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A Note on LP-Kenmotsu Manifolds Admitting Conformal Ricci-Yamabe Solitons

Mobin Ahmad[1,*], **Gazala**[1], **Maha Atif Al-Shabrawi**[2]

[1] *Department of Mathematics and statistics, Integral University, Kursi Road, Lucknow-226026, India*

[2] *Department of Mathematical Sciences, Umm Ul Qura University, Makkah, Saudi Arabia*

* *Corresponding author:* mobinahmad68@gmail.com

Abstract. In the current note, we study Lorentzian para-Kenmotsu (in brief, LP-Kenmotsu) manifolds admitting conformal Ricci-Yamabe solitons (CRYS) and gradient conformal Ricci-Yamabe soliton (gradient CRYS). At last by constructing a 5-dimensional non-trivial example we illustrate our result.

1. Introduction

As a generalization of the classical Ricci flow [8], the concept of conformal Ricci flow was introduced by Fischer [5], which is defined on an n-dimensional Riemannian manifold M by the equations

$$\frac{\partial g}{\partial t} = -2(S + \frac{g}{n}) - pg, \quad r(g) = -1,$$

where p defines a time dependent non-dynamical scalar field (also called the conformal pressure), g is the Riemannian metric; r and S represent the scalar curvature and the Ricci tensor of M, respectively. The term $-pg$ plays a role of constraint force to maintain r in the above equation.

In [1], the authors Basu and Bhattacharyya proposed the concept of conformal Ricci soliton on M and is defined by

$$\pounds_K g + 2S + (2\Lambda - (p + \frac{2}{n}))g = 0,$$

where $\pounds_K$ represents the Lie derivative operator along the smooth vector field K on M and $\Lambda \in \mathbb{R}$ (the set of real numbers).

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Very recently, a scalar combination of Ricci and Yamabe flows was proposed by the authors Güler and Crasmareanu [7], this advanced class of geometric flows called Ricci-Yamabe (RY) flow of type (σ, ρ) and is defined by

$$\frac{\partial}{\partial t}g(t) + 2\sigma S(g(t)) + \rho r(t)g(t) = 0, \quad g(0) = g_0$$

for some scalars σ and ρ. A solution to the RY flow is called Ricci-Yamabe soliton (RYS) if it depends only on one parameter group of diffeomorphism and scaling. A Riemannian (or semi-Riemannian) manifold M is said to have a RYS if [9, 10]

$$\pounds_K g + 2\sigma S + (2\Lambda - \rho r)g = 0. \tag{1.1}$$

A Riemannian (or semi-Riemannian) manifold M is said to have a conformal Ricci-Yamabe soliton (CRYS) if [20]

$$\pounds_K g + 2\sigma S + (2\Lambda - \rho r - (p + \frac{2}{n}))g = 0, \tag{1.2}$$

where $\sigma, \rho, \Lambda \in \mathbb{R}$.

If K is the gradient of a smooth function v on M, then (1.2) is called the gradient conformal Ricci-Yamabe soliton (gradient CRYS) and hence (1.2) turns to

$$\nabla^2 v + \sigma S + (\Lambda - \frac{\rho r}{2} - \frac{1}{2}(p + \frac{2}{n}))g = 0, \tag{1.3}$$

where $\nabla^2 v$ is the Hessian of v and is defined by $Hess v = \nabla\nabla v$.

A CRYS is said to be shrinking, steady or expanding if $\Lambda < 0, = 0$ or > 0, respectively. A CRYS is said to be a

- Conformal Ricci soliton if $\sigma = 1, \rho = 0$,
- Conformal Yamabe soliton if $\sigma = 0, \rho = 1$,
- Conformal Einstein soliton if $\sigma = 1, \rho = -1$.

As a continuation of this study, we tried to study CRYS and gradient CRYS in the frame-work of LP-Kenmotsu manifolds of dimension n. We recommend the papers [2–4, 6, 13–17] and the references therein for more details about the related studies.

