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On the Behavior of the Nonlinear Difference Equation $y_{n+1} = Ay_{n-1} + By_{n-3} + \frac{Cy_{n-1}+Dy_{n-3}}{Fy_{n-3}-E}$

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Abstract. The theory of difference equations got a significant position in the applicable analysis. Therefore, studying the qualitative behavior of the difference equations is a fruitful area of research that has increasingly attracted many researchers. In this paper, we demonstrate the stability and the existence of periodic solutions of the nonlinear difference equation. Moreover, we provide some numerical simulations to confirm our results.

1. Introduction

The major purpose of this study is to provide a substantial analysis on periodicity of solution, local asymptotic stability and global behavior of the following difference equations

$$y_{n+1} = Ay_{n-1} + By_{n-3} + \frac{Cy_{n-1} + Dy_{n-3}}{Fy_{n-3} - E}, \qquad n = 0, 1, \ldots \tag{1.1}$$

where the parameters A, B, C , and D are positive real numbers and the initial conditions y_{-3}, y_{-2}, y_{-1}, and y_0 are positive real.

The study of difference equations is of utmost importance in mathematical applications. These equations also naturally appear as discrete analogs and as numerical solutions of some dynamical systems of differential equations that illustrate several phenomena in physics, biology, ecology, engineering, economics, etc. [1–10]. The theory of difference equations occupied a central position in

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the applicable analysis. So, there is no doubt that the theory of discrete time equations will persist in playing an important role in mathematics. Therefore, it has been developing in terms of analysing the behavior and solving these equations. This progress can obviously be seen in the published studies, take for instance, Alharbi et al. [11] analysed the stability and the periodicity of solutions and explored the form of solution for a special case of the rational difference equation

$$Z_{n+1} = aZ_{n-5} - \frac{bZ_{n-5}}{cZ_{n-5} - dX_{n-11}}, \qquad n = 0, 1, ...$$

El-Dessoky [12] obtained the local and global stability of the positive solutions, the periodic behavior, and the boundedness character of the following difference equation

$$x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \frac{ax_{n-t}}{bx_{n-t} + c}, \quad n = 0, 1, ...$$

Elsayed et al. [13] investigated the stability and periodicity as well as obtaining the solutions of a higher-order difference equation

$$U_{n+1} = \frac{U_{n-9}U_{n-5}U_{n-1}}{U_{n-7}U_{n-3}(\pm 1 \pm U_{n-9}U_{n-5}U_{n-1})}, \qquad n = 0, 1, ...$$

In [14], Zayed et al. studied some qualitative properties of the solutions for the non-linear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k} + hx_{n-l}}{dx_{n-k} + ex_{n-l}}, \qquad n = 0, 1, ...$$

The boundedness solution, local stability, and global attractivity of the following second-order fractional equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_n x_{n-1} + x_{n-1}}, \qquad n = 0, 1, ...$$

are demonstrated in [15] by Kostrov et al.

Avotina [16] presented the periodic solution of three special cases of the rational difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \qquad n = 0, 1, ...$$

For more recent studies, we refer the reader to [17-43] and references cited therein.

2. Preliminaries and Notation

In this section, we introduce some definitions and theorems of the theory of difference equations that be utilized in our analysis. Assume that S be a continuously differentiable function such that $S : [a, b]^{k+1} \to [a, b]$, where $[a, b]$ is a real numbers interval and k is a positive integer. Then the difference equation

$$t_{n+1} = S(t_n, t_{n-1}, ..., t_{n-k}), \qquad n = 0, 1, 2, ... \tag{2.1}$$

has a unique solution $\{t_n\}_{n=-k}^{\infty}$ for all set of initial values $t_{-k}, t_{-k+1}, ..., t_0 \in [a, b]$. (Kocic and Ladas [5])

Definition 2.1. *(Equilibrium Point)*
A point $t^* \in [a, b]$ is called an equilibrium point of equation (2.1) if

$$t^* = S(t^*, t^*, ..., t^*).$$

That is, $t_n = t^*$ for all $n \geq 0$, is a solution of equation (2.1), or equivalently, t^* is a fixed point of S.

