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On Anti-Q-Fuzzy Deductive Systems of Hilbert Algebras

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Abstract. In this paper, the concept of anti-Q-fuzzy deductive systems concepts of Hilbert algebras are introduced and proved some results. Further, we discuss the relation between anti-Q-fuzzy deductive system and level subsets of a Q-fuzzy set. Anti Q-fuzzy deductive system is also applied in the Cartesian product of Hilbert algebras.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [19]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 4, 7]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [12–14]. The concept of Hilbert algebra was introduced in early 50-ties by L.Henkin and T.Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A.Horn and A.Diego from algebraic point of view. A.Diego proved (cf. [9] that Hilbert algebras form a variety which is locally finite. Hilbert algebras were

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treated by D.Busneag (cf. [5], [6]) and Y.B.Jun (cf. [15]) and some of their filters forming deductive systems were recognized. W.A.Dudek (cf. [11]) considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. In this paper, the concept of anti-Q-fuzzy deductive systems concepts of Hilbert algebras are introduced and proved some results. Further, we discuss the relation between anti-Q-fuzzy deductive system and level subsets of a Q-fuzzy set. Anti Q-fuzzy deductive system is also applied in the Cartesian product of Hilbert algebras.

2. Preliminaries

Definition 2.1. *[9] A Hilbert algebra is a triplet $H = (H, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation and 1 is fixed element of H such that the following axioms hold for each $x, y, z \in H$.*

(1) $x * (y * x) = 1$,
(2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(3) $x * y = 1$ *and* $y * x = 1$ *imply* $x = y$.

The following result was proved in [11].

Lemma 2.1. *Let $H = (H, \cdot, 1)$ be a Hilbert algebra and $x, y, z \in H$. Then*

(1) $x * x = 1$,
(2) $1 * x = x$,
(3) $x * 1 = 1$,
(4) $x * (y * z) = y * (x * z)$.

It is easily checked that in a Hilbert algebra H the relation $\leq y$ defined by $x \leq y \;\Leftrightarrow\; x * y = 1$ is a partial order on H with 1 as the largest element.

Definition 2.2. *[8] A nonempty subset I of a Hilbert algebra $H = (H, \cdot, 1)$ is called an ideal of H if*

(1) $1 \in I$,
(2) $x * y, q \in I$ *for all* $x \in H$, $y \in I$,
(3) $(y_2 * (y_1 * x)) * x \in I$ *for all* $x \in H$, $y_1, y_2 \in I$.

Definition 2.3. *[10] A fuzzy set μ in a Hilbert algebra H is said to be a fuzzy ideal of H if the following conditions are hold:*

(1) $\mu(1, q) \geq \mu(x, q)$ *for all* $x \in H$,
(2) $\mu(x * y, q) \geq \mu(y, q)$ *for all* $x, y \in H$,
(3) $\mu((y_1 * (y_2 * x, q), q) * x) \geq \min\{\mu(y_1, q), \mu(y_2, q)\}$ *for all* $x, y_1, y_2 \in H$.

Lemma 2.2. *Let μ be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) $1 - \max\{\mu(x), \mu(y)\} = \min\{1 - \mu(x), 1 - \mu(y)\}$,
(2) $1 - \min\{\mu(x), \mu(y)\} = \max\{1 - \mu(x), 1 - \mu(y)\}$.

Definition 2.4. *[16] A Q-fuzzy set in a nonempty set H (or a Q-fuzzy subset of H) is an arbitrary function $\mu : X \times Q \to [0,1]$, where Q is a nonempty set and $[0,1]$ is the unit segment of the real line.*

Definition 2.5. *[16] Let μ be a Q-fuzzy set in A. The Q-fuzzy set μ defined by $\overline{\mu}(x,q) = 1-\mu(x,q)$ for all $x \in A$ and $q \in Q$ is called the complement of μ in A.*

Remark 2.1. *For a Q-fuzzy set μ in A, we have $\mu = \overline{\overline{\mu}}$.*

Definition 2.6. *[17] Let $f : A \to B$ be a function and μ be a Q-fuzzy set in B. We define a new Q-fuzzy set in A by μ_f as $\mu(f(x),q)$ for all $x \in A$ and $q \in Q$.*

Definition 2.7. *[17] Let $f : A \to B$ be a bijection and μ_f be a Q-fuzzy set in A. We define a new Q-fuzzy set in B by μ as $\mu(y,q) = \mu_f(x,q)$, where $f(x) = y$ for all $y \in B$ and $q \in Q$.*

Definition 2.8. *[17] Let μ be a Q-fuzzy set in A and δ be a Q-fuzzy set in B. The Cartesian product $\mu \times \delta : (A \times B) \times Q \to [0,1]$ is defined by $(\mu \times \delta)((x,y),q) = \max\{\mu(x,q),\delta(y,q)\}$ for all $x \in A$, $y \in B$ and $q \in Q$. The dot product $\mu \cdot \delta : (A \times B) \times Q \to [0,1]$ is defined by $(\mu * \delta)((x,y),q) = \min\{\mu(x,q),\delta(y,q)\}$ for all $x \in A$, $y \in B$ and $q \in Q$.*

Lemma 2.3. *For any $a, b \in \mathbb{R}$ such that $a < b$, $a < \frac{b+a}{2} < b$.*

3. Anti-Q-fuzzy deductive systems

Definition 3.1. *A Q-fuzzy set μ in a Hilbert algebra H is said to be an anti-Q-fuzzy deductive system of H if the following conditions are hold:*

$$(\forall x \in H)\left(\ \mu(1,q) \leq \mu(x,q)\ \right), \tag{3.1}$$

$$(\forall\ x, y \in H)\left(\ \mu(y,q) \leq \max\{\mu(x*y,q), \mu(x,q)\}\ \right). \tag{3.2}$$

A Q-fuzzy set μ in a Hilbert algebra H is called an anti-Q-fuzzy deductive system of H if it is an anti-q-fuzzy deductive system of H for all $q \in Q$.

