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EIGENVALUES FOR ITERATIVE SYSTEMS OF (n, p)-TYPE FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we determine the eigenvalue intervals of $\lambda_1, \lambda_2, \cdots, \lambda_n$ for which the iterative system of (n, p)-type fractional order two-point boundary value problem has a positive solution by an application of Guo-Krasnosel'skii fixed point theorem on a cone.

1. INTRODUCTION

The study of fractional order differential equations has emerged as an important area of mathematics. It has wide range of applications in various fields of science and engineering such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity. Recently, much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point boundary value problems (BVPs) associated with ordinary and fractional order differential equations. To mention the related papers along these lines, we refer to Erbe and Wang [4], Davis, Henderson, Prasad and Yin [3] for ordinary differential equations, Henderson and Ntouyas [6, 7], Henderson, Ntouyas and Purnaras [8, 9] for systems of ordinary differential equations, Bai and Lu [1], Zhang [17], Kauffman and Mboumi [10], Benchohra, Henderson, Ntoyuas and Ouahab [2], Su and Zhang [16], Khan, Rehman and Henderson[11], Prasad and Krushna [15] for fractional order differential equations.

This paper concerned with determining the eigenvalues $\lambda_i,\ 1 \le i \le n$, for which there exist positive solutions for the iterative system of (n, p)-type fractional order boundary value problems

$$\left.\begin{aligned} D_{0^+}^{\alpha} y_i(t) + \lambda_i a_i(t) f_i(y_{i+1}(t)) = 0,\ 1 \le i \le n,\ 0 < t < 1,\\ y_{n+1}(t) = y_1(t),\ 0 < t < 1, \end{aligned}\right\} \tag{1.1}$$

$$y_i^{(j)}(0) = 0,\ 0 \le j \le n-2,\ y_i^{(p)}(1) = 0, \tag{1.2}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional order derivative, $n-1 < \alpha \le n$ and $n \ge 3,\ 1 \le p \le \alpha - 1$ is a fixed integer.

By a positive solution of the fractional order BVP (1.1)-(1.2), we mean $(y_1(t), y_2(t), \cdots, y_n(t)) \in \left(C^{[\alpha]+1}[0,1]\right)^n$ satisfying (1.1)-(1.2) with $y_i(t) \ge 0, i = 1, 2, 3, \cdots n$, for

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all $t \in [0,1]$ and $(y_1(t), y_2(t), \cdots, y_n(t)) \neq (0, 0, \cdots, 0)$.

We assume the following conditions hold throughout the paper:

(A1) $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $1 \leq i \leq n$,

(A2) $a_i : [0,1] \to \mathbb{R}^+$ is continuous and a_i does not vanish identically on any closed subinterval of $[0,1]$, for $1 \leq i \leq n$,

(A3) each of

$$f_{i0} = \lim_{x \to 0^+} \frac{f_i(x)}{x} \text{ and } f_{i\infty} = \lim_{x \to \infty} \frac{f_i(x)}{x},$$

for $1 \leq i \leq n$, exists as positive real numbers.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous BVP and estimate the bounds for the Green's function. In Section 3, we establish criteria to determine the eigenvalues for which the fractional order BVP (1.1)-(1.2) has at least one positive solution in a cone by using Guo-Krasnosel'skii fixed point theorem. In Section 4, as an application, we demonstrate our results with an example.

2. Green's function and Bounds

In this section, we construct the Green's function for the homogeneous BVP and estimate the bounds for the Green's function which are needed in establishing the main results.

Lemma 2.1. *If $h(t) \in C[0,1]$, then the fractional order BVP,*

$$D^{\alpha}_{0^+} y_1(t) + h(t) = 0, \ t \in (0,1), \tag{2.1}$$

$$y_1^{(j)}(0) = 0, \ 0 \leq j \leq n-2, \ y_1^{(p)}(1) = 0 \tag{2.2}$$

has a unique solution,

$$y_1(t) = \int_0^1 G(t,s) h(s) ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-p}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-p} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.3}$$

Proof. Assume that $y_1(t) \in C^{[\alpha]+1}[0,1]$ is a solution of fractional order BVP (2.1)-(2.2) and is uniquely expressed as

$$I^{\alpha}_{0^+} D^{\alpha}_{0^+} y_1(t) = -I^{\alpha}_{0^+} h(t)$$

$$y_1(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \cdots + c_n t^{\alpha-n}.$$

