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Hankel Determinant of Logarithmic Coefficients for Tilted Starlike Functions With Respect to Conjugate Points

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Abstract. The growth of the Hankel determinant whose elements are logarithmic coefficients for different subclasses of univalent functions has recently attracted considerable interest. In this paper, we obtain the bounds for the first four initial logarithmic coefficients for the subclass of starlike functions with respect to conjugate points in an open unit disk. Furthermore, we determine the upper bounds of the second Hankel determinant of logarithmic coefficients for this subclass. We also present some new consequences of our results.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in an open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ which satisfy $f(0) = f'(0) - 1 = 0$ and has the series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in E. \tag{1.1}$$

We also denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E.

Let P be the class of analytic functions $p(z)$ defined in E which satisfy $\operatorname{Re} p(z) > 0$ and has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \ z \in E. \tag{1.2}$$

This class is also known as the class of Carathéodory functions.

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We denote H as the class of Schwarz functions $\omega(z)$ defined in E which satisfy $\omega(0) = 0$ and $|\omega(z)| < 1$, and has the series representation

$$\omega(z) = \sum_{k=1}^{\infty} b_k z^k, \ z \in E. \tag{1.3}$$

If there exists a Schwarz function $\omega(z) \in H$ such that $f(z) = g(\omega(z))$ for all $z \in E$, then the analytic function $f(z)$ is subordinate to another analytic function $g(z)$ and is symbolically written as $f(z) \prec g(z)$. Furthermore, if $g(z)$ is univalent in E, then $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(E) = g(E)$.

In [1], El-Ashwah and Thomas introduced the class of functions that are starlike with respect to conjugate points. The class is denoted by ${S_C}^*$ which satisfies

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}\right\} > 0, \ z \in E. \tag{1.4}$$

In [2], Halim introduced the class ${S_C}^*(\delta)$ consisting of functions of the form (1.1) and satisfying

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}\right\} > \delta, 0 \leqslant \delta < 1, \ z \in E. \tag{1.5}$$

By means of subordination, Dahhar and Janteng [3] introduced the class ${S_C}^*(A, B)$ consisting of functions of the form (1.1) and satisfying

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1+Az}{1+Bz}, \ -1 \leqslant B < A \leqslant 1, \ z \in E. \tag{1.6}$$

From (1.6), it follows that $f(z) \in {S_C}^*(A, B)$ if and only if

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \frac{1+A\omega(z)}{1+B\omega(z)}, \ \omega(z) \in H. \tag{1.7}$$

In [4], Wahid et al. introduced the class ${S_C}^*(\alpha, \delta)$ consisting of functions of the form (1.1) and satisfying

$$\operatorname{Re}\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}\right\} > \delta, \ 0 \leqslant \delta < \ 1, \ |\alpha| < \frac{\pi}{2}, \ z \in E, \tag{1.8}$$

where $g(z) = \frac{f(z)+\overline{f(\overline{z})}}{2}$. The functions in the class ${S_C}^*(\alpha, \delta)$ are known as tilted starlike functions with respect to conjugate points of order δ. In terms of subordination, they defined the class ${S_C}^*(\alpha, \delta, A, B)$ which satisfies

$$\left\{e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}} \prec \frac{1+Az}{1+Bz}, \ -1 \leqslant B < A \leqslant 1, \ z \in E, \tag{1.9}$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

From (1.9), it follows that $f(z) \in {S_C}^*(\alpha, \delta, A, B)$ if and only if

$$\left\{e^{i\alpha}\frac{zf'(z)}{g(z)} - \delta - i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}} = \frac{1+A\omega(z)}{1+B\omega(z)}, \ \omega(z) \in H. \tag{1.10}$$

Remark 1.1. *It is observed that by choosing the specific values of the parameters α, δ, A and B in the class ${S_C}^*(\alpha, \delta, A, B)$ leads to the following classes:*

(a) If we let $\alpha = \delta = 0, A = 1$ and $B = -1$, then the class $S_C{}^*(\alpha, \delta, A, B)$ reduces to the class $S_C{}^*$ given in (1.4).