Example 3.1. *Let $H = \{1, x, y, z, 0\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	x	1	z
z	1	1	y	1

Then $(H, \cdot, 1)$ is a Hilbert algebra. Let $Q = \{q\}$. We define a q-fuzzy set μ as follows: $\mu(1,q) = 0.9, \mu(x,q) = 0.3, \mu(y,q) = 0.1, \mu(z,q) = 0.6$. Then μ is a Q-fuzzy ideal of H.

Lemma 3.1. *If μ is an anti-Q-fuzzy deductive system of a Hilbert algebra H, then*

$$(\forall\, x, y, z \in H)\left(\;\; z \leq x * y \;\Rightarrow \mu(y, q) \leq \max\{\mu(x, q), \mu(z, q)\} \;\right). \tag{3.3}$$

Proof. Let $x, y, z \in X$ such that $z \leq x * y$. Then $z \cdot (x * y) = 1$.

$$\begin{aligned} \mu(y, q) &\leq \max\{\mu(x * y, q), \mu(x, q)\} \\ &\leq \max\{\max\{\mu(z \cdot (x * y), q), \mu(z, q)\}, \mu(x, q)\} \\ &= \max\{\max\{\mu(1, q), \mu(z, q)\}, \mu(x, q)\} \\ &= \max\{\mu(x, q), \mu(z, q)\}. \end{aligned}$$

□

Lemma 3.2. *If μ is an anti-Q-fuzzy deductive system of a Hilbert algebra H, then*

$$(\forall\, x, y \in H)\left(\;\; x \leq y \;\Rightarrow \mu(x, q) \leq \mu(y, q) \;\right). \tag{3.4}$$

Proof. Let $x, y \in H$ be such that $x \leq y$. Then $x * y = 1$ and so $\mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\} = \max\{\mu_A(1, q), \mu_A(x, q)\} = \mu_A(x, q)$. □

Theorem 3.1. *If μ is an anti-Q-fuzzy deductive system of a Hilbert algebra H, then for any $x, a_1, a_2, ..., a_n \in X$, $(...((x * a_1) * a_2) \cdot ...) * a_n = 1$ implies $\mu(x, q) \leq \max\{\mu(a_1, q), \mu(a_2, q), \ldots \mu(a_n, q)\}$.*

Proof. Apply induction on n and apply Lemmas 3.1 and 3.2, the proof is clear. □

Theorem 3.2. *Every anti-Q-fuzzy deductive system of Hilbert algebra X is an anti-Q-fuzzy subalgebra of X.*

Proof. Let μ be an anti-Q-fuzzy deductive system of X. Since $y \leq x * y$ for all $x, y \in X$, from Lemma 3.2 that $\mu(y, q) \leq \mu(x * y, q)$ for all $q \in Q$. It follows from that $\mu(x * y, q) \leq \mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\} \leq \max\{\mu(x, q), \mu(y, q)\}$. Hence μ is an anti-Q-fuzzy subalgebra of X. □

Theorem 3.3. *If μ is a Q-fuzzy subalgebra of X such that $\mu(y, q) \geq \min\{\mu(x, q), \mu(z, q)\}$ for all $x, y, z \in X$ satisfying $x * y \leq z$, then μ is an anti-Q-fuzzy deductive system of X.*

Proof. Let μ be an anti-Q-fuzzy subalgebra of X. Then by definition $\mu(1, q) \leq \mu(z, q)$ for all $x \in X$ and $q \in Q$. Since $x \leq (x * y) * y$, then by hypothesis, $\mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\}$. Hence μ is an anti-Q-fuzzy deductive system of X. □

Proposition 3.1. *If $\{\mu_i : i \in \Delta\}$ is a family of anti-Q-fuzzy deductive systems of a Hilbert algebra H, then $\bigwedge\limits_{i \in \Delta} \mu_i$ is an anti-Q-fuzzy deductive system of H.*

Proof. Let $\{\mu_i : i \in \Delta\}$ be a family of anti-Q-fuzzy deductive systems of a Hilbert algebra H. Let $x \in H$ and $q \in Q$, we have $(\bigwedge_{i\in\Delta} \mu_i)(1, q) = \sup_{i\in\Delta}\{\mu_i(1, q)\} \leq \sup_{i\in\Delta}\{\mu_i(x, q)\} = (\bigwedge_{i\in\Delta} \mu_i)(x, q)$. Let $x, y \in H$ and $q \in Q$, we have

$$\begin{aligned}
(\bigwedge_{i\in\Delta} \mu_i)(y, q) &= \sup_{i\in\Delta}\{\mu_i(y, q)\} \\
&\leq \sup_{i\in\Delta}\{\max\{\mu_i(x * y, q), \mu_i(x, q)\}\} \\
&= \max\{\sup_{i\in\Delta} \mu_i(x * y, q), \sup_{i\in\Delta} \mu_i(x, q)\} \\
&= \max\{(\bigwedge_{i\in\Delta} \mu_i)(x * y, q), (\bigwedge_{i\in\Delta} \mu_i)(x, q)\}.
\end{aligned}$$