From $y_1^{(j)}(0) = 0, \ 0 \leq j \leq n-2$, we have $c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Then

$$y_1(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1},$$

$$y_1^{(p)}(t) = c_1 \prod_{i=1}^{p} (\alpha - i) t^{\alpha-1-p} - \prod_{i=1}^{p} (\alpha - i) \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1-p} h(s) ds.$$

From $y_1^{(p)}(1) = 0$, we have

$$c_1 \prod_{i=1}^{p}(\alpha - i) - \prod_{i=1}^{p}(\alpha - i)\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1-p}h(s)ds = 0.$$

Therefore, $c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1-p}h(s)ds$. Thus, the unique solution of (2.1)-(2.2) is

$$\begin{aligned} y_1(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1-p}h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds, \end{aligned}$$

where $G(t, s)$ is given in (2.3). □

Lemma 2.2. *The Green's function $G(t, s)$ satisfies the following inequalities,*

$$(i)\ G(t,s) \geq 0,\ \ for\ all\ (t,s) \in [0,1]\times[0,1],$$
$$(ii)\ G(t,s) \leq G(1,s),\ \ for\ all\ (t,s) \in [0,1]\times[0,1],$$
$$(iii)\ G(t,s) \geq \frac{1}{4^{\alpha-1}}G(1,s),\ \ for\ all\ (t,s) \in I\times[0,1],$$

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Proof. The Green's function $G(t, s)$ is given in (2.3). For $0 \leq t \leq s \leq 1$.
$G(t,s) = \dfrac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1-p}] \geq 0.$
For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} G(t,s) =& \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1-p} - (t-s)^{\alpha-1}] \\ \geq& \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1-p} - t^{\alpha-1}(1-s)^{\alpha-1}] \\ =& \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-1-p}][1-(1-s)^p] \geq 0. \end{aligned}$$

Hence the inequality (i) is proved. We prove the inequality (ii). For $0 \leq t \leq s \leq 1$,

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{\Gamma(\alpha)}[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1-p}] \geq 0.$$

For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} \frac{\partial}{\partial t}G(t,s) =& \frac{1}{\Gamma(\alpha)}[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1-p} - (\alpha-1)(t-s)^{\alpha-2}] \\ =& \frac{(\alpha-1)}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-2}(1-s)^{1-p} - (t-s)^{\alpha-2}\right] \\ \geq& \frac{(\alpha-1)}{\Gamma(\alpha)}\left[t^{\alpha-2}(1-s)^{\alpha-2}(1-s)^{1-p} - (t-ts)^{\alpha-2}\right] \\ =& \frac{(\alpha-1)}{\Gamma(\alpha)}\left[(1-s)^{1-p} - 1\right](t-ts)^{\alpha-2} \geq 0. \end{aligned}$$

Therefore $G(t,s)$ is increasing with respect to $t \in [0,1]$. Hence the inequality (ii) is proved. Now, we establish the inequality (iii). For $0 \le t \le s \le 1$ and $t \in I$,

$$\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-1-p}}{(1-s)^{\alpha-1-p}} = t^{\alpha-1} \ge \frac{1}{4^{\alpha-1}}.$$

For $0 \le s \le t \le 1$ and $t \in I$,

$$\begin{aligned}\frac{G(t,s)}{G(1,s)} =& \frac{t^{\alpha-1}(1-s)^{\alpha-1-p} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-1-p} - (1-s)^{\alpha-1}} \\ \ge& \frac{t^{\alpha-1}(1-s)^{\alpha-1-p} - (t-ts)^{\alpha-1}}{(1-s)^{\alpha-1-p} - (1-s)^{\alpha-1}} \\ =& t^{\alpha-1} \ge \frac{1}{4^{\alpha-1}}.\end{aligned}$$

Hence the inequality (iii) is proved. □

An n-tuple $(y_1(t), y_2(t), \cdots, y_n(t))$ is a solution of the BVP (1.1)-(1.2) if and only if $y_i(t) \in C^{[\alpha]+1}[0,1]$ satisfies the following equations

$$\begin{aligned} y_1(t) =& \lambda_1 \int_0^1 G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1,s_2)a_2(s_2)\cdots \\ & f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1},s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)ds_1 \end{aligned}$$

and

$$y_i(t) = \lambda_i \int_0^1 G(t,s)a_i(s)f_i(y_{i+1}(s))ds,\ 0 \le t \le 1,\ 2 \le i \le n,$$

where

$$y_{n+1}(t) = y_1(t),\ 0 \le t \le 1.$$

In establishing our main result, we will employ the following fixed point theorem due to Guo-Krasnosel'skii [5, 13].

