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# The Prominentness of Fuzzy GE-Filters in GE-Algebras

**Sun Shin Ahn**[1,*], **Rajab Ali Borzooei**[2], **Young Bae Jun**[3]

[1]*Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

[2]*Department of Mathematics, Shahid Beheshti University, Tehran 1983963113, Iran*

[3]*Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

[*]*Corresponding author:* sunshine@dongguk.edu

Abstract. Based on the concept of fuzzy points, the notion of a prominent fuzzy GE-filter is defined, and the various properties involved are investigated. The relationship between a fuzzy GE-filter and a prominent fuzzy GE-filter is discussed, and the characterization of a prominent fuzzy GE-filter is considered. The conditions under which a fuzzy GE-filter can be a prominent fuzzy GE-filter are explored, and conditions for the trivial fuzzy GE-filter to be a prominent fuzzy GE-filter are provided. The conditions under which the $\in_t$-set and $Q_t$-set can be prominent GE-filters are explored. Finally, the extension property for the prominent fuzzy GE-filter is discussed.

## 1. Introduction

Henkin and Scolem introduced the concept of Hilbert algebra in the implication investigation in intuitionistic logics and other nonclassical logics. Diego [6] established that Hilbert algebras form a locally finite variety. Later several researchers extended the theory on Hilbert algebras (see [4,5,7,8]). The notion of BE-algebra was introduced by Kim et al. [9] as a generalization of a dual BCK-algebra. Rezaei et al. [13] discussed relations between Hilbert algebras and BE-algebras. As a generalization of Hilbert algebras, Bandaru et al. [2] introduced the notion of GE-algebras, and investigated several properties. Bandaru et al. [3] introduced the concept of bordered GE-algebra and investigated its properties. Later, Ozturk et al. [10] introduced the concept of strong GE-filters, GE-ideals of bordered GE-algebras and investigated its properties. Song et al. [14] introduced the concept of Imploring GE-filters of GE-algebras and discussed its properties. Rezaei et al. [12] introduced the concept of

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prominent GE-filters in GE-algebras and discussed its properties. Bandaru et al. [1] discussed the fuzzy notion of GE-filters in GE-algebras.

The purpose of this paper is to define a prominent fuzzy GE-filter using the concept of fuzzy points and investigate the various properties involved. We consider the relationship between a fuzzy GE-filter and a prominent fuzzy GE-filter. We explore the conditions under which a fuzzy GE-filter can be a prominent fuzzy GE-filter. We discuss the characterization of a prominent fuzzy GE-filter. We provide conditions for the trivial fuzzy GE-filter to be a prominent fuzzy GE-filter. We explore the conditions under which the $\in_t$-set and $Q_t$-set can be prominent GE-filters. We finally discuss the extension property for the prominent fuzzy GE-filter.

## 2. Preliminaries

### 2.1. **Basics related to GE-algebras.**

**Definition 2.1** ( [2]). *By a GE-algebra we mean a set $X$ with a constant "1" and a binary operation "$*$" satisfying the following axioms:*

*(GE1)* $a * a = 1$,

*(GE2)* $1 * a = a$,

*(GE3)* $a * (b * c) = a * (b * (a * c))$

*for all $a, b, c \in X$.*

We denote the GE-algebra by $\mathbf{X} := (X, *, 1)$. A binary relation "$\leq$" in a GE-algebra $\mathbf{X} := (X, *, 1)$ is defined by:

$$(\forall x, y \in X)(x \leq y \;\Leftrightarrow\; x * y = 1). \tag{2.1}$$

**Definition 2.2** ( [2]). *A GE-algebra $\mathbf{X} := (X, *, 1)$ is said to be*

- *transitive if it satisfies:*

$$(\forall a, b, c \in X)\,(a * b \leq (c * a) * (c * b))\,. \tag{2.2}$$

- *commutative if it satisfies:*

$$(\forall a, b \in X)\,((a * b) * b = (b * a) * a)\,. \tag{2.3}$$

Note that every commutative GE-algebra is transitive and antisymmetric.