Hence $\bigwedge_{i\in\Delta} \mu_i$ is an anti-Q-fuzzy deductive system of a Hilbert algebra H. □

Proposition 3.2. *Every Q-fuzzy ideal of a Hilbert algebra H is an anti-Q-fuzzy deductive system of H.*

Proof. Let μ be an anti-Q-fuzzy ideal of H. If $y_1 = x * y$, $y_2 = x$, where $x, y \in H$ and $q \in Q$, then by (1), (2) of Lemma 2.1, we have $\mu(y, q) = \mu(1 * y, q) = \mu(((x * y, q) * (x * y, q)) * y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\}$. Hence μ be an anti-Q-fuzzy deductive system of H. □

Lemma 3.3. *[18] Let μ be a fuzzy set in A and for any $t \in [0, 1]$. Then the following properties hold:*

(1) $L(\mu, t) = U(\overline{\mu}, 1 - t)$,
(2) $L^-(\mu, t) = U^+(\overline{\mu}, 1 - t)$,
(3) $U(\mu, t) = L(\overline{\mu}, 1 - t)$,
(4) $U^+(\mu, t) = L^-(\overline{\mu}, 1 - t)$.

Lemma 3.4. *[18] Let μ be a Q-fuzzy set in A and for any $t \in [0, 1]$ and $q \in Q$. Then the following properties hold:*

(1) $L(\mu, t, q) = U(\overline{\mu}, 1 - t, q)$,
(2) $L^-(\mu, t, q) = U^+(\overline{\mu}, 1 - t, q)$,
(3) $U(\mu, t, q) = L(\overline{\mu}, 1 - t, q)$,
(4) $U^+(\mu, t, q) = L^-(\overline{\mu}, 1 - t, q)$.

Lemma 3.5. *[18] Let μ be a Q-fuzzy set in A and for any $t \in [0, 1]$ and $q \in Q$. Then the following properties hold:*

(1) $L(\mu, t) = \bigcap_{q\in Q} L(\mu, t, q)$,
(2) $L^-(\mu, t) = \bigcap_{q\in Q} L^-(\mu, t, q)$,
(3) $U(\mu, t) = \bigcap_{q\in Q} U(\mu, t, q)$,
(4) $U^+(\mu, t) = \bigcap_{q\in Q} U^+(\mu, t, q)$.

Definition 3.2. *Let μ be a q-fuzzy set of a Hilbert algebra H and $t \in [0,1]$. Then we define the sets $U(\mu,t) = \{x \in H : \mu(x,q) \geq t \; for \; all \; q \in Q\}$ and $U^{+}(\mu,t) = \{x \in H : \mu(x,q) > t \; for \; all \; q \in Q\}$ are called an upper α-level subset and an upper α-strong level subset of μ, respectively. The sets $L(\mu,t) = \{x \in H : \mu(x,q) \leq t \; for \; all \; q \in Q\}$ and $L^{-}(\mu,t) = \{x \in H : \mu(x,q) < t \; for \; all \; q \in Q\}$ are called a lower t-level subset and a lower t-strong level subset of μ, respectively. For any $q \in Q$, the sets $U(\mu,t,q) = \{x \in H : \mu(x,q) \geq t\}$ and $U^{+}(\mu,t,q) = \{x \in H : \mu(x,q) > t\}$ are called a q-upper t-level subset and a q-upper t-strong level subset of μ, respectively. The sets $L(\mu,t,q) = \{x \in H : \mu(x,q) \leq t\}$ and $L^{+}(\mu,t,q) = \{x \in H : \mu(x,q) < t\}$ are called a q-lower t-level subset and a q-lower t-strong level subset of μ, respectively.*

Theorem 3.4. *Let μ be a Q-fuzzy set in H. Then the following statements hold:*

(1) *μ is an anti-Q-fuzzy deductive system of H if and only if for any $t \in [0,1]$ and $q \in Q$, $L(\mu,t,q)$ is either empty or a deductive system of H.*
(2) *μ is an anti-Q-fuzzy deductive system of H if and only if for any $t \in [0,1]$ and $q \in Q$, $L^{-}(\mu,t,q)$ is either empty or a deductive system of H.*
(3) *$\overline{\mu}$ is an anti-Q-fuzzy deductive system of H if and only if for any $t \in [0,1]$ and $q \in Q$, $U(\mu,t,q)$ is either empty or a deductive system of H.*
(4) *$\overline{\mu}$ is an anti-Q-fuzzy deductive system of H if and only if for any $t \in [0,1]$ and $q \in Q$, $U^{+}(\mu,t,q)$ is either empty or a deductive system of H.*