2. Preliminaries

An n-dimensional differentiable manifold M with structure (φ, ζ, ν, g) is said to be a Lorentzian almost paracontact metric manifold, if it admits a $(1,1)$-tensor field φ, a contravariant vector field ζ, a 1-form ν and a Lorentzian metric g satisfying

$$\nu(\zeta) + 1 = 0, \tag{2.1}$$

$$\varphi^2 E = E + \nu(E)\zeta, \tag{2.2}$$

$$\varphi\zeta = 0, \quad \nu(\varphi E) = 0,$$

$$g(\varphi E, \varphi F) = g(E, F) + \nu(E)\nu(F),$$

$$g(E, \zeta) = \nu(E), \tag{2.3}$$

$$\varphi(E, F) = \varphi(F, E) = g(E, \varphi F)$$

for any vector fields $E, F \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on M.
If ζ is a killing vector field, the (para) contact structure is called a K-(para) contact. In such a case, we have

$$\nabla_E \zeta = \varphi E.$$

Recently, the authors Haseeb and Prasad defined and studied the following notion:

Definition 2.1. *A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmostu manifold if [11]*

$$(\nabla_E \varphi)F = -g(\varphi E, F)\zeta - \nu(F)\varphi E$$

for any E, F on M.

In an LP-Kenmostu manifold, we have

$$\nabla_E \zeta = -E - \nu(E)\zeta, \tag{2.4}$$

$$(\nabla_E \nu)F = -g(E, F) - \nu(E)\nu(F), \tag{2.5}$$

where ∇ denotes the Levi-Civita connection respecting to the Lorentzian metric g.
Furthermore, in an LP-Kenmotsu manifold, the following relations hold [11]:

$$g(R(E, F)G, \zeta) = \nu(R(E, F)G) = g(F, G)\nu(E) - g(E, G)\nu(F),$$

$$R(\zeta, E)F = -R(E, \zeta)F = g(E, F)\zeta - \nu(F)E,$$

$$R(E, F)\zeta = \nu(F)E - \nu(E)F,$$

$$R(\zeta, E)\zeta = E + \nu(E)\zeta, \tag{2.6}$$

$$S(E, \zeta) = (n-1)\nu(E), \ \ S(\zeta, \zeta) = -(n-1), \tag{2.7}$$

$$Q\zeta = (n-1)\zeta,$$

for any $E, F, G \in \chi(M)$, where R, S and Q represent the curvature tensor, the Ricci tensor and the Q Ricci operator, respectively.

Definition 2.2. *[19] An LP-Kenmotsu manifold M is said to be ν-Einstein manifold if its $S(\neq 0)$ is of the form*

$$S(E, F) = ag(E, F) + b\nu(E)\nu(F),$$

where a and b are smooth functions on M. In particular, if $b = 0$, then M is termed as an Einstein manifold.

Remark 2.1. *[12] In an LP-Kenmotsu manifold of n-dimension, S is of the form*

$$S(E,F)=(\frac{r}{n-1}-1)g(E,F)+(\frac{r}{n-1}-n)\nu(E)\nu(F), \tag{2.8}$$

where r is the scalar curvature of the manifold.

Lemma 2.1. *In an n-dimensional LP-Kenmotsu manifold, we have*

$$\zeta(r)=2(r-n(n-1)), \tag{2.9}$$

$$(\nabla_E Q)\zeta = QE-(n-1)E, \tag{2.10}$$

$$(\nabla_\zeta Q)E = 2QE-2(n-1)E, \tag{2.11}$$

for any E on M.

Proof. Equation (2.8) yields

$$QE=(\frac{r}{n-1}-1)E+(\frac{r}{n-1}-n)\nu(E)\zeta. \tag{2.12}$$

Taking the covariant derivative of (2.12) with respect to F and making use of (2.4) and (2.5), we lead to

$$(\nabla_F Q)E=\frac{F(r)}{n-1}(E+\nu(E)\zeta)-(\frac{r}{n-1}-n)(g(E,F)\zeta+\nu(E)F+2\nu(E)\nu(F)\zeta).$$

By contracting F in the foregoing equation and using trace $\{F\to(\nabla_F Q)E\}=\frac{1}{2}E(r)$, we find

$$\frac{n-3}{2(n-1)}E(r)=\{\frac{\zeta(r)}{n-1}-(r-n(n-1))\}\nu(E),$$

which by replacing E by ζ and using (2.1) gives (2.9). We refer the readers to see [13] for the proof of (2.10) and (2.11). □