**Proposition 2.1** ( [2]). *Every GE-algebra $\mathbf{X} := (X, *, 1)$ satisfies the following items.*

$$(\forall a \in X)\,(a * 1 = 1)\,. \tag{2.4}$$

$$(\forall a, b \in X)\,(a * (a * b) = a * b)\,. \tag{2.5}$$

$$(\forall a, b \in X)\,(a \leq b * a)\,. \tag{2.6}$$

$$(\forall a, b, c \in X)\,(a * (b * c) \leq b * (a * c))\,. \tag{2.7}$$

$$(\forall a \in X)\,(1 \leq a \;\Rightarrow\; a = 1)\,. \tag{2.8}$$

$$(\forall a, b \in X)\,(a \leq (a * b) * b)\,. \tag{2.9}$$

*If* $\mathbf{X} := (X, *, 1)$ *is transitive, then*

$$(\forall a, b, c \in X)\,(a \leq b \;\Rightarrow\; c * a \leq c * b,\; b * c \leq a * c)\,. \tag{2.10}$$

$$(\forall a, b, c \in X)\,(a * b \leq (b * c) * (a * c))\,. \tag{2.11}$$

$$(\forall a, b, c \in X)\,(a * b \leq (c * a) * (c * b))\,. \tag{2.12}$$

**Definition 2.3.** *A subset* $F$ *of a GE-algebra* $\mathbf{X} := (X, *, 1)$ *is called*

- *a GE-filter of* $\mathbf{X} := (X, *, 1)$ *(see [2]) if it satisfies:*

$$1 \in F, \tag{2.13}$$

$$(\forall a, b \in X)(a \in F,\; a * b \in F \;\Rightarrow\; b \in F). \tag{2.14}$$

- *a prominent GE-filter of* $\mathbf{X} := (X, *, 1)$ *(see [12]) if it satisfies* (2.13) *and*

$$(\forall a, b, c \in X)(a * (b * c) \in F,\; a \in F \;\Rightarrow\; ((c * b) * b) * c \in F). \tag{2.15}$$

**Lemma 2.1** ( [2])**.** *Every GE-filter* $F$ *of* $\mathbf{X} := (X, *, 1)$ *satisfies:*

$$(\forall x, y \in X)(x \leq y,\; x \in F \;\Rightarrow\; y \in F). \tag{2.16}$$

**Lemma 2.2** ( [12])**.** *Every prominent GE-filter is a GE-filter.*

2.2. **Basics related to fuzzy sets.** A fuzzy set $f$ in a set $X$ of the form

$$f(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support $a$ and value $t$ and is denoted by $\frac{a}{t}$.

For a fuzzy set $f$ in a set $X$ and $t \in (0, 1]$, we say that a fuzzy point $\frac{a}{t}$ is

(i) *contained* in $f$, denoted by $\frac{a}{t} \in f$, (see [11]) if $f(a) \geq t$.

(ii) *quasi-coincident* with $f$, denoted by $\frac{a}{t}\, q\, f$, (see [11]) if $f(a) + t > 1$.

If $\frac{a}{t}\, \alpha\, f$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $\frac{a}{t}\, \overline{\alpha}\, f$.

Given $t \in (0, 1]$ and a fuzzy set $f$ in a set $X$, consider the following sets

$$(f, t)_{\in} := \{x \in X \mid \tfrac{x}{t} \in f\} \text{ and } (f, t)_q := \{x \in X \mid \tfrac{x}{t}\, q\, f\}$$

which are called an $\in_t$-*set* and $Q_t$-*set* of $f$, respectively, in $X$.

**Definition 2.4** ( [1]). *A fuzzy set $f$ in a GE-algebra $\mathbf{X} := (X, *, 1)$ is called a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ if it satisfies:*

$$(\forall t \in (0,1])\,((f,t)_{\in} \neq \emptyset \;\Rightarrow\; 1 \in (f,t)_{\in})\,, \tag{2.17}$$

$$x * y \in (f, t_b)_{\in},\, x \in (f, t_a)_{\in} \;\Rightarrow\; y \in (f, \min\{t_a, t_b\})_{\in} \tag{2.18}$$

*for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.*

## 3. The Prominentness of Fuzzy GE-Filters

In what follows, let $\mathbf{X} := (X, *, 1)$ denote a GE-algebra unless otherwise specified.