Proof. (1). Assume that μ is an anti-Q-fuzzy deductive system of H. Then μ is an anti-Q-fuzzy deductive system of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(\mu,t,q) \neq \emptyset$ and let $x \in H$ be such that $x \in L(\mu,t,q)$. Then $\mu(x,q) \leq t$. Thus $\mu(1,q) = \mu(x * x,q) \leq \mu(x,q)$. Hence $\mu(1,q) \leq \mu(x,q) \leq t$, so $1 \in L(\mu,t,q)$. Let $x,y \in H$ and $q \in Q$ be such that $x \in L(\mu,t,q)$ and $x * y \in L(\mu,t,q)$. Then $\mu(x,q) \leq t$ and $\mu(x*y,q) \leq t$. Then we have $\mu(y,q) \leq \max\{\mu(x*y,q),\mu(x,q)\}$. Then $\mu(y,q) \leq \max\{\mu(x*y,q),\mu(x,q)\} \leq t$, so $y \in L(\mu,t,q)$. Hence $L(\mu,t,q)$ is a deductive system of H. Conversely, assume that every nonempty set $L(\mu,t,q)$ is a deductive system in H. If $\mu(1,q) \leq \mu(x,q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\mu(1,q) > \mu(x_0,q)$. Let $t = \frac{1}{2}(\mu(1,q) + \mu(x_0,q))$. Then $t \in [0,1]$ and by Lemma 2.8, we have $\mu(1,q) > t > \mu(x_0,q)$. Thus $x_0 \in L(\mu,t,q)$, so, $L(\mu,t,q) \neq \emptyset$. By assumption, we have $L(\mu,t,q)$ is a deductive system of H. It follows that $1 \in L(\mu,t,q)$, so $\mu(1,q) \leq t$, which is a contradicition. Hence $\mu(1,q) \leq \mu(x,q)$ for all $x \in H$ and $q \in Q$. Suppose that $\mu(y,q) \leq \max\{\mu(x*y,q),\mu(x,q)\}$ is not true for all $x,y \in H$ and $q \in Q$. Then there exist $u_0,v_0 \in H$ and $q \in Q$ such that $\mu(v_0,q) > \max\{\mu(u_0*v_0,q),\mu(u_0,q)\}$. Let $p = \frac{1}{2}(\mu(v_0,q) + \max\{\mu(u_0*v_0,q),\mu(u_0,q)\})$. Then $p \in [0,1]$ and we have $\mu(v_0,q) > p > \max\{\mu(u_0*v_0,q),\mu(u_0,q)\}$. Then $u_0,v_0 \in L(\mu,p,q)$, so, $L(\mu,p,q) \neq \emptyset$. By assumption, we have $L(\mu,p,q)$ is a deductive system of H. It follows that $v_0 \in L(\mu,p,q)$. Hence $\mu(v_0,q) \leq p$, which is a contradiction. Hence $\mu(y,q) \leq \max\{\mu(x*y,q),\mu(x,q)\}$ is true for all $x,y \in H$ and $q \in Q$. Hence μ is an anti-q-fuzzy deductive system of H for all $q \in Q$. Consequently

μ is an anti-Q-fuzzy deductive system of H.

(2). Assume that μ is an anti-Q-fuzzy deductive system of H. Then μ is an anti-Q-fuzzy deductive system of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L^-(\mu, t, q) \neq \emptyset$ and let $x \in H$ be such that $x \in L^-(\mu, t, q)$. Then $\mu(x, q) < t$. Thus $\mu(1, q) = \mu(x * x, q) \leq \mu(x, q)$. Hence $\mu(1, q) \leq \mu(x, q) < t$, so $1 \in L^-(\mu, t, q)$. Let $x, y \in H$ and $q \in Q$ be such that $x \in L^-(\mu, t, q)$ and $x * y \in L^-(\mu, t, q)$. Then $\mu(x, q) < t$ and $\mu(x * y, q) < t$. Then we have $\mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\}$. Then $\mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\} < t$, so $y \in L^-(\mu, t, q)$. Hence $L^-(\mu, t, q)$ is a deductive system of H. Conversely, assume that every nonempty set $L^-(\mu, t, q)$ is a deductive system in H. If $\mu(1, q) \leq \mu(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\mu(1, q) > \mu(x_0, q)$. Let $t' = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t' \in [0,1]$ and we have $\mu(1, q) > t' > \mu(x_0, q)$. Thus $x_0 \in L^-(\mu, t', q)$, so, $L^-(\mu, t', q) \neq \emptyset$. By assumption, we have $L^-(\mu, t', q)$ is an ideal of H. It follows that $1 \in L^-(\mu, t', q)$, so $\mu(1, q) < t'$, which is a contradicition. Hence $\mu(1, q) \leq \mu(x, q)$ for all $x \in H$ and $q \in Q$. Suppose that $\mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\}$ is not true for all $x, y \in H$ and $q \in Q$. Then there exist $u_0, v_0 \in H$ and $q \in Q$ such that $\mu(v_0, q) > \max\{\mu(u_0 * v_0, q), \mu(u_0, q)\}$. Let $p' = \frac{1}{2}(\mu(v_0, q) + \max\{\mu(u_0 * v_0, q), \mu(u_0, q)\})$. Then $p' \in [0,1]$ and we have $\mu(v_0, q) > p' > \max\{\mu(u_0 * v_0, q), \mu(u_0, q)\}$. Then $u_0, v_0 \in L^-(\mu, p', q)$, so, $L^-(\mu, p', q) \neq \emptyset$. By assumption, we have $L^-(\mu, p', q)$ is a deductive system of H. It follows that $v_0 \in L^-(\mu, p', q)$. Hence $\mu(v_0, q) < p'$, which is a contradiction. Hence $\mu(y, q) \leq \max\{\mu(x * y, q), \mu(x, q)\}$ is true for all $x, y \in H$ and $q \in Q$. Hence μ is an anti-q-fuzzy deductive system of H for all $q \in Q$. Consequently μ is an anti-Q-fuzzy deductive system of H.