Theorem 2.3. [5, 13] *Let X be a Banach Space, $P \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous operator such that either*

(i) $\| Tu \| \le \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \ge \| u \|,\ u \in P \cap \partial\Omega_2$, *or*
(ii) $\| Tu \| \ge \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \le \| u \|,\ u \in P \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Positive Solutions in a Cone

In this section, we establish criteria to determine the eigenvalues for which the fractional order BVP (1.1)-(1.2) has at least one positive solution in a cone.

Let $X = \{x : x \in C[0,1]\}$ be the Banach space equipped with the norm

$$\|x\| = \max_{0 \le t \le 1} |x(t)|.$$

Define a cone $P \subset X$ by

$$P = \Big\{x \in X \mid x(t) \ge 0 \text{ on } [0,1] \text{ and } \min_{t \in I} x(t) \ge \frac{1}{4^{\alpha-1}}\|x\|\Big\}.$$

Now, we define an integral operator $T : P \to X$, for $y_1 \in P$, by

$$\begin{aligned} Ty_1(t) = \lambda_1 \int_0^1 G(t, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2) \cdots \\ f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1. \end{aligned} \tag{3.1}$$

Notice from $(A1)$, $(A2)$ and Lemma 2.2 that, for $y_1 \in P$, $Ty_1(t) \geq 0$ on $[0, 1]$. And also, we have

$$\begin{aligned} Ty_1(t) \leq \lambda_1 \int_0^1 G(1, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2) \cdots \\ f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1 \end{aligned}$$

so that

$$\begin{aligned} \|Ty_1\| \leq \lambda_1 \int_0^1 G(1, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2) \cdots \\ f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1. \end{aligned} \tag{3.2}$$

Next, if $y_1 \in P$, we have from Lemma 2.2 and (3.2) that

$$\begin{aligned} \min_{t\in I} Ty_1(t) &= \min_{t\in I} \lambda_1 \int_0^1 G(t, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2) \cdots \\ &\quad f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1 \\ &\geq \lambda_1 \frac{1}{4^{\alpha-1}} \int_0^1 G(1, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2) \cdots \\ &\quad f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1 \\ &\geq \frac{1}{4^{\alpha-1}} \|Ty_1\|. \end{aligned}$$

Therefore,

$$\min_{t\in I} Ty_1(t) \geq \frac{1}{4^{\alpha-1}} \|Ty_1\|.$$

Hence, $Ty_1 \in P$ and so $T : P \to P$. Further, the operator T is a completely continuous operator by an application of the Arzela-Ascoli Theorem.

Now, we seek suitable fixed point of T belonging to the cone P. For our first result, we define positive numbers N_1 and N_2, by

$$N_1 = \max_{1\leq i\leq n} \left\{ \left[\frac{1}{4^{\alpha-1}} \int_{s\in I} G(1, s)a_i(s)ds f_{i\infty} \right]^{-1} \right\}$$

and

$$N_2 = \min_{1\leq i\leq n} \left\{ \left[\int_0^1 G(1, s)a_i(s)ds f_{i0} \right]^{-1} \right\}.$$

Theorem 3.1. *Assume that the conditions* (*A*1)-(*A*3) *are satisfied. Then, for each* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *satisfying*

$$N_1 < \lambda_i < N_2,\ 1 \le i \le n, \tag{3.3}$$

there exists an n-tuple $(y_1, y_2, \cdots, y_n)$ *satisfying* (1.1)-(1.2) *such that* $y_i(t) > 0$, $1 \le i \le n$ *on* $(0, 1)$.

Proof. Let λ_i, $1 \le i \le n$ be given as in (3.3). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1\le i\le n}\left\{\left[\frac{1}{4^{\alpha-1}}\int_{s\in I} G(1,s)a_i(s)ds(f_{i\infty}-\epsilon)\right]^{-1}\right\} \le \min_{1\le i\le n}\lambda_i$$

and

$$\max_{1\le i\le n}\lambda_i \le \min_{1\le i\le n}\left\{\left[\int_0^1 G(1,s)a_i(s)ds(f_{i0}+\epsilon)\right]^{-1}\right\}.$$

We seek fixed point of the completely continuous operator $T : P \to P$ defined by (3.1). Now, from the definitions of f_{i0}, $1 \le i \le n$, there exists an $H_1 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \le (f_{i0} + \epsilon)x,\ 0 < x \le H_1.$$