**Definition 3.1.** *A fuzzy set $f$ in $X$ is called a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ if it satisfies (2.17) and*

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0,1]) \left( \begin{array}{l} x * (y * z) \in (f, t_b)_{\in},\, x \in (f, t_a)_{\in} \;\Rightarrow \\ ((z * y) * y) * z \in (f, \min\{t_a, t_b\})_{\in} \end{array} \right). \tag{3.1}$$

**Example 3.1.** *Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ be a set with a binary operation "$*$" given by Table 1.*

Table 1. Cayley table for the binary operation "$*$"

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 1 | 1 | 1 | 4 | 6 | 6 | 1 |
| 3 | 1 | 2 | 1 | 5 | 5 | 5 | 7 |
| 4 | 1 | 1 | 3 | 1 | 1 | 1 | 1 |
| 5 | 1 | 2 | 1 | 1 | 1 | 1 | 7 |
| 6 | 1 | 2 | 3 | 1 | 1 | 1 | 1 |
| 7 | 1 | 2 | 3 | 6 | 5 | 6 | 1 |

*Then $\mathbf{X} := (X, *, 1)$ is a GE-algebra (see [12]). Define a fuzzy set $f$ in $X$ as follows:*

$$f : X \to [0,1],\;\; x \mapsto \begin{cases} 0.85 & \text{if } x \in \{1,2,3,7\}, \\ 0.37 & \text{otherwise.} \end{cases}$$

*It is routine to verify that $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$.*

We discuss the relationship between a fuzzy GE-filter and a prominent fuzzy GE-filter.

**Theorem 3.1.** *Every prominent fuzzy GE-filter is a fuzzy GE-filter.*

*Proof.* Let $f$ be a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $x \in (f, t_a)_{\in}$ and $x * y \in (f, t_b)_{\in}$. Then $x * (1 * y) = x * y \in (f, t_b)_{\in}$ by (GE2), and so

$y = ((y * 1) * 1) * y \in (f, t_b)_\in$ by (GE1), (GE2), (2.4) and (3.1). Hence $f$ is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. □

The following example shows that the converse of Theorem 3.1 may not be true.

**Example 3.2.** *Consider the GE-algebra* $\mathbf{X} := (X, *, 1)$ *in Example 3.1 and let* $f$ *be a fuzzy set in* $X$ *defined by*

$$f : X \to [0, 1],\ x \mapsto \begin{cases} 0.79 & \text{if } x \in \{1, 3, 7\}, \\ 0.46 & \text{otherwise.} \end{cases}$$

*It is routine to verify that* $f$ *is a fuzzy GE-filter of* $\mathbf{X} := (X, *, 1)$*. But it is not a prominent fuzzy GE-filter of* $\mathbf{X} := (X, *, 1)$ *since* $3 \in (f, 0.67)_\in$ *and* $3 * (4 * 2) = 1 \in (f, 0.62)_\in$*, but* $((2 * 4) * 4) * 2 = 2 \notin (f, 0.62)_\in = (f, \min\{0.67, 0.62\})_\in$.

We explore the conditions under which a fuzzy GE-filter can be a prominent fuzzy GE-filter.

**Theorem 3.2.** *Given a fuzzy GE-filter* $f$ *of* $\mathbf{X} := (X, *, 1)$*, it is a prominent fuzzy GE-filter of* $\mathbf{X} := (X, *, 1)$ *if and only if it satisfies:*

$$(\forall x, y \in X)(\forall t \in (0, 1])(x * y \in (f, t)_\in \ \Rightarrow\ ((y * x) * x) * y \in (f, t)_\in). \tag{3.2}$$

*Proof.* Assume that $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ and let $x, y \in X$ and $t \in (0, 1]$ be such that $x * y \in (f, t)_\in$. Then $1 * (x * y) = x * y \in (f, t)_\in$ by (GE2). Since $1 \in (f, t)_\in$, it follows from (3.1) that $((y * x) * x) * y \in (f, t)_\in$.