(3). Assume that $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H. Then $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $U(\mu, t, q) \neq \emptyset$ and let $x \in U(\mu, t, q)$. Then $\mu(x, q) \geq t$. Thus $\overline{\mu}(1, q) = \overline{\mu}(x * x, q) \leq \overline{\mu}(x, q)$. Then $1 - \mu(1, q) \leq 1 - \mu(x, q)$, so $\mu(1, q) \geq \mu(x, q) \geq t$. Hence $1 \in U(\mu, t, q)$. Let $x, y \in H$ be such that $x \in U(\mu, t, q)$ and $x * y \in U(\mu, t, q)$. Then $\mu(x, q) \geq t$ and $\mu(x * y, q) \geq t$. Then we have $\overline{\mu}(y, q) \leq \max\{\overline{\mu}(x * y, q), \overline{\mu}(x, q)\}$. Then

$$\begin{aligned}
\overline{\mu}(y, q) &\leq \max\{\overline{\mu}(x * y, q), \overline{\mu}(x, q)\} \\
1 - \mu(y, q) &\leq \max\{1 - \mu(x * y, q), 1 - \mu(x, q)\} \\
1 - \mu(y, q) &\leq 1 - \min\{\mu(x * y, q), \mu(x, q)\} \\
\mu(y, q) &\geq \min\{\mu(x * y, q), \mu(x, q)\} \geq t.
\end{aligned}$$

Thus $y \in U(\mu, t, q)$. Hence $U(\mu, t, q)$ is a deductive system of H. Conversely, assume that every nonempty set $U(\mu, t, q)$ is a deductive system in H. If $\overline{\mu}(1, q) \leq \overline{\mu}(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\overline{\mu}(1, q) > \overline{\mu}(x_0, q)$. Then

$$\begin{aligned}
\overline{\mu}(1, q) &> \overline{\mu}(x_0, q) \\
1 - \mu(1, q) &> 1 - \mu(x_0, q) \\
\mu(1, q) &< \mu(x_0, q).
\end{aligned}$$

Let $t = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t \in [0, 1]$ and we have $\mu(1, q) < t < \mu(x_0, q)$. Thus $x_0 \in U(\mu, t, q)$, that is $U(\mu, t, q) \neq \emptyset$. By assumption, we have $U(\mu, t, q)$ is a deductive system of H. It follows that $1 \in U(\mu, t, q)$, so $\mu(1, q) \geq t$, which is a contradiction. Hence $\overline{\mu}(1, q) \leq \overline{\mu}(x, q)$ for all $x \in H$ and $q \in Q$. Suppose that $\overline{\mu}(y, q) \leq \max\{\overline{\mu}(x * y, q), \overline{\mu}(x, q)\}$ is not true for all $x, y \in H$. Then there exist $u_0, v_0 \in H$ and $q \in Q$ such that $\overline{\mu}(v_0, q) > \max\{\overline{\mu}(u_0 * v_0, q), \overline{\mu}(u_0, q)\}$. Then

$$\begin{aligned} \overline{\mu}(v_0, q) &> \max\{\overline{\mu}(u_0 * v_0, q), \overline{\mu}(u_0, q)\} \\ 1 - \mu(v_0, q) &> \max\{1 - \mu(u_0 * v_0, q), 1 - \mu(u_0, q)\} \\ 1 - \mu(v_0, q) &> 1 - \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\} \\ \mu(v_0, q) &< \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\}. \end{aligned}$$

Taking $p = \frac{1}{2}(\mu(v_0, q) + \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\})$. Then $p \in [0, 1]$ and we have $\mu(v_0, q) < p < \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\}$. Thus $\mu(u_0 * v_0, q) > p$ and $\mu(u_0, q) > p$, so $u_0 * v_0, u_0, \in U(\mu, p, q)$, so $U(\mu, p, q) \neq \emptyset$. By assumption, we have $U(\mu, p, q)$ is a deductive system of H. It follows that $v_0 \in U(\mu, p, q)$, so $\mu(v_0, q) \geq p$, a contradiction. Hence $\overline{\mu}(y, q) \leq \max\{\overline{\mu}(x*y, q), \overline{\mu}(x, q)\}$ is true for all $x, y \in H$ and $q \in Q$. Hence $\overline{\mu}$ is an anti-q-fuzzy deductive system of H for all $q \in Q$. Consequently $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H.