(b) If we let $\alpha = 0, A = 1$ and $B = -1$, then the class $S_C{}^*(\alpha, \delta, A, B)$ reduces to the class $S_C{}^*(\delta)$ given in (1.5).

(c) If we replace $\alpha = \delta = 0$, then the class $S_C{}^*(\alpha, \delta, A, B)$ reduces to the class $S_C{}^*(A, B)$ given in (1.6).

(d) If we replace $A = 1$ and $B = -1$, then the class $S_C{}^*(\alpha, \delta, A, B)$ reduces to the class $S_C{}^*(\alpha, \delta)$ given in (1.8).

Aside from that, many interesting results especially related to coefficients of $f(z) \in S$ for various subclasses of starlike functions with respect to symmetric points, symmetric conjugate points and conjugate points were obtained by several authors. We may point interested readers to recent advances in these subclasses and their further results which point in a different direction than the current study, see for example, coefficient estimates [3,5,6], Fekete-Szegö inequality [7], Hankel determinant [8–12], Toeplitz determinant [13] and Zalcman coefficient functional [8,14].

The logarithmic coefficients γ_n, $n \geqslant 1$ for a function $f(z) \in S$ of the form (1.1) play an important role in Milin's conjecture [15,16], Brennan's conjecture [17] and can also be used to find estimations for the coefficients of an inverse function. It is given in the series representation

$$F_f(z) := \log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n, \; z \in E. \tag{1.11}$$

By differentiating (1.11) and comparing the coefficients of z^n, the logarithmic coefficients γ_n, $n = 1, 2, 3, 4$ are given as follows:

$$\gamma_1 = \frac{1}{2}a_2, \tag{1.12}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}{a_2}^2\right), \tag{1.13}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}{a_2}^3\right) \tag{1.14}$$

and

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2 a_4 + {a_2}^2 a_3 - \frac{1}{2}{a_3}^2 - \frac{1}{4}{a_2}^4\right). \tag{1.15}$$

The growth of the inequalities problems related to the upper bound of the Hankel determinant has been studied for different subclasses of $\mathcal{A}$. The Hankel determinant was defined by Pommerenke [18,19] for a function $f(z) \in S$ of the form (1.1) which is given by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \tag{1.16}$$

where $n,\ q \in \mathbb{N}$. It is very useful for example in the theory of singularities [20] and in the study of power series with integral coefficients.

Recently, Kowalczyk and Lecko [21, 22] proposed the study of the Hankel determinant whose elements are logarithmic coefficients of $f(z) \in S$ which is given by

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \tag{1.17}$$

The results of $H_{q,n}(F_f/2)$ broaden the knowledge of logarithmic coefficients for different subclasses of S. In particular, for values of $q = 2,\ n = 1$ and $q = 2,\ n = 2$, respectively, we have

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - {\gamma_2}^2 \tag{1.18}$$

and

$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - {\gamma_3}^2. \tag{1.19}$$

The problem of finding the upper bounds of $|\gamma_n|$ and $|H_{q,n}(F_f/2)|$ has been considered for some subclasses of univalent functions. Some significant contributions have been obtained recently to these problems; see for instance [8, 23–31]. However, as far as we know, no one has used the coefficients of logarithmic functions to obtain the bound for the second Hankel determinant for the classes ${S_C}^*$, ${S_C}^*(\delta)$, ${S_C}^*(A, B)$ and ${S_C}^*(\alpha, \delta)$. Thus, in this paper, we continue the research dealing with the logarithmic coefficients and the Hankel determinant of logarithmic coefficients for the class ${S_C}^*(\alpha, \delta, A, B)$ introduced in (1.9). Our main aim is to obtain the upper bounds of the logarithmic coefficients $|\gamma_n|$, $n = 1, 2, 3, 4$ and the second Hankel determinants of logarithmic coefficients, i.e., $|H_{2,1}(F_f/2)|$ and $|H_{2,2}(F_f/2)|$. Furthermore, we give several new consequences of our results based on the special choices of the involved parameters.

2. Preliminary results

In this section, we present some lemmas which will be used to prove our main results.