Conversely, let $f$ be a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ that satisfies the condition (3.2). Let $x, y, z \in X$ and $t_a, t_b \in (0, 1]$ be such that $x * (y * z) \in (f, t_b)_\in$ and $x \in (f, t_a)_\in$. Then $y * z \in (f, \min\{t_a, t_b\})_\in$ by (2.18), and so $((z * y) * y) * z \in (f, \min\{t_a, t_b\})_\in$ by (3.2). Therefore $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. □

**Lemma 3.1** ( [1]). *Every fuzzy GE-filter* $f$ *of* $X$ *satisfies:*

$$(\forall x, y \in X)(\forall t_a \in (0, 1])\left(x \leq y,\ x \in (f, t_a)_\in \ \Rightarrow\ y \in (f, t_a)_\in\right), \tag{3.3}$$

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0, 1])\begin{pmatrix} z \leq y * x,\ y \in (f, t_b)_\in,\ z \in (f, t_a)_\in \\ \Rightarrow\ x \in (f, \min\{t_a, t_b\})_\in \end{pmatrix}. \tag{3.4}$$

**Theorem 3.3.** *In a commutative GE-algebra, every fuzzy GE-filter is a prominent fuzzy GE-filter.*

*Proof.* Let $f$ be a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. It is sufficient to show that $f$ satisfies the condition (3.1). Let $x, y, z \in X$ and $t_a, t_b \in (0, 1]$ be such that $x * (y * z) \in (f, t_b)_\in$ and

$x \in (f, t_a)_\in$. Using (2.3), (2.7), and (2.12), we have

$$\begin{aligned} 1 &= ((z * y) * y) * ((y * z) * z) \\ &\leq (y * z) * (((z * y) * y) * z) \\ &\leq (x * (y * z)) * (x * (((z * y) * y) * z)) \\ &\leq x * ((x * (y * z)) * (((z * y) * y) * z)), \end{aligned}$$

and so $x * ((x * (y * z)) * (((z * y) * y) * z)) = 1$, i.e., $x \leq (x * (y * z)) * (((z * y) * y) * z)$. It follows from Lemma 3.1 that $((z * y) * y) * z \in (f, \min\{t_a, t_b\})_\in$. Therefore $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. □

**Theorem 3.4.** *A fuzzy set $f$ in $X$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ if and only if it satisfies:*

$$(\forall x \in X)(f(1) \geq f(x)). \tag{3.5}$$

$$(\forall x, y, z \in X)(f(((z * y) * y) * z) \geq \min\{f(x), f(x * (y * z))\}). \tag{3.6}$$

*Proof.* Assume that $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. Suppose there exists $a \in X$ such that $f(1) < f(a)$. Let $t_0 = \frac{1}{2}(f(1) + f(a))$. Then $f(1) < t_0$ and $0 < t_0 < f(a) \leq 1$. Hence $a \in (f, t_0)_\in$ and so $(f, t_0)_\in \neq \emptyset$. Thus $1 \in (f, t_0)_\in$, that is, $f(1) \geq t_0$, which is contradiction. Hence $f(1) \geq f(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $f(x) = t_1$ and $f(x * (y * z)) = t_2$. Then $x \in (f, t_1)_\in$ and $x * (y * z) \in (f, t_2)_\in$. It follows from (3.1) that $((z * y) * y) * z \in (f, \min\{t_1, t_2\})_\in$. Hence $f(((z * y) * y) * z) \geq \min\{t_1, t_2\} = \min(f(x), f(x * (y * z)))$.

Conversely, assume that $f$ satisfies (3.5) and (3.6). Let $t \in (0, 1]$ and $x \in (f, t)_\in$. Then $f(x) \geq t$ and hence $f(1) \geq f(x) \geq t$. Thus $1 \in (f, t)_\in$. Let $x, y, z \in X$ be such that $x \in (f, t_1)_\in$ and $x * (y * z) \in (f, t_2)_\in$. Then $f(x) \geq t_1$ and $f(x * (y * z)) \geq t_2$. Therefore $f(((z * y) * y) * z) \geq \min\{f(x), f(x * (y * z))\} \geq \min\{t_1, t_2\}$ by (3.6). Hence $((z * y) * y) * z \in (f, \min\{t_1, t_2\})_\in$. Thus $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. □

**Theorem 3.5.** *Given an element $b \in X$, define a fuzzy set $f_b$ in $X$ as follows:*

$$f_b : X \to [0, 1], \; x \mapsto \begin{cases} t_1 & \text{if } x \in \vec{b}, \\ t_2 & \text{otherwise.}, \end{cases}$$