(4). Assume that $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H. Then $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H for all $q \in Q$. Let $q \in Q$ and $t \in [0, 1]$ be such that $U^+(\mu, t, q) \neq \emptyset$ and let $x \in U^+(\mu, t, q)$. Then $\mu(x, q) > t$. Thus $\overline{\mu}(1, q) = \overline{\mu}(x*x, q) \leq \overline{\mu}(x, q)$. Then $1 - \mu(1, q) \leq 1 - \mu(x, q)$, so $\mu(1, q) \geq \mu(x, q) > t$. Hence $1 \in U^+(\mu, t, q)$. Let $x, y \in H$ be such that $x \in U^+(\mu, t, q)$ and $x*y \in U^+(\mu, t, q)$. Then $\mu(x*y, q) > t$ and $\mu(x, q) > t$. Then we have $\overline{\mu}(y, q) \leq \max\{\overline{\mu}(x*y, q), \overline{\mu}(x, q)\}$. Then

$$\begin{aligned} \overline{\mu}(y, q) &\leq \min\{\overline{\mu}(x * y, q), \overline{\mu}(x, q)\} \\ 1 - \mu(y, q) &\leq \max\{1 - \mu(x * y, q), 1 - \mu(x, q)\} \\ 1 - \mu(y, q) &\leq 1 - \min\{\mu(x * y, q), \mu(x, q)\} \\ \mu(y, q) &\geq \min\{\mu(x * y, q), \mu(x, q)\} > t. \end{aligned}$$

Thus $y \in U^+(\mu, t, q)$. Hence $U^+(\mu, t, q)$ is a deductive system of H. Conversely, assume that every nonempty set $U^+(\mu, t, q)$ is a deductive system in H. If $\overline{\mu}(1, q) \leq \overline{\mu}(x, q)$ is not true for all $x \in H$ and $q \in Q$. Then there exist $x_0 \in H$ and $q \in Q$ such that $\overline{\mu}(1, q) > \overline{\mu}(x_0, q)$. Then

$$\begin{aligned} \overline{\mu}(1, q) &> \overline{\mu}(x_0, q) \\ 1 - \mu(1, q) &> 1 - \mu(x_0, q) \\ \mu(1, q) &< \mu(x_0, q). \end{aligned}$$

Let $t = \frac{1}{2}(\mu(1, q) + \mu(x_0, q))$. Then $t \in [0, 1]$ and we have $\mu(1, q) < t < \mu(x_0, q)$. Thus $x_0 \in U^+(\mu, t, q)$, that is $U^+(\mu, t, q) \neq \emptyset$. By assumption, we have $U^+(\mu, s, q)$ is a deductive system of H. It follows that $1 \in U^+(\mu, t, q)$, so $\mu(1, q) > t$, which is a contradiction. Hence $\overline{\mu}(1, q) \leq \overline{\mu}(x, q)$ for all $x \in H$ and $q \in Q$. Suppose that $\overline{\mu}(y, q) \leq \max\{\overline{\mu}(x * y, q), \overline{\mu}(x, q)\}$ is not true for all $x, y \in H$.

Then there exist $u_0, v_0 \in H$ and $q \in Q$ such that $\overline{\mu}(v_0, q) > \max\{\overline{\mu}(u_0 * v_0, q), \overline{\mu}(u_0, q)\}$. Then

$$\begin{array}{rcl} \overline{\mu}(v_0, q) & > & \max\{\overline{\mu}(u_0 * v_0, q), \overline{\mu}(u_0, q)\} \\ 1 - \mu(v_0, q) & > & \max\{1 - \mu(u_0 * v_0, q), 1 - \mu(u_0, q)\} \\ 1 - \mu(v_0, q) & > & 1 - \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\} \\ \mu(v_0, q) & < & \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\}. \end{array}$$

Taking $p = \frac{1}{2}(\mu(v_0, q) + \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\})$. Then $p \in [0, 1]$ and we have $\mu(v_0, q) < p < \min\{\mu(u_0 * v_0, q), \mu(u_0, q)\}$. Thus $\mu(u_0 * v_0, q) > p$ and $\mu(u_0, q) > p$, so $u_0 * v_0, u_0 \in U^+(\mu, p, q)$, so $U^+(\mu, p, q) \neq \emptyset$. By assumption, we have $U^+(\mu, p, q)$ is a deductive system of H. It follows that $v_0 \in U^+(\mu, p, q)$, so $\mu(v_0, q) \geq p$, a contradiction. Hence $\overline{\mu}(y, q) \leq \max\{\overline{\mu}(x * y, q), \overline{\mu}(y, q)\}$ is true for all $x, y \in H$ and $q \in Q$. Hence $\overline{\mu}$ is an anti-q-fuzzy deductive system of H for all $q \in Q$. Then $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H. □

Corollary 3.1. *Let μ be a Q-fuzzy set in H. Then the following statements hold:*

(1) *μ is an anti-Q-fuzzy deductive system of H, then for any $t \in [0, 1]$, $L(\mu, t)$ is either empty or a deductive system of H.*
(2) *μ is an anti-Q-fuzzy deductive system of H, then for any $t \in [0, 1]$, $L^-(\mu, t)$ is either empty or a deductive system of H.*
(3) *$\overline{\mu}$ is an anti-Q-fuzzy deductive system of H, then for any $t \in [0, 1]$, $U(\mu, t)$ is either empty or a deductive system of H.*
(4) *$\overline{\mu}$ is an anti-Q-fuzzy deductive system of H, for any $t \in [0, 1]$, $U^+(\mu, t)$ is either empty or a deductive system of H.*

Proof. (1). Assume that μ is an anti-Q-fuzzy deductive system of H. Then we have for any $t \in [0, 1]$ and $q \in Q$. Let $L(\mu, t, q)$ is either empty or a deductive system of H. Let $t \in [0, 1]$. If $L(\mu, t, q) = \emptyset$ for some $q \in Q$, it follows that $L(\mu, t) = \bigcap_{q \in Q} L(\mu, t, q)$. If $L(\mu, t, q) \neq \emptyset$ for all $q \in Q$, it follows from that $L(\mu, t, q)$ is a deductive system of H for all $q \in Q$. Then we have $L(\mu, t) = \bigcap_{q \in Q} L(\mu, t, q)$ is an ideal of H.