Definition 2.2. *(Stability)*
The equilibrium point t^* of equation 2.1 is said to be

- *Locally stable if, for every $\alpha > 0$, there exists $\beta > 0$ such that for every $t_{-k}, t_{-k+1}, ..., t_{-1}, t_0 \in [a, b]$ with*

$$|t_{-k} - t^*| + |t_{-k+1} - t^*| + ... + |t_0 - t^*| < \beta,$$

we have

$$|t_n - t^*| < \alpha \qquad \forall n \geq -k.$$

- *Locally asymptotically stable if t^* is locally stable solution of equation 2.1 and there exists $\mu > 0$ such that for every $t_{-k}, t_{-k+1}, ..., t_{-1}, t_0 \in [a, b]$ with*

$$|t_{-k} - t^*| + |t_{-k+1} - t^*| + ... + |t_0 - t^*| < \mu,$$

we have

$$\lim_{n\to\infty} t_n = t^*.$$

- *Global attractor if, for every $t_{-k}, t_{-k+1}, ..., t_{-1}, t_0 \in [a, b]$ we have*

$$\lim_{n\to\infty} t_n = t^*.$$

- *globally asymptotically stable if t^* is locally stable, and also a global attractor of equation 2.1*
- *unstable if t^* is not locally stable of equation 2.1.*

Definition 2.3. *(Periodicity)*
A sequence $\{t_n\}_{n=-k}^{\infty}$ is a periodic solution with period q if $t_{n+q} = t_n$ for all $n \geq -k$.

Definition 2.4. *(Linearised Equation)*
The linearized equation of the difference equation (2.1) about the equilibrium t^* is the linear difference equation

$$X_{n+1} = \sum_{i=0}^{k-1} \frac{\partial S(t^*, t^*, ..., t^*)}{\partial t_{n-i}} X_{n-i} \tag{2.2}$$

Now, suppose that the characteristic equation associated with (2.2) is

$$Q(\zeta) = Q_0\zeta^k + Q_1\zeta^{k-1} + ... + Q_{k-1}\zeta + Q_k = 0\,. \tag{2.3}$$

Theorem A [8]

Assume that $Q_i \in R$, where $i = 1, 2, 3, ..., K$ and $k \in \{0,1,2,3,..\}$. Then

$$\sum_{i=1}^{k} |Q_i| < 1 \ .$$

is a sufficient condition for the asymptotic stability of the following the difference equation

$$X_{n+k} + Q_1 X_{n+k-1} + ... + Q_k X_n = 0 \ .$$

Theorem B [9]

Assume that h is a continuous function such that $h : [\alpha, \beta]^{s+1} \to [\alpha, \beta]$, where k is a positive integer and $[\alpha, \beta]$ is a real numbers interval.And consider the difference equation

$$t_{n+1} = h(t_n, t_{n-1}, ..., t_{n-k}), \qquad n = 0, 1, 2, ... \tag{2.4}$$

Now, let h satisfies the following

(1) For all $1 \leq i \leq k+1$ where i is an integer , the function $h(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for each $z_1, z_2, ..., z_{k+1}$.

(2) Assume (m, M) is a solution of the the system

$$m = h(m_1, m_2, ..., m_{k+1}) \ ,$$

$$M = h(M_1, M_2, ..., M_{k+1}) \ .$$

Then M=m, fer each $(i = 1, 2, ..., k+1)$ we set

$$m_i = \begin{cases} m, & \text{if } h \text{ is non-decreasing in } z_i \\ M, & \text{if } h \text{ is non-inceasing in } z_i, \end{cases}$$

and

$$M_i = \begin{cases} M, & \text{if } h \text{ is non-decreasing in } z_i \\ m, & \text{if } h \text{ is non-inceasing in } z_i, \end{cases}$$

$$m = M \ .$$

So, there exists a unique fixed point t^* of the equation (2.4) and any solution of (2.4) converges to t^*

3. The Local Stability Analysis

In this section, we calculate the equilibrium points of equation (1.1). Moreover, the local stability of these equilibrium points will be investigated.