*where $\vec{b} := \{x \in X \mid b \leq x\}$ and $t_1 > t_2$ in $(0, 1]$. Then $f_b$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ if and only if $\mathbf{X} := (X, *, 1)$ satisfies:*

$$(\forall x, y, z \in X)(x \in \vec{b},\, x * (y * z) \in \vec{b} \;\Rightarrow\; ((z * y) * y) * z \in \vec{b}). \tag{3.7}$$

*Proof.* Assume that $f_b$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ and let $x, y, z \in X$ be such that $x \in \vec{b}$ and $x * (y * z) \in \vec{b}$. Then $f_b(x) = t_1 = f_b(x * (y * z))$, which implies from (3.6) that

$$f_b(((z * y) * y) * z) \geq \min\{f_b(x), f_b(x * (y * z))\} = t_1.$$

Hence $f_b(((z*y)*y)*z) = t_1$, and thus $((z*y)*y)*z \in \vec{b}$.

Conversely, suppose that $\mathbf{X} := (X, *, 1)$ satisfies the condition (3.7). Since $1 \in \vec{b}$, we get $f_b(1) = t_1 \geq f_b(x)$ for all $x \in X$. For every $x, y, z \in X$, if $x \notin \vec{b}$ or $x*(y*z) \notin \vec{b}$, then $f_b(x) = t_2$ or $f_b(x*(y*z)) = t_2$. Hence

$$f_b(((z*y)*y)*z) \geq t_2 = \min\{f_b(x), f_b(x*(y*z))\}.$$

If $x \in \vec{b}$ and $x*(y*z) \in \vec{b}$, then $f_b(x) = t_1$ and $f_b(x*(y*z)) = t_1$. Thus

$$f_b(((z*y)*y)*z) = t_1 = \min\{f_b(x), f_b(x*(y*z))\}.$$

Therefore $f_b$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ by Theorem 3.4. □

Consider a fuzzy set $f$ in $X$ which is given by

$$f : X \to [0,1],\ x \mapsto \begin{cases} t_1 & \text{if } x = 1, \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $(0, 1]$. It is clear that $f$ is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, which is called the *trivial fuzzy GE-filter* of $\mathbf{X} := (X, *, 1)$. But it is not a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ as seen in the following example.

**Example 3.3.** *Consider the GE-algebra $\mathbf{X} := (X, *, 1)$ in Example 3.1 and let $f$ be a fuzzy set in $X$ defined by*

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.83 & \text{if } x = 1, \\ 0.57 & \text{otherwise.} \end{cases}$$

*Then $f$ is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, but it is not a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ since $1 \in (f, 0.69)_\in$ and $1*(4*2) = 1 \in (f, 0.64)_\in$, but $((2*4)*4)*2 = 2 \notin (f, \min\{0.69, 0.64\})_\in$.*

We provide conditions for the trivial fuzzy GE-filter to be a prominent fuzzy GE-filter.

**Theorem 3.6.** *In a commutative GE-algebra, the trivial fuzzy GE-filter is a prominent fuzzy GE-filter.*

*Proof.* Let $f$ be the trivial fuzzy GE-filter of a commutative GE-algebra $\mathbf{X} := (X, *, 1)$. Then

$$(f, t)_\in = \begin{cases} \emptyset & \text{if } t \in (t_1, 1], \\ \{1\} & \text{if } t \in (t_2, t_1], \\ X & \text{if } t \in (0, t_2]. \end{cases}$$

It is sufficient to show that $(f, t)_\in = \{1\}$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$. Let $x, y, z \in X$ be such that $x \in \{1\}$ and $x*(y*z) \in \{1\}$. Using (GE2), (2.3) and (GE1), we get $y*z = 1$, and thus $((z*y)*y)*z = ((y*z)*z)*z = (1*z)*z = z*z = 1 \in \{1\}$. Hence $(f, t)_\in = \{1\}$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$, and therefore $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ by Theorem **??**. □

We explore the conditions under which the $\in_t$-set and $Q_t$-set can be prominent GE-filters.