(2). Similarly to as in the proof of (1).

(3). Assume that $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H. Then we have for any $t \in [0, 1]$ and $q \in Q$. Let $U(\mu, t, q)$ is either empty or a deductive system of H. Let $t \in [0, 1]$. If $U(\mu, t, q) = \emptyset$ for some $q \in Q$, it follows that $U(\mu, t) = \bigcap_{q \in Q} U(\mu, t, q)$. If $U(\mu, t, q) \neq \emptyset$ for all $q \in Q$, it follows that $U(\mu, t, q)$ is a deductive system of H for all $q \in Q$. Then we have $U(\mu, t) = \bigcap_{q \in Q} U(\mu, t, q)$ is a deductive system of H.

(4). Similarly to as in the proof of (3). □

Corollary 3.2. *Let D be a deductive system of H. Then the following statements hold:*

(1) *for any* $k \in (0, 1]$*, there exists an anti-*Q*-fuzzy deductive system* μ *of* H *such that* $L(\mu, t) = D$ *for all* $t < k$ *and* $L(\mu, t) = H$ *for all* $t \geq k$*,*

(2) *for any* $k \in (0, 1]$*, there exists an anti-*Q*-fuzzy deductive system* γ *of* H *such that* $U(\overline{\gamma}, t) = D$ *for all* $t > k$ *and* $U(\overline{\gamma}, t) = H$ *for all* $t \leq k$*.*

Proof. (1). Let μ be a Q-fuzzy set in H defined by, for all $q \in Q$ $\mu(x, q) = \begin{cases} 0 & \text{if } x \in D \\ k & \text{if } x \notin D. \end{cases}$ Case 1 : To show that $L(\mu, t) = D$ for all $t < k$, let $t \in [0, 1]$ be such that $t < k$. Let $x \in L(\mu, t)$. Then $\mu(x, q) \leq t < k$ for all $q \in Q$. Thus $\mu(x, q) \neq k$ for all $q \in Q$, so $\mu(x, q) = 0$ for all $q \in Q$. Then $x \in D$, so $L(\mu, t) \subseteq D$. Now, let $x \in D$. Then $\mu(x, q) = 0 \leq t$ for all $q \in Q$. Thus $x \in L(\mu, t)$, so $D \subseteq L(\mu, t)$. Hence $L(\mu, t) = D$ for all $t < k$.

Case 2 : To show that $L(\mu, t) = H$ for all $t \geq k$, let $t \in [0, 1]$ be such that $t \geq k$. Clearly, $L(\mu, t) \subseteq H$. Let $x \in H$. Then for all $q \in Q$, we define $\mu(x, q) = \begin{cases} 0 < t & \text{if } x \in D \\ k \leq t & \text{if } x \notin D. \end{cases}$ Then $x \in L(\mu, t)$, so $H \subseteq L(\mu, t)$. Hence $L(\mu, t) = H$ for all $t \geq k$. We claim that $L(\mu, t, q) = L(\mu, t, q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain $x \in L(\mu, t, q) \Leftrightarrow \mu(x, q) \leq t \Leftrightarrow \mu(x, q') \leq t \Leftrightarrow x \in L(\mu, t, q')$. Hence $L(\mu, t, q) = L(\mu, t, q')$ for all $q, q' \in Q$. Then we have $L(\mu, t) = \bigcap_{q \in Q} L(\mu, t, q)$. By the claim, we have $L(\mu, t) = L(\mu, t, q)$ for all $q \in Q$. Since $L(\mu, t, q) = L(\mu, t) = D$ for all $t < k$ and $L(\mu, t) = L(\mu, t) = H$ for all $t \geq k$, it follows that $\overline{\mu}$ is an anti-Q-fuzzy deductive system of H.

(2). Let μ be a Q-fuzzy set in H defined by, for all $q \in Q$ $\mu(x, q) = \begin{cases} 1 & \text{if } x \in D \\ k & \text{if } x \notin D. \end{cases}$ Case 1 : To show that $U(\mu, t) = D$ for all $t > k$, let $t \in [0, 1]$ be such that $t > k$. Let $x \in U(\mu, t)$. Then $\mu(x, q) \geq t > k$ for all $q \in Q$. Then $\mu(x, q) \neq k$ for all $q \in Q$, so $\mu(x, q) = 1$ for all $q \in Q$. Thus $x \in I$, so $U(\mu, t) \subseteq D$. Now, let $x \in D$. Then $\mu(x, q) = 1 \geq t$ for all $q \in Q$. Then $x \in U(\mu, t)$, so $D \subseteq U(\mu, t)$. Hence $U(\mu, t) = D$ for all $t > k$.