Lemma 2.1. ([15]) *Let $p(z) \in P$ of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$. Then*

$$|p_n| \leqslant 2,\ \ n \geqslant 1.$$

The inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$.

Lemma 2.2. ([32]) *Let $p(z) \in P$ of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2max\{1, |2\mu - 1|\},\ \ 1 \leqslant k \leqslant n - 1.$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

Lemma 2.3. ([33]) *Let $p(z) \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then*

$$\left| I{c_1}^3 - Xc_1c_2 + Vc_3 \right| \leqslant 2\left| I \right| + 2\left| X - 2I \right| + 2\left| I - X + V \right|,$$

where I, X and V are real numbers.

3. Main results

In this section, we find the estimate for initial logarithmic coefficients for functions belonging to the class ${S_C}^*(\alpha, \delta, A, B)$. Furthermore, we obtain the upper bounds of the second Hankel determinant of logarithmic coefficients for the case of $q = 2$ and $n = 1$, and $q = 2$ and $n = 2$ for functions from the class ${S_C}^*(\alpha, \delta, A, B)$.

Theorem 3.1. *If $f(z) \in {S_C}^*(\alpha, \delta, A, B)$ and has the series representation (1.1), then*

$$|\gamma_1| \leqslant \frac{T}{2}, \quad |\gamma_2| \leqslant \frac{T}{4}, \quad |\gamma_3| \leqslant \frac{T}{6} \text{ and } |\gamma_4| \leqslant \frac{T(1+2\Upsilon)}{8},$$

where $T = (A - B)t_{\alpha\delta}$, $t_{\alpha\delta} = \cos\alpha - \delta$ and $\Upsilon = 1 + B$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in {S_C}^*(\alpha, \delta, A, B)$. The coefficients a_n, $n = 2, 3, 4, 5$ are given by [14]

$$a_2 = \frac{k_1\xi}{2}, \tag{3.1}$$

$$a_3 = \frac{\xi}{8}\left[2k_2 + {k_1}^2(\xi - \Upsilon)\right], \tag{3.2}$$

$$a_4 = \frac{\xi}{48}\left[8k_3 + k_1k_2(6\xi - 8\Upsilon) + {k_1}^3\left(\xi^2 - 3\Upsilon\xi + 2\Upsilon^2\right)\right] \tag{3.3}$$

and

$$\begin{aligned} a_5 = \frac{\xi}{384} &\left[48k_4 + k_1k_3(32\xi - 48\Upsilon) + {k_2}^2(12\xi - 24\Upsilon) + {k_1}^4\left(\xi^3 - 6\Upsilon\xi^2 + 11\Upsilon^2\xi - 6\Upsilon^3\right)\right. \\ &\left. + {k_1}^2k_2\left(12\xi^2 - 44\Upsilon\xi + 36\Upsilon^2\right)\right], \end{aligned} \tag{3.4}$$

where $\xi = Te^{-i\alpha}$, $T = (A - B)t_{\alpha\delta}$, $t_{\alpha\delta} = \cos\alpha - \delta$ and $\Upsilon = 1 + B$.
Using (3.1)−(3.4), from (1.12)−(1.15), respectively, we obtain

$$\gamma_1 = \frac{k_1\xi}{4}, \tag{3.5}$$

$$\begin{aligned} \gamma_2 &= \frac{1}{2}\left[\frac{\xi}{8}\left(2k_2 + {k_1}^2(\xi - \Upsilon)\right) - \frac{{k_1}^2\xi^2}{8}\right] \\ &= \frac{\xi}{16}\left(2k_2 - {k_1}^2\Upsilon\right), \end{aligned} \tag{3.6}$$