**Theorem 3.7.** *Given a fuzzy set $f$ in $X$, its $\in_t$-set $(f,t)_\in$ is a prominent GE-filter of $X$ for all $t \in (0.5, 1]$ if and only if $f$ satisfies:*

$$(\forall x \in X)(f(x) \leq \max\{f(1), 0.5\}), \tag{3.8}$$

$$(\forall x, y \in X)(\min\{f(x), f(x*(y*z))\} \leq \max\{f(((z*y)*y)*z), 0.5\}). \tag{3.9}$$

*Proof.* Assume that the $\in_t$-set $(f,t)_\in$ of $f$ is a prominent GE-filter of $X$ for all $t \in (0.5, 1]$. If there exists $a \in X$ such that $f(a) \nleq \max\{f(1), 0.5\}$, then $t := f(a) \in (0.5, 1]$, $\frac{a}{t} \in f$ and $\frac{1}{t} \overline{\in} f$, that is, $a \in (f,t)_\in$ and $1 \notin (f,t)_\in$. This is a contradiction, and thus $f(x) \leq \max\{f(1), 0.5\}$ for all $x \in X$. If (3.9) is not valid, then

$$\min\{f(a), f(a*(b*c))\} > \max\{f(((c*b)*b)*c), 0.5\}$$

for some $a, b, c \in X$. If we take $t := \min\{f(a), f(a*(b*c))\}$, then $t \in (0.5, 1]$, $\frac{a}{t} \in f$ and $\frac{a*(b*c)}{t} \in f$. Hence $a \in (f,t)_\in$ and $a*(b*c) \in (f,t)_\in$, which imply that $((c*b)*b)*c \in (f,t)_\in$. Thus $\frac{((c*b)*b)*c}{t} \in f$, and so $f(((c*b)*b)*c) \geq t > 0.5$ which is a contradiction. Therefore

$$\min\{f(x), f(x*(y*z))\} \leq \max\{f(((z*y)*y)*z), 0.5\}$$

for all $x, y \in X$.

Conversely, suppose that $f$ satisfies (3.8) and (3.9). Let $(f,t)_\in \neq \emptyset$ for all $t \in (0.5, 1]$. Then there exists $a \in (f,t)_\in$ and thus $\frac{a}{t} \in f$, i.e., $f(a) \geq t$. It follows from (3.8) that $\max\{f(1), 0.5\} \geq f(a) \geq t > 0.5$. Thus $\frac{1}{t} \in f$, i.e., $1 \in (f,t)_\in$. Let $t \in (0.5, 1]$ and $x, y, z \in X$ be such that $x \in (f,t)_\in$ and $x*(y*z) \in (f,t)_\in$. Then $\frac{x}{t} \in f$ and $\frac{x*(y*z)}{t} \in f$, that is, $f(x) \geq t$ and $f(x*(y*z)) \geq t$. Using (3.9), we get

$$\max\{f(((z*y)*y)*z), 0.5\} \geq \min\{f(x), f(x*(y*z))\} \geq t > 0.5$$

and so $\frac{((z*y)*y)*z}{t} \in f$, i.e., $((z*y)*y)*z \in (f,t)_\in$. Therefore $(f,t)_\in$ is a prominent GE-filter of $X$ for all $t \in (0.5, 1]$. □

**Lemma 3.2** ( [1]). *A fuzzy set $f$ in $X$ is a fuzzy GE-filter of $X$ if and only if the nonempty $\in_t$-set $(f,t)_\in$ of $f$ in $X$ is a GE-filter of $X$ for all $t \in (0, 1]$.*

**Lemma 3.3** ( [12]). *Let $F$ be a GE-filter of $\mathbf{X} := (X, *, 1)$. Then it is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ if and only if it satisfies:*

$$(\forall x, y \in X)(x*y \in F \;\Rightarrow\; ((y*x)*x)*y \in F). \tag{3.10}$$

**Theorem 3.8.** *A fuzzy set $f$ in $X$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ if and only if the nonempty $\in_t$-set $(f,t)_\in$ of $f$ in $X$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$.*

*Proof.* Assume that $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. Then $f$ is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ (see Theorem 3.1), and so the nonempty $\in_t$-set $(f, t)_\in$ of $f$ in $X$ is a GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$ by Lemma 3.2. Let $x, y \in X$ and $t \in (0, 1]$ be such that $x * y \in (f, t)_\in$. Since $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, it follows from (3.2) that $((y * x) * x) * y \in (f, t)_\in$, and therefore $(f, t)_\in$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$ by Lemma 3.3.