Theorem 3.1. *The non-linear difference equation (1.1) has two equilibrium points* $y_1^* = 0$ *and* $y_2^* = \frac{C+D}{F(1-A-B)} + \frac{E}{F}$.

Proof. Equation (1.1) can be written as

$$y^*(1-A-B)=\frac{Cy^*+Dy^*}{Fy^*-E}$$

or

$$Fy^{2*}(1-A-B)-Ey^*(1-A-B)-Cy^*-Dy^*=0,$$

then,

$$Fy^{2*}(1-A-B)-y^*(E(1-A-B)+C+D)=0.$$

So, The difference equation (1.1) has two equilibrium points

$$y_1^*=0, \qquad y_2^*=\frac{C+D}{F(1-A-B)}+\frac{E}{F}.$$

Theorem 3.2. *The first equilibrium point $y_1^*=0$ of the difference equation (1.1) is locally asymptotically stable if*

$$|-C-D|<E(1-A-B).$$

Proof. Suppose that $g(0,\infty)^3\rightarrow(0,\infty)$ is a function defined as follows

$$g(u,w)=Au+Bw+\frac{Cu+Dw}{Fw-E}. \tag{3.1}$$

Differentiating $g(u,w)$ with respect to u and w. We get

$$g_u=A+\frac{C}{Fw-E}, \qquad g_w=B-\frac{(FCu+DE)}{(Fw-E)^2},$$

substituting $y_1^*=0$ into g_u, and g_w. We get

$$g_u(y_1^*,y_1^*)=A-\frac{C}{E}=-P_1, \qquad g_w(y_1^*,y_1^*)=B-\frac{D}{E}=-P_2.$$

Hence, the linearized equation of (1.1) about the equilibrium point y_1^* is

$$Z_{n+1}+P_1Z_{n-1}+P_2Z_{n-3}=0\ . \tag{3.2}$$

It follows by **Theorem A** that the fixed point ,y_1^*, of equation (1.1) is locally asymptotically stable if

$$|P_1|+|P_2|<1\ .$$

So,

$$|A-\frac{C}{E}|+|B-\frac{D}{E}|<1,$$

this implies,

$$|AE-C+BE-D|<E.$$

Thus, the first equilibrium point y_1^* is locally asymptotically stable if

$$|-C-D|<E(1-A-B).$$

The proof is completed.

Theorem 3.3. *Suppose that*

$$|C\alpha-(C+E\alpha)\alpha|<C+D-A-B.$$

Where $\alpha=(1-A-B)$, then the second equilibrium point y_2^* of equation (1.1) is locally asymptotically stable.

Proof. Substituting $y_2^*=\frac{C+D}{F\alpha}+\frac{E}{F}$ into g_u, and g_w. We get

$$g_u(y_2^*,y_2^*)=A+\frac{C\alpha}{C+D}=-Q_1,$$

$$g_w(y_2^*,y_2^*)=B-\frac{(C+E\alpha)\alpha}{C+D}=-Q_2.$$

Where $\alpha=(1-A-B)$.
So, the linearized equation of (1.1) about the equilibrium point y_2^* is

$$Z_{n+1}+Q_1Z_{n-1}+Q_2Z_{n-3}=0\,. \tag{3.3}$$

It can be shown by **Theorem A** that the fixed point y_2^* of the difference equation (1.1) is locally asymptotically stable if

$$|Q_1|+|Q_2|<1\,.$$

So,

$$|A+\frac{C\alpha}{C+D}|+|B-\frac{(C+E\alpha)\alpha}{C+D}|<1,$$

thus,

$$|A+C\alpha+B-(C+E\alpha)\alpha|<C+D.$$

Therefore, the second equilibrium y_2^* is locally asymptotically stable if

$$|C\alpha-(C+E\alpha)\alpha|<C+D-A-B.$$

The proof is completed.

4. Global Behaviour Analysis

We dedicate this section to showing the case under which the equilibrium points y^* of equation (1.1) are asymptotically globally stable.