Remark 2.2. *From the equation (2.9), it is noticed that if an n-dimensional LP-Kenmotsu manifold possesses the constant scalar curvature, then $r=n(n-1)$ and hence (2.8) reduces to $S(E,F)=(n-1)g(E,F)$. Thus the manifold under consideration is an Einstein manifold.*

3. CRYS on LP-Kenmotsu Manifolds

Let the metric of an n-dimensional LP-Kenmotsu manifold be a conformal Ricci-Yamabe soliton, thus (1.2) holds. By differentiating (1.2) covariantly with resprct to G, we have

$$(\nabla_G \pounds_K g)(E,F) \;=\; -2\sigma(\nabla_G S)(E,F)+\rho(Gr)g(E,F). \tag{3.1}$$

Since $\nabla g=0$, then the following formula [18]

$$(\pounds_K\nabla_E g-\nabla_E\pounds_K g-\nabla_{[K,E]}g)(F,G)=-g((\pounds_K\nabla)(E,F),G)-g((\pounds_K\nabla)(E,G),F)$$

turns to

$$(\nabla_E\pounds_K g)(F,G)=g((\pounds_K\nabla)(E,F),G)+g((\pounds_K\nabla)(E,G),F).$$

Since the operator $\pounds_K\nabla$ is symmetric, therefore we have

$$2g((\pounds_K\nabla)(E,F),G)=(\nabla_E\pounds_K g)(F,G)+(\nabla_F\pounds_K g)(E,G)-(\nabla_G\pounds_K g)(E,F),$$

which by using (3.1) takes the form

$$\begin{aligned}2g((\pounds_K\nabla)(E,F),G) &= -2\sigma[(\nabla_E S)(F,G)+(\nabla_F S)(G,E)-(\nabla_G S)(E,F)]\\ &\quad +\rho[(Er)g(F,G)+(Fr)g(G,E)-(Gr)g(E,F)].\end{aligned}\tag{3.2}$$

Putting $F=\zeta$ in (3.2) and using (2.3), we find

$$\begin{aligned}2g((\pounds_K\nabla)(E,\zeta),G) &= -2\sigma[(\nabla_E S)(\zeta,G)+(\nabla_\zeta S)(G,E)-(\nabla_G S)(E,\zeta)]\\ &\quad +\rho[(Er)\nu(G)+2(r-n(n-1))g(E,G)-(Gr)\nu(E)].\end{aligned}\tag{3.3}$$

By virtue of (2.10) and (2.11), (3.3) leads to

$$\begin{aligned}2g((\pounds_K\nabla)(E,\zeta),G) &= -4\sigma[S(E,G)-(n-1)g(E,G)]\\ &\quad +\rho[(Er)\nu(G)+2(r-n(n-1))g(E,G)-(Gr)\nu(E)].\end{aligned}$$

By eliminating G from the foregoing equation, we have

$$\begin{aligned}2(\pounds_K\nabla)(F,\zeta) &= \rho g(Dr,F)\zeta-\rho(Dr)\nu(F)-4\sigma QF\\ &\quad +[4\sigma(n-1)+2\rho(r-n(n-1))]F.\end{aligned}\tag{3.4}$$

If we take r as constant, then from (2.9) it follows that $r=n(n-1)$, and hence (3.4) reduces to

$$(\pounds_K\nabla)(F,\zeta) = -2\sigma QF+2\sigma(n-1)F.\tag{3.5}$$

Taking covariant derivative of (3.5) with respect to E, we have

$$\begin{aligned}(\nabla_E\pounds_K\nabla)(F,\zeta) &= (\pounds_K\nabla)(F,E)-2\sigma\nu(E)[QF-(n-1)F]\\ &\quad - 2\sigma(\nabla_E Q)F.\end{aligned}\tag{3.6}$$