Conversely, suppose that the nonempty $\in_t$-set $(f, t)_\in$ of $f$ in $X$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$. Then $(f, t)_\in$ is a GE-filter of $\mathbf{X} := (X, *, 1)$ by Lemma 2.2, and thus $f$ is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ by Lemma 3.2. Let $x, y \in X$ and $t \in (0, 1]$ be such that $x * y \in (f, t)_\in$. Then $((y * x) * x) * y \in (f, t)_\in$ by Lemma 3.3. It follows from Theorem 3.2 that $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. □

**Theorem 3.9.** *If $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, then the nonempty $Q_t$-set $(f, t)_q$ of $f$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$.*

*Proof.* Let $f$ be a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ and assume that $(f, t)_q \neq \emptyset$ for all $t \in (0, 1]$. Then there exists $a \in (f, t)_q$, and so $\frac{a}{t}\, q\, f$, i.e., $f(a) + t > 1$. Hence $f(1) + t \geq f(a) + t > 1$, i.e., $1 \in (f, t)_q$. Let $x, y, z \in X$ be such that $x \in (f, t)_q$ and $x * (y * z) \in (f, t)_q$. Then $\frac{x}{t}\, q\, f$ and $\frac{x*(y*z)}{t}\, q\, f$, that is, $f(x) + t > 1$ and $f(x * (y * z)) + t > 1$. It follows from (3.6) that

$$\begin{aligned} f(((z * y) * y) * z) + t &\geq \min\{f(x), f(x * (y * z))\} + t \\ &= \min\{f(x) + t, f(x * (y * z)) + t\} > 1. \end{aligned}$$

Hence $\frac{((z*y)*y)*z}{t}\, q\, f$, and therefore $((z * y) * y) * z \in (f, t)_q$. Consequently, $(f, t)_q$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$. □

We finally discuss the extension property for the prominent fuzzy GE-filter.

**Question.** *Let $f$ and $g$ be fuzzy GE-filters of $\mathbf{X} := (X, *, 1)$ such that $f \subseteq g$, that is, $f(x) \leq g(x)$ for all $x \in X$. If $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, then is $g$ also a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$?*

The example below provides a negative answer to the Question.

**Example 3.4.** *Let $X = \{1, 2, 3, 4, 5, 6\}$ be a set with a binary operation "$*$" given by Table 2. Then $\mathbf{X} := (X, *, 1)$ is a GE-algebra (see [12]). Define a fuzzy set $f$ in $X$ as follows:*

$$f : X \to [0, 1],\ x \mapsto \begin{cases} 0.65 & \text{if } x = 1, \\ 0.37 & \text{otherwise.} \end{cases}$$

Table 2. Cayley table for the binary operation "$*$"

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 1 | 1 | 3 | 4 | 3 | 1 |
| 3 | 1 | 6 | 1 | 1 | 6 | 6 |
| 4 | 1 | 2 | 1 | 1 | 2 | 2 |
| 5 | 1 | 1 | 1 | 4 | 1 | 1 |
| 6 | 1 | 1 | 3 | 4 | 3 | 1 |

*It is routine to verify that $f$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$. Now, we define a fuzzy set $g$ in $X$ as follows:*

$$g : X \to [0, 1],\ x \mapsto \begin{cases} 0.73 & \text{if } x = 1, \\ 0.67 & \text{if } x \in \{2, 6\}, \\ 0.48 & \text{otherwise.} \end{cases}$$

*Then $f(x) \leq g(x)$ for all $x \in X$, that is, $f \subseteq g$, and $g$ is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. Since $4 * 5 = 2 \in (g, 0.61)_\in$ and $((5 * 4) * 4) * 5 = 5 \notin (g, 0.61)_\in$, we know that $g$ is not a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ by Theorem 3.2.*

We provide conditions for the answer of Question above to be positive.