Theorem 4.1. *The equilibrium points y^* of the difference equation (1.1) is globally asymptotically stable if*

i $AE+BF+D>C+BE+E$
ii $E+C+D>F$

Proof. Suppose that k and r be real numbers and assume $g(k,r)^2 \to (k,r)$ is a function that defined by

$$g(u,w) = Au + Bw + \frac{Cu + Dw}{Fw - E}. \tag{4.1}$$

Now, we consider two cases.

Case i. Suppose that $g(u,w)$ is increasing in u and w.

Then, assume (ζ,ρ) is a solution of the following system

$$\zeta = g(\zeta,\zeta)\ ,$$

$$\rho = g(\rho,\rho)\ .$$

So,

$$\zeta = A\zeta + B\zeta + \frac{C\zeta + D\zeta}{F\zeta - E}\ ,$$

$$\rho = A\rho + B\rho + \frac{C\rho + D\rho}{F\rho - E}\ ,$$

this gives,

$$F\zeta^2(1 - A - B) - E\zeta(1 - A - B) = \zeta(C + D), \tag{4.2}$$

$$F\rho^2(1 - A - B) - E\rho(1 - A - B) = \rho(C + D), \tag{4.3}$$

after subtracting (4.3) from (4.2). We get

$$(\zeta^2 - \rho^2)F(1 - A - B) - (\zeta - \rho)E(1 - A - B) - (\zeta - \rho)(C + D) = 0, \tag{4.4}$$

this implies,

$$(\zeta - \rho)\{(\zeta + \rho)F(1 - A - B) - E(1 - A - B) - (C + D)\} = 0. \tag{4.5}$$

Thus, when $E + C + D > F$,

$$\zeta = \rho\ .$$

It follows by **Theorem B** that y^* is globally asymptotically stable. The proof is completed.

Case ii. Suppose that $g(u,w)$ is increasing in u and it is decreasing in w.

Then, assume (ζ,ρ) is a solution of the following system

$$\zeta = g(\zeta,\rho)\ ,$$

$$\rho = g(\rho,\zeta)\ .$$

So,

$$\zeta = A\zeta + B\rho + \frac{C\zeta + D\rho}{F\rho - E}\ ,$$

$$\rho = A\rho + B\zeta + \frac{C\rho + D\zeta}{F\zeta - E}\ ,$$

this implies,

$$\zeta(1 - A)(F\rho - E) - B\rho(F\rho - E) - C\zeta - D\rho = 0, \tag{4.6}$$

$$\rho(1-A)(F\zeta - E) - B\zeta(F\zeta - E) - C\rho - D\zeta = 0. \tag{4.7}$$

Now, subtracting (4.7) from (4.6). We get

$$(\zeta - \rho)\{AE - E + BF(\zeta + \rho) - BE - C + D\} = 0. \tag{4.8}$$

Therefore, when $AE + BF + D > C + BE + E$

$$\zeta = \rho\ .$$

It can be shown by **Theorem B** that y^* is globally asymptotically stable. The proof is completed.

5. Existence of Periodic Solutions

This section discusses the existence of periodic behavior of the nonlinear difference equation (1.1). The following theorem states the necessary and sufficient conditions that assure Eq.(1.1) has periodic behavior of prime period two.

Theorem 5.1. *The difference equation (1.1) has solution of period two if and only if*

$$E(1-A-B) + C + D \neq 0 \tag{5.1}$$

Proof. Assume that equation (1.1) has a solution of period two

$$\ldots, \alpha, \beta, \alpha, \beta, \ldots$$

with $\alpha \neq \beta$

$$\alpha = A\alpha + B\alpha + \frac{C\alpha + D\alpha}{F\alpha - E}\ ,$$

$$\beta = A\beta + B\beta + \frac{C\beta + D\beta}{F\beta - E}\ .$$

So,

$$F\alpha^2(1-A-B) - E\alpha(1-A-B) = \alpha(C+D)\ , \tag{5.2}$$

$$F\beta^2(1-A-B) - E\beta(1-A-B) = \beta(C+D)\ . \tag{5.3}$$

Subtracting (5.3) from (5.2) gives

$$F(1-A-B)(\alpha^2 - \beta^2) - E(1-A-B)(\alpha - \beta) = (C+D)(\alpha - \beta)\ ,$$