Again from [18], we have

$$(\pounds_K R)(E,F)G=(\nabla_E\pounds_K\nabla)(F,G)-(\nabla_F\pounds_K\nabla)(E,G),$$

which by putting $G=\zeta$ and using (3.6) takes the form

$$\begin{aligned}(\pounds_K R)(E,F)\zeta &= 2\sigma\nu(F)(QE-(n-1)E)-2\sigma\nu(E)(QF-(n-1)F)\\ &\quad -2\sigma((\nabla_E Q)F-(\nabla_F Q)E).\end{aligned}\tag{3.7}$$

Putting $F=\zeta$ in (3.7) then using (2.1), (2.2), (2.10) and (2.11), we arrive at

$$(\pounds_K R)(E,\zeta)\zeta=0.\tag{3.8}$$

The Lie derivative of (2.6) along K leads to

$$(\pounds_K R)(E,\zeta)\zeta-g(E,\pounds_K\zeta)\zeta+2\nu(\pounds_K\zeta)E=-(\pounds_K\nu)(E)\zeta.\tag{3.9}$$

From (3.8) and (3.9), we have

$$(\pounds_K \nu)(E)\zeta = -2\nu(\pounds_K \zeta)E + g(E, \pounds_K \zeta)\zeta. \tag{3.10}$$

Taking the Lie derivative of $g(E, \zeta) = \nu(E)$, we find

$$(\pounds_K \nu)(E) = g(E, \pounds_K \zeta) + (\pounds_K g)(E, \zeta). \tag{3.11}$$

By putting $F = \zeta$ in (1.2) and using (2.7), we have

$$(\pounds_K g)(E, \zeta) = -\{2\sigma(n-1) + 2\Lambda - \rho n(n-1) - (p + \frac{2}{n})\}\nu(E), \tag{3.12}$$

where $r = n(n-1)$ being used.

Taking the Lie derivative of $g(\zeta, \zeta) = -1$ along K we lead to

$$(\pounds_K g)(\zeta, \zeta) = -2\nu(\pounds_K \zeta). \tag{3.13}$$

From (3.12) and (3.13), we find

$$\nu(\pounds_K \zeta) = -\{\sigma(n-1) + \Lambda - \frac{\rho n(n-1)}{2} - \frac{1}{2}(p + \frac{2}{n})\}. \tag{3.14}$$

Now combining the equations (3.10), (3.11), (3.12) and (3.14), we find

$$\Lambda = \frac{\rho n(n-1)}{2} - \sigma(n-1) + \frac{1}{2}(p + \frac{2}{n}). \tag{3.15}$$

Thus we have

Theorem 3.1. *Let (M, g) be an n-dimensional LP-Kenmotsu manifold admitting CRYS with constant scalar curvature tensor, then* $\Lambda = \frac{\rho n(n-1)}{2} - \sigma(n-1) + \frac{1}{2}(p + \frac{2}{n})$.

Corollary 3.1. *Let the metric of n-dimensional LP-Kenmotsu manifold is CRYS. Then we have*

Values of σ, ρ	*Soliton type*	*Soliton constant*	*CRYS to be expanding, shrinking or steady*
$\sigma = 1, \rho = 0$	*conformal Ricci soliton*	$\Lambda = \frac{1}{2}(p+\frac{2}{n})-(n-1)$	*CRYS is shrinking, steady and expanding if* $p > \frac{2(n^2-n-1)}{n}$, $p = \frac{2(n^2-n-1)}{n}$ *and* $p < \frac{2(n^2-n-1)}{n}$, *resp.*
$\sigma = 0, \rho = 1$	*conformal Yamabe soliton*	$\Lambda = \frac{1}{2}(p + \frac{2}{n}) + \frac{n(n-1)}{2}$	*CRYS is shrinking, steady and expanding if* $p < \frac{-(n^3-n^2+2)}{n}$, $p = \frac{-(n^3-n^2+2)}{n}$ *and* $p > \frac{-(n^3-n^2+2)}{n}$, *resp.*
$\sigma = 1, \rho = -1$	*conformal Einstein soliton*	$\Lambda = \frac{1}{2}(p + \frac{2}{n}) - \frac{(n-1)(n+2)}{2}$	*CRYS is shrinking, steady and expanding if* $p < \frac{(n+1)(n^2-2)}{n}$, $p = \frac{(n+1)(n^2-2)}{n}$ *and* $p > \frac{(n+1)(n^2-2)}{n}$, *resp.*