Let $y_1 \in P$ with $\|y_1\| = H_1$. We first have from Lemma 2.2 and the choice of ϵ, for $0 \le s_{n-1} \le 1$,

$$\begin{aligned}\lambda_n &\int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\\ &\le \lambda_n \int_0^1 G(1, s_n)a_n(s_n)(f_{n0}+\epsilon)y_1(s_n)ds_n\\ &\le \lambda_n \int_0^1 G(1, s_n)a_n(s_n)ds_n(f_{n0}+\epsilon)\|y_1\|\\ &\le \|y_1\| = H_1.\end{aligned}$$

It follows in a similar manner from Lemma 2.2 and the choice of ϵ that, for $0 \le s_{n-2} \le 1$,

$$\begin{aligned}\lambda_{n-1} &\int_0^1 G(s_{n-2}, s_{n-1})a_{n-1}(s_{n-1})\\ &f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big)ds_{n-1}\\ &\le \lambda_{n-1}\int_0^1 G(s_{n-1}, s_{n-1})a_{n-1}(s_{n-1})ds_{n-1}(f_{n-1,0}+\epsilon)\|y_1\|\\ &\le \|y_1\| = H_1.\end{aligned}$$

Continuing with this bootstrapping argument, we have, for $0 \le t \le 1$,

$$\begin{aligned}\lambda_1 \int_0^1 G(t, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2)\cdots\\ f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)ds_1 \le H_1,\end{aligned}$$

so that, for $0 \le t \le 1$,

$$Ty_1(t) \le H_1.$$

Hence, $\|Ty_1\| \le H_1 = \|y_1\|$. If we set $\Omega_1 = \{x \in X \mid \|x\| < H_1\}$, then

$$\|Ty_1\| \le \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_1. \tag{3.4}$$

Next, from the definitions of $f_{i\infty}$, $1 \le i \le n$, there exists $\overline{H}_2 > 0$ such that, for each $1 \le i \le n$, $f_i(x) \ge (f_{i\infty} - \epsilon)x$, $x \ge \overline{H}_2$. Choose $H_2 = \max\{2H_1, 4^{\alpha-1}\overline{H}_2\}$. Let $y_1 \in P$ and $\|y_1\| = H_2$. Then,

$$\min_{t\in I} y_1(t) \ge \frac{1}{4^{\alpha-1}}\|y_1\| \ge \overline{H}_2.$$

Then, from Lemma 2.2 and choice of ϵ, for $0 \le s_{n-1} \le 1$, we have that

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n \\
&\ge \lambda_n \int_{s\in I} G(1, s_n)a_n(s_n)f_n(y_1(s_n))ds_n \\
&\ge \frac{1}{4^{\alpha-1}}\lambda_n \int_{s\in I} G(1, s_n)a_n(s_n)(f_{n\infty} - \epsilon)y_1(s_n)ds_n \\
&\ge \frac{1}{4^{\alpha-1}}\lambda_n \int_{s\in I} G(1, s_n)a_n(s_n)ds_n(f_{n\infty} - \epsilon)\|y_1\| \\
&\ge \|y_1\| = H_2.
\end{aligned}$$

It follows in a similar manner from Lemma 2.2 and choice of ϵ, for $0 \le s_{n-2} \le 1$,

$$\begin{aligned}
\lambda_{n-1} &\int_0^1 G(s_{n-2}, s_{n-1})a_{n-1}(s_{n-1}) \\
&f_{n-1}\Big(\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n\Big)ds_{n-1} \\
&\ge \frac{1}{4^{\alpha-1}}\lambda_{n-1} \int_{s\in I} G(1, s_{n-1})a_{n-1}(s_{n-1})ds_{n-1}(f_{n-1,\infty} - \epsilon)\|y_1\| \\
&\ge \|y_1\| = H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we have

$$\begin{aligned}
\lambda_1 \int_0^1 G(t, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1, s_2)a_2(s_2)\cdots \\
f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)ds_1 \ge H_2,
\end{aligned}$$

so that

$$Ty_1(t) \ge H_2 = \|y_1\|.$$

Hence, $\|Ty_1\| \ge \|y_1\|$. So if we set $\Omega_2 = \{x \in X \mid \|x\| < H_2\}$, then

$$\|Ty_1\| \ge \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_2. \tag{3.5}$$