this implies,

$$F(1-A-B)(\alpha + \beta) - E\alpha(1-A-B) = (C+D)\ .$$

Consequently,

$$\alpha + \beta = \frac{E(1-A-B) + C + D}{F(1-A-B)}\ . \tag{5.4}$$

Again, adding (5.2) and (5.3). We get

$$F(1-A-B)(\alpha^2+\beta^2)=\{E(1-A-B)+(C+D)\}(\alpha+\beta)\ . \tag{5.5}$$

By using (5.4), (5.5) , and the relation $(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$, we obtain

$$F(1-A-B)\{(\alpha+\beta)^2-2\alpha\beta\}=\{E(1-A-B)+(C+D)\}(\alpha+\beta)\ ,$$

then,

$$2F(1-A-B)\alpha\beta=F(1-A-B)(\alpha+\beta)^2-\{E(1-A-B)+(C+D)\}(\alpha+\beta)\ ,$$

$$2F(1-A-B)\alpha\beta=\frac{(E(1-A-B)+C+D)^2}{F(1-A-B)}-\{E(1-A-B)+C+D\}(\frac{E(1-A-B)+C+D}{F(1-A-B)})\ .$$

Thus,

$$\alpha\beta=0. \tag{5.6}$$

Therefore, it follows from equations (20) and (22) that α and β are the two distinct roots of the quadratic equation

$$X^2-(\alpha+\beta)X+\alpha\beta=0. \tag{5.7}$$

That is,

$$X^2-(\frac{E(1-A-B)+C+D}{F(1-A-B)})X=0,$$

then,

$$F(1-A-B)X^2-(E(1-A-B)+C+D)X=0,$$

so,

$$(E(1-A-B)+C+D)^2>0.$$

For $(E(1-A-B)+C+D)\neq 0$, the condition (5.1) holds.
On the other side, suppose that condition (5.1) is true. We will demonstrate that equation (1.1) has a prime period two solution.
Set

$$y_{-3}=y_{-1}=p=\frac{E(1-A-B)+C+D}{F(1-A-B)}$$

and

$$y_{-2}=y_0=q=0.$$

Now, we want to show that

$$y_1=p,\quad and\quad y_2=0.$$

It follows from equation (1.1) that

$$y_1=Ap+Bp+\frac{Cp+Dp}{Fp-E},$$

so,

$$y_1=(A+B)\Big(\frac{E(1-A-B)+C+D}{F(1-A-B)}\Big)+\frac{(C+D)(\frac{E(1-A-B)+C+D}{F(1-A-B)})}{F(\frac{E(1-A-B)+C+D}{F(1-A-B)})-E},$$

$$= (A+B)\Big(\frac{E(1-A-B)+C+D}{F(1-A-B)}\Big) + \frac{(C+D)(E(1-A-B)+C+D}{F(C+D)},$$
$$= (A+B)\Big(\frac{E(1-A-B)+C+D}{F(1-A-B)}\Big) + \frac{(E(1-A-B)+C+D}{F},$$
$$= \Big(\frac{E(1-A-B)+C+D}{F}\Big)\Big(1+\frac{A+B}{(1-A-B)}\Big) = \frac{E(1-A-B)+C+D}{F(1-A-B)} = p,$$
$$y_2 = Aq + Bq + \frac{Cq+Dq}{Fq-E} = 0 = q.$$

So, by induction we get

$$y_{2n} = q \quad \text{and} \quad y_{2n+1} = p \qquad \text{for all } n \geq -3.$$

Hence, equation (1.1) has the prime period two solution p and q. Where p and q are the distinct roots of the quadratic equation (5.7).

6. Numerical Examples

In this part, we provide some examples that verify our analytical results. MATLAB programming is used to show numerically the behavior of the nonlinear difference equation(1.1).

