $M_n(F)$ being an operator space, $\mathscr{L}_1(M,\varphi,E;M_n(F))$ is well-defined as a Banach space and so the norm $\|\cdot\|_n$ is complete for all $n\in\mathbb{N}^*$. Let us prove that the two conditions of Ruan theorem are realised:
Let $n,m\in\mathbb{N}^*$, $u\in M_n(\mathscr{L}_1(M,\varphi,E;F))$, $v\in M_m(\mathscr{L}_1(M,\varphi,E;F))$ and $\alpha,\beta\in M_n$.

First condition:

$$\begin{aligned}
\|u\oplus v\|_{n+m} &= \|u\oplus v\|_{\mathscr{L}_1(M,\varphi,E;M_{n+m}(F))}\\
&= \sup\{\|(u\oplus v)(x,y)\|_{M_{n+m}(F)} : \varphi(|x|)\le 1, \|y\|_E\le 1\}\\
&= \sup\{\max\left\{\|u(x,y)\|_{M_n(F)}, \|v(x,y)\|_{M_m(F)}\right\} : \varphi(|x|)\le 1, \|y\|_E\le 1\}\\
&= \sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\left\{\max\left\{\|u(x,y)\|_{M_n(F)}, \|v(x,y)\|_{M_m(F)}\right\}\right\}\\
&= \max\left\{\sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\left\{\|u(x,y)\|_{M_n(F)}\right\}, \sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\left\{\|v(x,y)\|_{M_m(F)}\right\}\right\}\\
&= \max\left\{\|u\|_{\mathscr{L}_1(M,\varphi,E;M_n(F))}, \|v\|_{\mathscr{L}_1(M,\varphi,E;M_m(F))}\right\}
\end{aligned}$$

Second condition:

$$\begin{aligned}
\|\alpha u\beta\|_n &= \|\alpha u\beta\|_{\mathscr{L}_1(M,\varphi,E;M_n(F))}\\
&= \sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\|(\alpha u\beta)(x,y)\|_{M_n(F)}\\
&= \sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\|\alpha(u(x,y))\beta\|_{M_n(F)}
\end{aligned}$$

$u(x,y)$ being in $M_n(F)$, where F is an operator space, according to the second condition of Ruan theorem, we have obviously

$$\|\alpha(u(x,y))\beta\|_{M_n(F)}\le\|\alpha\|\|u(x,y)\|_{M_n(F)}\|\beta\|.$$

So

$$\begin{aligned}
\|\alpha u\beta\| &\le \sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\|\alpha\|\|u(x,y)\|_{M_n(F)}\|\beta\|\\
&\le \|\alpha\|\left(\sup_{\varphi(|x|)\le 1,\|y\|_E\le 1}\|u(x,y)\|_{M_n(F)}\right)\|\beta\|\\
&\le \|\alpha\|\|u\|_n\|\beta\|.
\end{aligned}$$

Using the same method, we give to $\mathscr{L}_\infty(M,\varphi,E;F)$ an operator space structure by identifying $M_n(\mathscr{L}_\infty(M,\varphi,E;F))$ with $\mathscr{L}_\infty(M,\varphi,E;M_n(F))$.

In the following, $\mathscr{L}_1(M,\varphi,E;F)$ and $\mathscr{L}_\infty(M,\varphi,E;F)$ are viewed as operator spaces and by interpolation, we define for $1<p<\infty$, the operator space:

$$\mathscr{L}_p(M,\varphi,E;F)=(\mathscr{L}_\infty(M,\varphi,E;F),\mathscr{L}_1(M,\varphi,E;F))_\theta$$

where $\theta=\dfrac{1}{p}$.

The following theorem is the analogous of Pisier's Theorem that we've recalled in the previous sequence (Theorem 2.5).

Theorem 3.15. *Let (M,φ) be any n.c.p. space. Let E be an operator space. If E' has the $ORNP_q$ with $1<p<\infty$ and $q=\frac{p}{p-1}$, then we have a completely isometric identity*

$$\mathscr{L}_p(M,\varphi,E;\mathbb{C})=L_q(M,\varphi,E').$$

Proof.

$$\begin{aligned}\mathscr{L}_p(M,\varphi,E;\mathbb{C}) &= \mathcal{L}(L_p(M,\varphi,E),\mathbb{C})\\ &= (L_p(M,\varphi,E))'\\ &= (L_q(M,\varphi,E')\end{aligned}$$

□

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