$$\begin{aligned}\gamma_3 &= \frac{1}{2}\left[\frac{\xi}{48}\left(8k_3 + k_1k_2\left(6\xi - 8\Upsilon\right) + {k_1}^3\left(\xi^2 - 3\Upsilon\xi + 2\Upsilon^2\right)\right) - \frac{k_1\xi^2}{16}\left(2k_2 + {k_1}^2\left(\xi - \Upsilon\right)\right) + \frac{{k_1}^3\xi^3}{24}\right] \\ &= \frac{\xi}{48}\left({k_1}^3\Upsilon^2 - 4k_1k_2\Upsilon + 4k_3\right)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\gamma_4 = \frac{1}{2}\Bigg[&\frac{\xi}{384}\left(48k_4 + k_1k_3\left(32\xi - 48\Upsilon\right) + {k_2}^2\left(12\xi - 24\Upsilon\right) + {k_1}^4\left(\xi^3 - 6\Upsilon\xi^2 + 11\Upsilon^2\xi - 6\Upsilon^3\right)\right. \\ &\left. + {k_1}^2k_2\left(12\xi^2 - 44\Upsilon\xi + 36\Upsilon^2\right)\right) - \frac{k_1\xi^2}{96}\left(8k_3 + k_1k_2\left(6\xi - 8\Upsilon\right) + {k_1}^3\left(\xi^2 - 3\Upsilon\xi + 2\Upsilon^2\right)\right) \\ &+ \frac{{k_1}^2\xi^3}{32}\left(2k_2 + {k_1}^2\left(\xi - \Upsilon\right)\right) - \frac{\xi^2}{128}\left(4{k_2}^2 + 4{k_1}^2k_2\left(\xi - \Upsilon\right) + {k_1}^4(\xi - \Upsilon)^2\right) - \frac{{k_1}^4\xi^4}{64}\Bigg] \\ = \;&\frac{\xi}{128}\left(8k_4 - 4{k_2}^2\Upsilon - {k_1}^4\Upsilon^3 + 6{k_1}^2k_2\Upsilon^2 - 8k_1k_3\Upsilon\right).\end{aligned} \tag{3.8}$$

For γ_1, implementing Lemma 2.1 in (3.5), we obtain

$$|\gamma_1| \leqslant \frac{T}{2}.$$

For γ_2, γ_3 and γ_4, we can write (3.6)–(3.8), respectively, as

$$\gamma_2 = \frac{\xi}{8}\left(k_2 - \mu {k_1}^2\right), \tag{3.9}$$

$$\gamma_3 = \frac{\xi}{48}\left(I{k_1}^3 - Xk_1k_2 + Vk_3\right) \tag{3.10}$$

and

$$\gamma_4 = \frac{\xi}{128}\left(8\left(k_4 - \mu {k_2}^2\right) - k_1\left(I^*{k_1}^3 - X^*k_1k_2 + V^*k_3\right)\right), \tag{3.11}$$

where $\mu = \frac{\Upsilon}{2}$, $I = \Upsilon^2$, $X = 4\Upsilon$, $V = 4$, $I^* = \Upsilon^3$, $X^* = 6\Upsilon^2$ and $V^* = 8\Upsilon$.
Implementing Lemma 2.2 in (3.9), Lemma 2.3 in (3.10) and both Lemma 2.2 and Lemma 2.3 in (3.11), and application of triangle inequality, respectively, we get

$$|\gamma_2| \leqslant \frac{T}{4},$$

$$|\gamma_3| \leqslant \frac{T}{6}$$

and

$$|\gamma_4| \leqslant \frac{T\left(1 + 2\Upsilon\right)}{8}.$$

This completes the proof. $\square$

Theorem 3.2. *If $f(z) \in S_C^*(\alpha, \delta, A, B)$ and has the series representation (1.1), then*

$$|H_{2,1}(F_f/2)| \leqslant \frac{7T^2}{48},$$

where $T = (A-B)t_{\alpha\delta}$, $t_{\alpha\delta} = \cos\alpha - \delta$ and $\Upsilon = 1 + B$.