4. Gradient CRYS on LP-Kenmotsu Manifolds

Let M be an n-dimensional LP-Kenmotsu manifold with g as a gradient CRYS. Then equation (1.3) can be written as

$$\nabla_E Dv + \sigma QE + (\Lambda - \frac{\rho r}{2} - \frac{1}{2}(p + \frac{2}{n}))E = 0, \tag{4.1}$$

for all vector fields E on M, where D denotes the gradient operator of g. Taking the covariant derivative of (4.1) with respect to F, we have

$$\begin{aligned} \nabla_F \nabla_E Dv &= -\sigma\{(\nabla_F Q)E + Q(\nabla_F E)\} + \rho\frac{F(r)}{2}E \\ &\quad -(\Lambda - \frac{\rho r}{2} - \frac{1}{2}(p + \frac{2}{n}))\nabla_F E. \end{aligned} \tag{4.2}$$

Interchanging E and F in (4.2), we lead to

$$\begin{aligned} \nabla_E \nabla_F Dv &= -\sigma\{(\nabla_E Q)F + Q(\nabla_E F)\} + \rho\frac{E(r)}{2}F \\ &\quad -(\Lambda - \frac{\rho r}{2} - \frac{1}{2}(p + \frac{2}{n}))\nabla_E F. \end{aligned} \tag{4.3}$$

By making use of (4.1)-(4.3), we find

$$R(E,F)Dv = \sigma\{(\nabla_F Q)E - (\nabla_E Q)F\} + \frac{\rho}{2}\{E(r)F - F(r)E\}. \tag{4.4}$$

Now from (2.8), we find

$$QE = (\frac{r}{n-1} - 1)E + (\frac{r}{n-1} - n)\nu(E)\zeta,$$

which on taking covariant derivative with repect to F leads to

$$\begin{aligned} (\nabla_F Q)E &= \frac{F(r)}{n-1}(E + \nu(E)\zeta) - (\frac{r}{n-1} - n)(g(E,F)\zeta \\ &\quad +2\nu(E)\nu(F)\zeta + \nu(E)F). \end{aligned} \tag{4.5}$$

By using (4.5) in (4.4), we have

$$\begin{aligned} R(E,F)Dv &= \frac{(n-1)\rho - 2\sigma}{2(n-1)}\{E(r)F - F(r)E\} + \frac{\sigma}{n-1}\{F(r)\nu(E)\zeta - E(r)\nu(F)\zeta\} \\ &\quad -\sigma(\frac{r}{n-1} - n)(\nu(E)F - \nu(F)E). \end{aligned} \tag{4.6}$$

Contracting forgoing equation along E gives

$$\begin{aligned} S(F,Dv) &= -\{\frac{(n-1)^2\rho - 2\sigma(n-2)}{2(n-1)}\}F(r) \\ &\quad +\frac{\sigma(n-3)(r - n(n-1))}{n-1}\nu(F). \end{aligned} \tag{4.7}$$

From the equation (2.8), we have

$$S(F,Dv) = (\frac{r}{n-1} - 1)F(v) + (\frac{r}{n-1} - n)\nu(F)\zeta(v). \tag{4.8}$$

Now by equating (4.7) and (4.8), then putting $F=\zeta$ and using (2.1), (2.9), we find

$$\zeta(v)=\frac{r-n(n-1)}{n-1}\{\sigma-(n-1)\rho\}. \tag{4.9}$$

Taking the inner product of (4.6) with ζ, we get

$$F(v)\nu(E)-E(v)\nu(F)=\frac{\rho}{2}\{E(r)\nu(F)-F(r)\nu(E)\},$$

which by replacing E by ζ then using (2.9) and (4.9), we infer

$$F(v)=-\frac{\sigma(r-n(n-1))}{n-1}\nu(F)-\frac{\rho}{2}F(r). \tag{4.10}$$

If we take r as constant, then from Remark 2.5, we get $r=n(n-1)$. Thus (4.10) leads to $F(v)=0$. This implies that v is constant. Thus the soliton under the consideration is trivial. Hence we state:

Theorem 4.1. *If the metric of an n-dimensional LP-Kenmotsu manifold of constant scalar curvature tensor admitting a special type of vector field is gradient CRYS, then the soliton is trivial.*

For v constant, (1.3) turns to

$$\sigma QE=-(\Lambda-\frac{\rho r}{2}-\frac{1}{2}(p+\frac{2}{n}))E,$$

which leads to

$$S(E,F)=-\frac{1}{\sigma}(\Lambda-\frac{\rho n(n-1)}{2}-\frac{1}{2}(p+\frac{2}{n}))g(E,F),\quad \sigma\neq 0. \tag{4.11}$$

By putting $E=F=\zeta$ in (4.11) and using (2.7), we obtain

$$\Lambda=\frac{\rho n(n-1)}{2}-\sigma(n-1)+\frac{1}{2}(p+\frac{2}{n}). \tag{4.12}$$

Corollary 4.1. *If an n-dimensional LP-Kenmotsu manifold admits a gradient CRYS with the constant scalar curvature, then the manifold under the consideration is an Einstein manifold and* $\Lambda=\frac{\rho n(n-1)}{2}-\sigma(n-1)+\frac{1}{2}(p+\frac{2}{n})$.

5. Example

We consider the 5-dimensional manifold $M^5=\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5 : x_5>0\}$, where (x_1,x_2,x_3,x_4,x_5) are the standard coordinates in $\mathbb{R}^5$. Let ϱ_1, ϱ_2, ϱ_3, ϱ_4 and ϱ_5 be the vector fields on M^5 given by

$$\varrho_1=e^{x_5}\frac{\partial}{\partial x_1},\ \varrho_2=e^{x_5}\frac{\partial}{\partial x_2},\ \varrho_3=e^{x_5}\frac{\partial}{\partial x_3},\ \varrho_4=e^{x_5}\frac{\partial}{\partial x_4},\ \varrho_5=\frac{\partial}{\partial x_5}=\zeta,$$

which are linearly independent at each point of M^5. Let g be the Lorentzian metric defined by

$$g(\varrho_i,\varrho_i)=1,\quad \text{for}\quad 1\leq i\leq 4\quad \text{and}\quad g(\varrho_5,\varrho_5)=-1,$$

$$g(\varrho_i,\varrho_j)=0,\quad \text{for}\quad i\neq j,\quad 1\leq i,j\leq 5.$$

Let ν be the 1-form defined by $\nu(E) = g(E, \varrho_5) = g(E, \zeta)$ for all $E \in \chi(M^5)$, and let φ be the $(1,1)$-tensor field defined by

$$\varphi\varrho_1 = -\varrho_2,\ \varphi\varrho_2 = -\varrho_1,\ \varphi\varrho_3 = -\varrho_4,\ \varphi\varrho_4 = -\varrho_3,\ \varphi\varrho_5 = 0.$$

By applying linearity of φ and g, we have

$$\nu(\zeta) = g(\zeta,\zeta) = -1,\ \varphi^2 E = E + \nu(E)\zeta \text{ and } g(\varphi E, \varphi F) = g(E,F) + \nu(E)\nu(F)$$

for all $E, F \in \chi(M^5)$. Thus for $\varrho_5 = \zeta$, the structure (φ, ζ, ν, g) defines a Lorentzian almost paracontact metric structure on M^5. Then we have

$$[\varrho_i, \varrho_j] = -\varrho_i, \quad \text{for} \quad 1 \le i \le 4, j = 5,$$

$$[\varrho_i, \varrho_j] = 0, \quad \text{otherwise.}$$

By using Koszul's formula, we can easily find we obtain

$$\nabla_{\varrho_i}\varrho_j = \begin{cases} -\varrho_5, & 1 \le i = j \le 4, \\ -\varrho_i, & 1 \le i \le 4, j = 5, \\ 0, & otherwise. \end{cases}$$

Also one can easily verify that

$$\nabla_E \zeta = -E - \eta(E)\zeta \quad \text{and} \quad (\nabla_E \varphi)F = -g(\varphi E, F)\zeta - \nu(F)\varphi E.$$

Therefore, the manifold is an LP-Kenmotsu manifold.