Example 6.1. Figure 1 shows the behavior of Eq.(1.1) tends to the first equilibrium point $y_1^* = 0$ when the parameters and the initial values are $A = 0.1$, $B = 0.2$, $C = 1$, $D = 2$, $F = 4$, $E = 6$, $y_{-3} = -3$, $y_{-2} = 2$, $y_{-1} = -0.5$, and $y_0 = 1$.

Example 6.2. Figure 2 presents the behavior of Eq.(1) approaches to the second equilibrium point

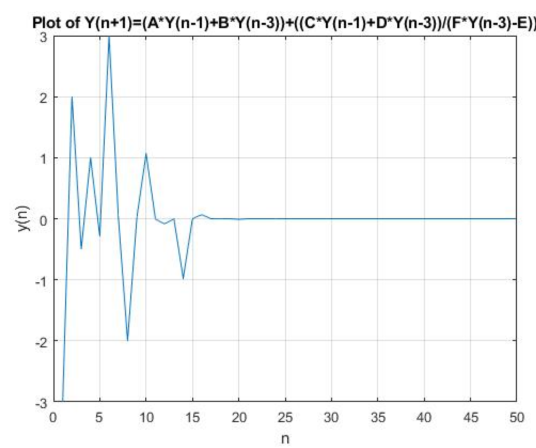

Figure 1. The Behaviour of Equation (1.1)

$y_2^* = 0$ when we assume the parameters and the initial values that $A = 0.6$, $B = 0.2$, $C = 3$, $D = 4$, $F = 6$, $E = 5$, $y_{-3} = 5$, $y_{-2} = -3$, $y_{-1} = 1$, and $y_0 = 4$.

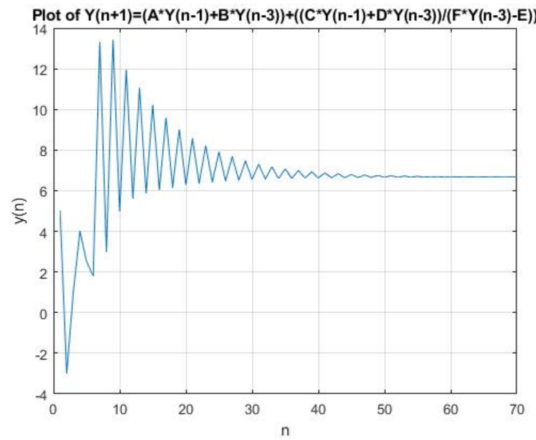

Figure 2. The Behaviour of Equation (1.1)

Example 6.3. The unstable behavior of Eq.(1.1) is shown in figure 3. we assume the parameters and the initial values that $A = 0.1$, $B = 0.2$, $C = 5$, $D = 8$, $F = 0.4$, $E = 6$, $y_{-3} = 1$, $y_{-2} = -6$, $y_{-1} = 3$, and $y_0 = -4$.

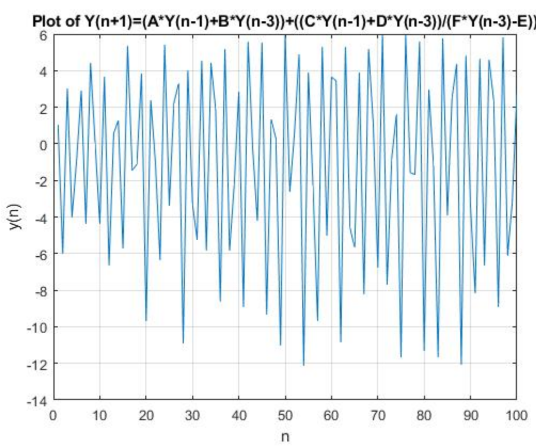

Figure 3. The Behaviour of Equation (1.1)

Example 6.4. In figure 4, The global stability behavior of Eq.(1.1) is shown. It is clear that the behavior of Eq.(1.1) tends to the fixed point y_1^* as n goes to ∞ under the following the initial conditions and the parameters $A = 0.1$, $B = 0.2$, $C = 1$, $D = 2$, $F = 4$, $E = 8$, $y_{-3} = 1$, $y_{-2} = -6$, $y_{-1} = 3$, and $y_0 = -4$.