**Theorem 3.10.** (Extension property for the prominent fuzzy GE-filter) *Let $f$ and $g$ be fuzzy GE-filters of a transitive GE-algebra $\mathbf{X} := (X, *, 1)$ such that $f \subseteq g$, that is, $f(x) \leq g(x)$ for all $x \in X$. If $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, then so is $g$.*

*Proof.* If $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$, then it is a fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ by Theorem 3.1 and $(f, t)_\in$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ for all $t \in (0, 1]$ by Theorem 3.8. Let $a := x * y \in (g, t)_\in$ for all $x, y \in X$ and $t \in (0, 1]$. Then $1 \in (f, t)_\in$ by (2.17) and $1 = a * (x * y) \leq x * (a * y)$ by (GE1) and (2.7). Hence $x * (a * y) \in (f, t)_\in$ by (3.3). Using assumption and Theorem 3.2 induces

$$(((a * y) * x) * x) * (a * y) \in (f, t)_\in \subseteq (g, t)_\in.$$

Since $(((a * y) * x) * x) * (a * y) \leq a * ((((a * y) * x) * x) * y)$ by (2.7) and $(g, t)_\in$ is a GE-filter of $\mathbf{X} := (X, *, 1)$, we have $a * ((((a * y) * x) * x) * y) \in (g, t)_\in$ by Lemma 2.1. Hence $(((a * y) * x) * x) * y \in (g, t)_\in$ by (2.14). Since $y \leq a * y$ by (2.6), we have

$$(((a * y) * x) * x) * y \leq ((y * x) * x) * y$$

by running (2.10) three times. It follows from Lemma 2.1 that $((y * x) * x) * y \in (g, t)_\in$. Hence $(g, t)_\in$ is a prominent GE-filter of $\mathbf{X} := (X, *, 1)$ by Lemma 3.3, and therefore $g$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ by Theorem 3.8. □

**Corollary 3.1.** *Let $\mathbf{X} := (X, *, 1)$ be a transitive GE-algebra. Then the trivial fuzzy GE-filter $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$ if and only if every fuzzy GE-filter is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$.*

**Corollary 3.2.** *In a commutative GE-algebra, every fuzzy GE-filter is a prominent fuzzy GE-filter.*

The following example describes the extension property for the prominent fuzzy GE-filter.

**Example 3.5.** *Let $X = \{1, 2, 3, 4, 5, 6\}$ be a set with a binary operation "$*$" given by Table 3.*

Table 3. Cayley table for the binary operation "$*$"

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 1 | 1 | 3 | 4 | 4 | 6 |
| 3 | 1 | 2 | 1 | 5 | 5 | 6 |
| 4 | 1 | 1 | 1 | 1 | 1 | 6 |
| 5 | 1 | 1 | 1 | 1 | 1 | 6 |
| 6 | 1 | 2 | 3 | 4 | 5 | 1 |

*Then $\mathbf{X} := (X, *, 1)$ is a GE-algebra (see [12]). Define a fuzzy set $f$ in $X$ as follows:*

$$f : X \to [0, 1],\ x \mapsto \begin{cases} 0.59 & \text{if } x \in \{1, 2, 3\}, \\ 0.36 & \text{otherwise.} \end{cases}$$

*Then $f$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$. If we take a fuzzy set $g$ in $X$ defined as follows:*

$$g : X \to [0, 1],\ x \mapsto \begin{cases} 0.69 & \text{if } x \in \{1, 2, 3, 6\}, \\ 0.56 & \text{otherwise,} \end{cases}$$

*then $f \subseteq g$ and $g$ is a prominent fuzzy GE-filter of $\mathbf{X} := (X, *, 1)$.*

## 4. Conclusion

Using the concept of fuzzy points, we have introduced the notion of a prominent fuzzy GE-filter in GE-algebras, and have investigated the various properties involved. We have considered the relationship between a fuzzy GE-filter and a prominent fuzzy GE-filter, and have discussed the characterization of a prominent fuzzy GE-filter. We have explored the conditions under which a fuzzy GE-filter can be a prominent fuzzy GE-filter. We have provided conditions for the trivial fuzzy GE-filter to be a prominent fuzzy GE-filter, and have explored the conditions under which the $\in_t$-set and $Q_t$-set can be prominent GE-filters. We finally have discussed the extension property for the prominent fuzzy GE-filter.

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