Proof. Using (3.5)–(3.7), from (1.18), we have

$$H_{2,1}(F_f/2) = \frac{k_1\xi^2}{192}\left(k_1{}^3\Upsilon^2 - 4k_1k_2\Upsilon + 4k_3\right) - \frac{\xi^2}{256}\left(4k_2{}^2 - 4k_1{}^2k_2\Upsilon + k_1{}^4\Upsilon^2\right). \tag{3.12}$$

Hence, simplifying (3.12), we can write it as

$$H_{2,1}(F_f/2) = \frac{\xi^2}{768}\left(k_1\left(Ik_1{}^3 - Xk_1k_2 + V^{**}k_3\right) - 12k_2{}^2\right), \tag{3.13}$$

where $I = \Upsilon^2$, $X = 4\Upsilon$ and $V^{**} = 16$.

Thus, applying Lemma 2.1 and Lemma 2.3, and by triangle inequality implies that

$$|H_{2,1}(F_f/2)| \leqslant \frac{7T^2}{48}.$$

This completes the proof. □

Theorem 3.3. *If $f(z) \in S_C^*(\alpha, \delta, A, B)$ and has the series representation (1.1), then*

$$|H_{2,2}(F_f/2)| \leqslant \frac{T^2\left(2\Upsilon^2 + 9\Upsilon + 17\right)}{288},$$

where $T = (A-B)t_{\alpha\delta}$, $t_{\alpha\delta} = \cos\alpha - \delta$ and $\Upsilon = 1 + B$.

Proof. Substituting (3.9)–(3.11) in (1.19) and after simplification, we get

$$\begin{aligned} H_{2,2}(F_f/2) &= \frac{\xi^2}{2048}\left(2k_2 - k_1{}^2\Upsilon\right)\left(8k_4 - 4k_2{}^2\Upsilon - k_1{}^4\Upsilon^3 + 6k_1{}^2k_2\Upsilon^2 - 8k_1k_3\Upsilon\right) \\ &\quad - \frac{\xi^2}{2304}\left(k_1{}^3\Upsilon^2 - 4k_1k_2\Upsilon + 4k_3\right)^2 \\ &= \frac{\xi^2}{18432}\left(k_1{}^6\Upsilon^4 - 8k_1{}^4k_2\Upsilon^3 + 8k_1{}^3k_3\Upsilon^2 + 144k_2k_4 - 72k_1{}^2k_4\Upsilon - 72k_2{}^3\Upsilon\right. \\ &\quad \left. + 16k_1{}^2k_2{}^2\Upsilon^2 - 128k_3{}^2 + 112k_1k_2k_3\Upsilon\right). \end{aligned} \tag{3.14}$$

By rearranging the terms in (3.14), we may write

$$\begin{aligned} H_{2,2}(F_f/2) = \frac{\xi^2}{18432}&\left[k_1{}^3\Upsilon^2\left(Ik_1{}^3 - V^*k_1k_2 + X^*k_3\right) + 144k_4\left(k_2 - \mu k_1{}^2\right) - 72k_2{}^2\Upsilon\left(k_2 - \nu k_1{}^2\right)\right. \\ &\left. - 128k_3\left(k_3 - \eta k_1k_2\right)\right], \end{aligned} \tag{3.15}$$

where $I = \Upsilon^2$, $X^* = 8$, $V^* = 8\Upsilon$, $\mu = \frac{\Upsilon}{2}$, $\nu = \frac{2\Upsilon}{9}$ and $\eta = \frac{7\Upsilon}{8}$.

Thus, implementing Lemma 2.2 and Lemma 2.3, and by triangle inequality, (3.15) yields

$$|H_{2,2}(F_f/2)| \leqslant \frac{T^2\left(2\Upsilon^2 + 9\Upsilon + 17\right)}{288}.$$

This completes the proof. □

Upon choosing the specific values of the parameters α, δ, A and B in Theorem 3.1, Theorem 3.2 and Theorem 3.3, respectively, we get the following consequences:

Corollary 3.1.