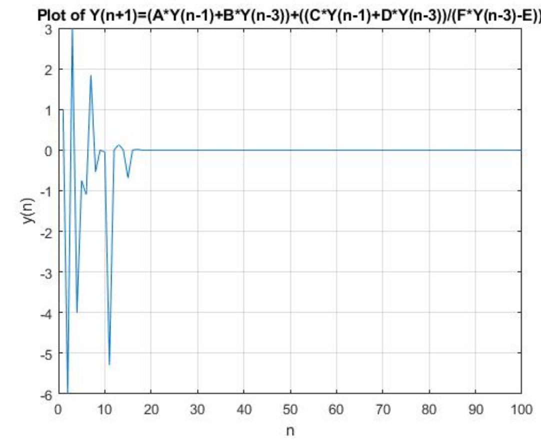

Figure 4. The Behaviour of Equation (1.1)

Example 6.5. Figure 5 demonstrates the global stability behavior of the fixed point y_2^* when the initial conditions and the parameters are $A = 0.2$, $B = 0.16$, $C = 0.123$, $D = 14$, $F = 0.5$, $E = 5$, $y_{-3} = 5$, $y_{-2} = -3$, $y_{-1} = 1$, and $y_0 = -4$.

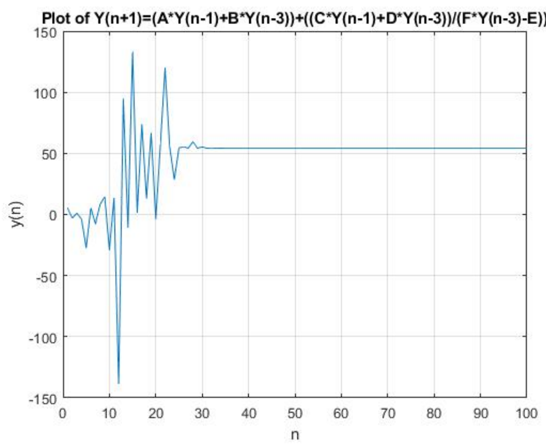

Figure 5. The Behaviour of Equation (1.1)

Example 6.6. Figure 6 shows that Eq.(1.1) has a prime period two solution when the initial conditions and the parameters are $A = 0.2$, $B = 0.1$, $C = 0.2$, $D = 3$, $F = 1$, $E = 0.6$, $y_{-3} = p$, $y_{-2} = q$, $y_{-1} = p$, and $y_0 = q$ where p and q satisfied **Theorem 5.1**

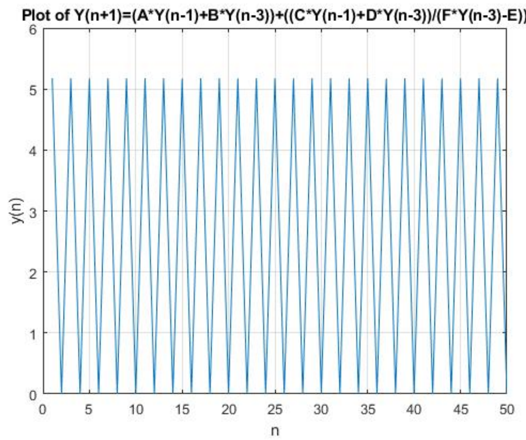

Figure 6. The Behaviour of Equation (1.1)

7. Conclusion

This study discusses the dynamics of the nonlinear difference equation (1.1). In section 3 we illustrated that when the local stability condition in **Theorem 3.2** is satisfied, the behavior tends to the stability state of the equilibrium point $y_1^* = 0$. While, the equilibrium y_2^* will be locally asymptotically stable when $|C\alpha - (C + E\alpha)\alpha| < C + D - A - B$. The global solution of the equilibrium points conditions is shown in section 4. Section 5 discussed the necessary and sufficient conditions to obtain the periodic solution of equation (1). For confirmation of our theoretical analysis, we presented some numerical examples in section 6, and figures 1-6 verified the results.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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