From the above results, we can easily obtain the non-vanishing components of R as follows:

$$R(\varrho_1,\varrho_2)\varrho_1 = -\varrho_2,\ R(\varrho_1,\varrho_2)\varrho_2 = \varrho_1,\ R(\varrho_1,\varrho_3)\varrho_1 = -\varrho_3,\ R(\varrho_1,\varrho_3)\varrho_3 = \varrho_1,$$

$$R(\varrho_1,\varrho_4)\varrho_1 = -v_4,\ R(\varrho_1,\varrho_4)\varrho_4 = \varrho_1,\ R(\varrho_1,\varrho_5)\varrho_1 = -\varrho_5,\ R(\varrho_1,\varrho_5)\varrho_5 = -\varrho_1,$$

$$R(\varrho_2,\varrho_3)\varrho_2 = -\varrho_3,\ R(\varrho_2,\varrho_3)\varrho_3 = \varrho_2,\ R(\varrho_2,\varrho_4)\varrho_2 = -\varrho_4,\ R(\varrho_2,\varrho_4)\varrho_4 = \varrho_2,$$

$$R(\varrho_2,\varrho_5)\varrho_2 = -\varrho_5,\ R(\varrho_2,\varrho_5)\varrho_5 = -\varrho_2,\ R(\varrho_3,\varrho_4)\varrho_3 = -\varrho_4,\ R(\varrho_3,\varrho_4)\varrho_4 = \varrho_3,$$

$$R(\varrho_3,\varrho_5)\varrho_3 = -\varrho_5,\ R(\varrho_3,\varrho_5)\varrho_5 = -\varrho_3,\ R(\varrho_4,\varrho_5)\varrho_4 = -\varrho_5,\ R(\varrho_4,\varrho_5)\varrho_5 = -\varrho_4.$$

Also, we calculate the Ricci tensors as follows:

$$S(\varrho_1,\varrho_1) = S(\varrho_2,\varrho_2) = S(\varrho_3,\varrho_3) = S(\varrho_4,\varrho_4) = 4, \quad S(\varrho_5,\varrho_5) = -4.$$

Therefore, we have

$$r = S(\varrho_1,\varrho_1) + S(\varrho_2,\varrho_2) + S(\varrho_3,\varrho_3) + S(\varrho_4,\varrho_4) - S(\varrho_5,\varrho_5) = 20.$$

Now by taking $Dv = (\varrho_1 v)\varrho_1 + (\varrho_2 v)\varrho_2 + (\varrho_3 v)\varrho_3 + (\varrho_4 v)\varrho_4 + (\varrho_5 v)\varrho_5$, we have

$$\begin{aligned} \nabla_{\varrho_1} Dv &= (\varrho_1(\varrho_1 v) - (\varrho_5 v))\varrho_1 + (\varrho_1(\varrho_2 v))\varrho_2 + (\varrho_1(\varrho_3 v))\varrho_3 + (\varrho_1(\varrho_4 v))\varrho_4 \\ &\quad + (\varrho_1(\varrho_5 v) - (\varrho_1 v))\varrho_5, \end{aligned}$$

$$\begin{aligned}\nabla_{\varrho_2}Dv &= (\varrho_2(\varrho_1 v))\varrho_1 + (\varrho_2(\varrho_2 v) - (\varrho_5 v))\varrho_2 + (\varrho_2(\varrho_3 v))\varrho_3 + (\varrho_2(\varrho_4 v))\varrho_4\\ &\quad +(\varrho_2(\varrho_5 v) - (\varrho_2 v))\varrho_5,\\ \nabla_{\varrho_3}Dv &= (\varrho_3(\varrho_1 v))\varrho_1 + (\varrho_3(\varrho_2 v))\varrho_2 + (\varrho_3(\varrho_3 v) - (\varrho_5 v))\varrho_3 + (\varrho_3(\varrho_4 v))\varrho_4\\ &\quad +(\varrho_3(\varrho_5 v) - (\varrho_3 v))\varrho_5,\\ \nabla_{\varrho_4}Dv &= (\varrho_4(\varrho_1 v))\varrho_1 + (\varrho_4(\varrho_2 v))\varrho_2 + (\varrho_4(\varrho_3 v))\varrho_3 + (\varrho_4(\varrho_4 v) - (\varrho_5 v))\varrho_4\\ &\quad +(\varrho_4(\varrho_5 v) - (\varrho_4 v))\varrho_5,\end{aligned}$$