(a) Let $S_C^*(0,0,1,-1) \equiv S_C^*$. *Then we have*

$$|\gamma_1| \leqslant 1, \quad |\gamma_2| \leqslant \frac{1}{2}, \quad |\gamma_3| \leqslant \frac{1}{3} \text{ and } |\gamma_4| \leqslant \frac{1}{4}.$$

(b) Let $S_C^*(0,\delta,1,-1) \equiv S_C^*(\delta)$. *Then we have*

$$|\gamma_1| \leqslant (1-\delta), \quad |\gamma_2| \leqslant \frac{(1-\delta)}{2}, \quad |\gamma_3| \leqslant \frac{(1-\delta)}{3} \text{ and } |\gamma_4| \leqslant \frac{(1-\delta)}{4}.$$

(c) Let $S_C^*(0,0,A,B) \equiv S_C^*(A,B)$. *Then we have*

$$|\gamma_1| \leqslant \frac{(A-B)}{2}, \quad |\gamma_2| \leqslant \frac{(A-B)}{4}, \quad |\gamma_3| \leqslant \frac{(A-B)}{6} \text{ and } |\gamma_4| \leqslant \frac{(A-B)(1+2\Upsilon)}{8}.$$

(d) Let $S_C^*(\alpha,\delta,1,-1) \equiv S_C^*(\alpha,\delta)$. *Then we have*

$$|\gamma_1| \leqslant t_{\alpha\delta}, \quad |\gamma_2| \leqslant \frac{t_{\alpha\delta}}{2}, \quad |\gamma_3| \leqslant \frac{t_{\alpha\delta}}{3} \text{ and } |\gamma_4| \leqslant \frac{t_{\alpha\delta}}{4}.$$

Corollary 3.2.

(a) Let $S_C^*(0,0,1,-1) \equiv S_C^*$. *Then we have*

$$|H_{2,1}(F_f/2)| \leqslant \frac{7}{12}.$$

(b) Let $S_C^*(0,\delta,1,-1) \equiv S_C^*(\delta)$. *Then we have*

$$|H_{2,1}(F_f/2)| \leqslant \frac{7(1-\delta)^2}{12}.$$

(c) Let $S_C^*(0,0,A,B) \equiv S_C^*(A,B)$. *Then we have*

$$|H_{2,1}(F_f/2)| \leqslant \frac{7(A-B)^2}{48}.$$

(d) Let $S_C^*(\alpha,\delta,1,-1) \equiv S_C^*(\alpha,\delta)$. *Then we have*

$$|H_{2,1}(F_f/2)| \leqslant \frac{7{t_{\alpha\delta}}^2}{12}.$$

Corollary 3.3.

(a) Let $S_C^*(0,0,1,-1) \equiv S_C^*$. *Then we have*

$$|H_{2,2}(F_f/2)| \leqslant \frac{17}{72}.$$

(b) Let $S_C^*(0,\delta,1,-1) \equiv S_C^*(\delta)$. *Then we have*

$$|H_{2,2}(F_f/2)| \leqslant \frac{17(1-\delta)^2}{72}.$$

(c) Let $S_C^*(0,0,A,B) \equiv S_C^*(A,B)$. *Then we have*

$$|H_{2,2}(F_f/2)| \leqslant \frac{(A-B)^2\left(2\Upsilon^2+9\Upsilon+17\right)}{288}.$$

(d) Let ${S_C}^*(\alpha,\delta,1,-1)\equiv {S_C}^*(\alpha,\delta)$*. Then we have*

$$|H_{2,2}(F_f/2)|\leqslant \frac{17{t_{\alpha\delta}}^2}{72}.$$

4. Conclusion

In this paper, we have obtained the upper bounds of the initial logarithmic coefficients and the second Hankel determinant of logarithmic coefficients for functions from the class ${S_C}^*(\alpha,\delta,A,B)$. It is shown in corollaries that the obtained results lead to new results for some existing subclasses, i.e., ${S_C}^*$, ${S_C}^*(\delta)$, ${S_C}^*(A,B)$ and ${S_C}^*(\alpha,\delta)$. Corollary 3.1(a) also coincides with the inequality $|\gamma_n|\leqslant \frac{1}{n}$, $n\geqslant 1$ that holds for the well-known class of starlike functions S^*. The results obtained could provide an opportunity for researchers to further investigate the third and fourth-order Hankel determinants of logarithmic coefficients, including other inequalities problems related to logarithmic coefficients for this class as well as other subclasses of S.

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