Applying Theorem 2.3 to (3.4) and (3.5), we obtain that T has a fixed point $y_1 \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Setting $y_1 = y_{n+1}$, we obtain a positive solution $(y_1, y_2, \cdots, y_n)$ of (1.1)-(1.2) given iteratively by

$$y_i(t) = \lambda_i \int_0^1 G(t, s)a_i(s)f_i(y_{i+1}(s))ds, \ i = n, n-1, \cdots, 1.$$

The proof is completed. □

Prior to our next result, we define the positive numbers N_3 and N_4 by

$$N_3 = \max_{1\le i\le n}\left\{\left[\frac{1}{4^{\alpha-1}}\int_{s\in I} G(1,s)a_i(s)ds f_{i0}\right]^{-1}\right\}$$

and

$$N_4 = \min_{1\le i\le n}\left\{\left[\int_0^1 G(1,s)a_i(s)ds f_{i\infty}\right]^{-1}\right\}.$$

Theorem 3.2. *Assume that the conditions* (*A*1)-(*A*3) *are satisfied. Then, for each* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *satisfying*

$$N_3 < \lambda_i < N_4,\ 1 \le i \le n, \tag{3.6}$$

there exists an n*-tuple* $(y_1, y_2, \cdots, y_n)$ *satisfying* (1.1)-(1.2) *such that* $y_i(t) > 0$, $1 \le i \le n$ *on* $(0, 1)$.

Proof. Let λ_i, $1 \le i \le n$ be given as in (3.6). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1\le i\le n}\left\{\left[\frac{1}{4^{\alpha-1}}\int_{s\in I} G(1,s)a_i(s)ds(f_{i0}-\epsilon)\right]^{-1}\right\} \le \min_{1\le i\le n}\lambda_i$$

and

$$\max_{1\le i\le n}\lambda_i \le \min_{1\le i\le n}\left\{\left[\int_0^1 G(1,s)a_i(s)ds(f_{i\infty}+\epsilon)\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.1). From the definition of f_{i0}, $1 \le i \le n$ there exists $\overline{H}_3 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \ge (f_{i0} - \epsilon)x,\ 0 < x \le \overline{H}_3.$$

Also, from the definitions of f_{i0}, it follows that $f_{i0}(0) = 0,\ 1 \le i \le n$, and so there exist $0 < K_n < K_{n-1} < \cdots < K_2 < \overline{H}_3$ such that

$$\lambda_i f_i(t) \le \frac{K_{i-1}}{\int_0^1 G(1,s)a_i(s)ds},\ t \in [0, K_i],\ 3 \le i \le n,$$

and

$$\lambda_2 f_2(t) \le \frac{\overline{H}_3}{\int_0^1 G(1,s)a_2(s)ds},\ t \in [0, K_2].$$

Choose $y_1 \in P$ with $\|y_1\| = K_n$. Then, we have

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1}, s_n)a_n(s_n)f_n(y_1(s_n))ds_n \\
&\le \lambda_n \int_0^1 G(1, s_n)a_n(s_n)f_n(y_1(s_n))ds_n \\
&\le \frac{\int_0^1 G(1,s_n)a_n(s_n)K_{n-1}ds_n}{\int_0^1 G(1,s_n)a_n(s_n)ds_n} \\
&\le K_{n-1}.
\end{aligned}$$

Continuing with this bootstrapping argument, it follows that

$$\lambda_2 \int_0^1 G(1,s_2)a_2(s_2)f_2\Big(\lambda_3 \int_0^1 G(s_2,s_3)a_3(s_3)\cdots f_n(y_1(s_n))ds_n\Big)\cdots ds_3\Big)ds_2 \le \overline{H}_3.$$

Then,

$$\begin{aligned} Ty_1(t) &= \lambda_1 \int_0^1 G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^1 G(s_1,s_2)a_2(s_2)\cdots \\ &\quad f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)ds_1 \\ &\ge \frac{1}{4^{\alpha-1}}\lambda_1 \int_{s\in I} G(1,s_1)a_1(s_1)(f_{10}-\epsilon)\|y_1\|ds_1 \ge \|y_1\|. \end{aligned}$$