$$\nabla_{\varrho_5}Dv = (\varrho_5(\varrho_1 v))\varrho_1 + (\varrho_5(\varrho_2 v))\varrho_2 + (\varrho_5(\varrho_3 v))\varrho_3 + (\varrho_5(\varrho_4 v))\varrho_4 + (\varrho_5(\varrho_5 v))\varrho_5.$$

Thus by virtue of (4.1), we obtain

$$\begin{cases} \varrho_1(\varrho_1 v) - \varrho_5 v = -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ \varrho_2(\varrho_2 v) - \varrho_5 v = -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ \varrho_3(\varrho_3 v) - \varrho_5 v = -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ \varrho_4(\varrho_4 v) - \varrho_5 v = -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ \varrho_5(\varrho_5 v) = -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ \varrho_1(\varrho_2 v) = \varrho_1(\varrho_3 v) = \varrho_1(\varrho_4 v) = 0,\\ \varrho_2(\varrho_1 v) = \varrho_2(\varrho_3 v) = \varrho_2(\varrho_4 v) = 0,\\ \varrho_3(\varrho_1 v) = \varrho_3(\varrho_2 v) = \varrho_3(\varrho_4 v) = 0,\\ \varrho_4(\varrho_1 v) = \varrho_4(\varrho_2 v) = \varrho_4(\varrho_3 v) = 0,\\ \varrho_1(\varrho_5 v) - (\varrho_1 v) = \varrho_2(\varrho_5 v) - (\varrho_2 v) = 0,\\ \varrho_3(\varrho_5 v) - (\varrho_3 v) = \varrho_4(\varrho_5 v) - (\varrho_4 v) = 0. \end{cases} \tag{5.1}$$

Thus the equations in (5.1) are respectively amounting to

$$\begin{aligned} e^{2x_5}\frac{\partial^2 v}{\partial x_1^2} - \frac{\partial v}{\partial x_5} &= -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ e^{2x_5}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial v}{\partial x_5} &= -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ e^{2x_5}\frac{\partial^2 v}{\partial x_3^2} - \frac{\partial v}{\partial x_5} &= -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ e^{2x_5}\frac{\partial^2 v}{\partial x_4^2} - \frac{\partial v}{\partial x_5} &= -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})),\\ \frac{\partial^2 v}{\partial x_5^2} &= -(\Lambda + 4\sigma - 10\rho - \frac{1}{2}(p + \frac{2}{5})), \end{aligned}$$

$$\frac{\partial^2 v}{\partial x_1\partial x_2} = \frac{\partial^2 v}{\partial x_1\partial x_3} = \frac{\partial^2 v}{\partial x_1\partial x_4} = \frac{\partial^2 v}{\partial x_2\partial x_3} = \frac{\partial^2 v}{\partial x_2\partial x_4} = \frac{\partial^2 v}{\partial x_3\partial x_4} = 0,$$

$$e^{x_5}\frac{\partial^2 v}{\partial x_5 \partial x_1}+\frac{\partial v}{\partial x_1}=e^{x_5}\frac{\partial^2 v}{\partial x_5 \partial x_2}+\frac{\partial v}{\partial x_2}=e^{x_5}\frac{\partial^2 v}{\partial x_5 \partial x_3}+\frac{\partial v}{\partial x_3}=e^{x_5}\frac{\partial^2 v}{\partial x_5 \partial x_4}+\frac{\partial v}{\partial x_4}=0.$$

From the above equations it is observed that v is constant for $\Lambda = -4\sigma + 10\rho + \frac{1}{2}(p + \frac{2}{5})$. Hence equation (4.1) is satisfied. Thus, g is a gradient RYS with the soliton vector field $K = Dv$, where v is constant and $\Lambda = -4\sigma + 10\rho + \frac{1}{2}(p + \frac{2}{5})$. Thus, Theorem 4.1 is verified.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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