So, $\|Ty_1\| \ge \|y_1\|$. If we set $\Omega_1 = \{x \in X \mid \|x\| < K_n\}$, then

$$\|Ty_1\| \ge \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_1. \tag{3.7}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \le i \le n$, is unbounded at ∞. For each $1 \le i \le n$, set

$$f_i^*(x) = \sup_{0\le s\le x} f_i(s).$$

Then, it is straightforward that, for each $1 \le i \le n$, f_i^* is a nondecreasing real-valued function, $f_i \le f_i^*$ and

$$\lim_{x\to\infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \le i \le n$, there exists $\overline{H}_4$ such that, for each $1 \le i \le n$,

$$f_i^*(x) \le (f_{i\infty}+\epsilon)x, \;\; x \ge \overline{H}_4.$$

It follows that there exists $H_4 = \max\{2\overline{H}_3, \overline{H}_4\}$ such that, for each $1 \le i \le n$,

$$f_i^*(x) \le f_i^*(H_4), \;\; 0 < x \le H_4.$$

Choose $y_1 \in P$ with $\|y_1\| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned} Ty_1(t) &= \lambda_1 \int_0^1 G(t,s_1)a_1(s_1)f_1(\lambda_2\cdots)ds_1 \\ &\le \lambda_1 \int_0^1 G(t,s_1)a_1(s_1)f_1^*(\lambda_2\cdots)ds_1 \\ &\le \lambda_1 \int_0^1 G(1,s_1)a_1(s_1)f_1^*(H_4)ds_1 \\ &\le \lambda_1 \int_0^1 G(1,s_1)a_1(s_1)ds_1(f_{1\infty}+\epsilon)H_4 \\ &\le H_4 = \|y_1\|, \end{aligned}$$

and so $\|Ty_1\| \le \|y_1\|$. So, if we let $\Omega_2 = \{x \in X \mid \|x\| < H_4\}$, then

$$\|Ty_1\| \le \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_2. \tag{3.8}$$

Applying Theorem 2.3 to (3.7)-(3.8), we obtain that T has a fixed point $y_1 \in P \cap (\overline{\Omega}_2\backslash\Omega_1)$, which in turn with $y_1 = y_{n+1}$, yields an n-tuple $(y_1, y_2, \cdots, y_n)$

satisfying the BVP (1.1)-(1.2) for the chosen values of λ_i, $1 \le i \le n$. The proof is thus completed. □

4. Example

In this section, as an application, we demonstrate our results with an example. Consider the fractional order boundary value problem

$$\left.\begin{aligned} D_{0^+}^{2.5}y_1(t) + \frac{\lambda_1}{1+t}y_2(46 - 27.5e^{-2y_2})(500 - 487e^{-3y_2}) = 0, t \in (0,1),\\ D_{0^+}^{2.5}y_2(t) + \frac{\lambda_2}{1+t}y_3(37 - 25.5e^{-5y_3})(400 - 368e^{-y_3}) = 0, t \in (0,1),\\ D_{0^+}^{2.5}y_3(t) + \frac{\lambda_3}{1+t}y_1(79 - 75e^{-y_1})(800 - 749.5e^{-2y_1}) = 0, t \in (0,1),\end{aligned}\right\} \tag{4.1}$$

$$y_i(0) = 0,\ y_i'(0) = 0 \text{ and } y_i'(1) = 0,\ i = 1,2,3. \tag{4.2}$$

The Green's function $G(t,s)$ of corresponding homogeneous BVP is given by

$$G(t,s) = \begin{cases} \frac{t^{1.5}(1-s)^{0.5}}{\Gamma(2.5)}, & 0 \le t \le s \le 1,\\ \frac{t^{1.5}(1-s)^{0.5}-(t-s)^{1.5}}{\Gamma(2.5)}, & 0 \le s \le t \le 1.\end{cases}$$

By direct calculations, we found that

$$f_{10} = 299, f_{20} = 368, f_{30} = 202,$$

$$f_{1\infty} = 23000, f_{2\infty} = 14800, f_{3\infty} = 63200,$$

$$\begin{aligned} N_1 = \max\Big\{ &\left[(0.25)^{1.5}\int_{0.25}^{0.75} G(1,s)a_1(s)ds(23000)\right]^{-1},\\ &\left[(0.25)^{1.5}\int_{0.25}^{0.75} G(1,s)a_2(s)ds(14800)\right]^{-1},\\ &\left[(0.25)^{1.5}\int_{0.25}^{0.75} G(1,s)a_3(s)ds(63200)\right]^{-1}\Big\},\\ = \max&\{0.0009634, 0.0014972, 0.0003506\} = 0.0014972.\end{aligned}$$

Similarly, $N_2 = \min\{0.0307737, 0.0250037, 0.0455512\} = 0.0250037$. Applying Theorem 3.1, we get an optimal eigenvalue interval $0.0014972355 < \lambda_i < 0.0250037$, for $i = 1,2,3$ in which the fractional order BVP (4.1)-(4.2) has at least one positive solution.

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