Case 2 : To show that $U(\mu, t) = H$ for all $t \leq k$, let $t \in [0, 1]$ be such that $t \leq k$. Clearly, $U(\mu, t) \subseteq H$. Let $x \in H$. Then for all $q \in Q$, $\mu(x, q) = \begin{cases} 1 > t & \text{if } x \in D \\ k \geq t & \text{if } x \notin D. \end{cases}$ Then $x \in U(\mu, t)$, so $H \subseteq U(\mu, t)$. Hence $U(\mu, t) = H$ for all $t \leq k$. We claim that $U(\mu, t, q) = U(\mu, t, q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain $x \in U(\mu, t, q) \Leftrightarrow \mu(x, q) \geq t \Leftrightarrow \mu(x, q') \geq t \Leftrightarrow x \in U(\mu, t, q')$. Hence $U(\mu, t, q) = U(\mu, t, q')$ for all $q, q' \in Q$. Then $U(\mu, t) = \bigcap_{q \in Q} U(\mu, t, q)$. By the claim, we have $U(\mu, t) = U(\mu, t, q)$ for all $q \in Q$. Since $U(\mu, t, q) = U(\mu, t) = D$ for all $t > k$ and $U(\mu, t) = U(\mu, t) = H$ for all $t \leq k$, it follows that μ is an anti-Q-fuzzy ideal of H. Then $L(\overline{\overline{\mu}}, t) = L(\mu, t) = D$ for all $t > k$ and $L(\overline{\overline{\mu}}, t) = L(\mu, t) = H$ for all $t \leq k$. Let $\overline{\mu} = \theta$. Then θ is an anti-Q-fuzzy ideal of H such that $L(\overline{\mu}, t) = L(\mu, t) = D$ for all $t > k$ and $L(\overline{\mu}, t) = L(\mu, t) = H$ for all $t \leq k$. □

Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras A mapping $f : A \to B$ is called a homomorphism if $f(x * y) = f(x) \star f(y)$ for all $x, y \in A$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1_A) = 1_B$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras.

Theorem 3.5. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be a homomorphism. If μ is an anti-q-fuzzy deductive system of B, then μ_f is also a q-fuzzy deductive system of A.*

Proof. Assume that μ be an anti-q-fuzzy deductive system of B. Let $x \in A$. Then $\mu_f(1_A, q) = \mu(f(1_A), q) = \mu(1_B, q) \leq \mu(f(x), q) = \mu_f(x, q)$. Let $x, y \in A$. Then $\mu_f(y, q) = \mu(f(y), q) \leq \max\{\mu(f(x*y), q), \mu(f(x), q)\} = \max\{\mu_f(x*y, q), \mu_f(x, q)\}$. Hence μ_f is an anti-q-fuzzy deductive system of A. □

Corollary 3.3. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be a homomorphism. If μ is an anti-Q-fuzzy deductive system of B, then μ_f is also an anti-Q-fuzzy deductive system of A.*

Theorem 3.6. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be a isomorphism. If μ_f is an anti-q-fuzzy deductive system of A, then μ is also an anti-q-fuzzy deductive system of B.*

Proof. Assume that μ_f be an anti-q-fuzzy deductive system of A. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$, we have $\mu(1_B, q) = \mu(y \star 1_B, q) = \mu(f(x) \star f(1_A), q) = \mu(f(x * 1_A), q) = \mu_f(x * 1_A, q) = \mu_f(1_A, q) \leq \mu_f(x, q) \leq \mu(f(x), q) = \mu(y, q)$. Let $x, y \in B$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. It follows that $\mu(y, q) = \mu(f(b), q) = \mu_f(b, q) \leq \max\{\mu_f(a*b, q), \mu_f(a, q)\} = \max\{\mu(f(a*b), q), \mu(f(a), q)\} = \max\{\mu(f(a) \star f(b), q), \mu(f(a), q)\} = \max\{\mu(x \star y, q), \mu(x, q)\}$. Hence μ is an anti-q-fuzzy deductive system of B. □

Corollary 3.4. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be a isomorphism. If μ_f is an anti-Q-fuzzy deductive system of A, then μ is also an anti-Q-fuzzy deductive system of B.*

Lemma 3.6. *For any $a, b, c, d \in \mathbb{R}$, the following properties hold:*

(1) $\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}$
(2) $\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}$.

Remark 3.1. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras. Then $A \times B$ is a Hilbert algebra defined by $(x, y) \diamond (u, v) = (x * u, y \star v)$ for every $x, y \in A$ and $u, v \in B$, then clearly $(A \times B, \diamond, (1_A, 1_B))$ is a Hilbert algebra.*

Theorem 3.7. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras. If μ is an anti-q-fuzzy deductive system of A and δ is an anti-q-fuzzy deductive system of B, then $\mu \times \delta$ is an anti-q-fuzzy deductive system of $A \times B$.*

Proof. Assume that μ is an anti-q-fuzzy deductive system of A and δ is an anti-q-fuzzy deductive system B. Let $(x, y) \in A \times B$. Then $(\mu \times \delta)((1_A, 1_B), q) = \max\{\mu(1_A, q), \delta(1_B, q)\} \leq$

$\max\{\mu(x,q),\delta(y,q)\}\} = (\mu\times\delta)((x,y),q)$. Let $(x_1,y_1),(x_2,y_2)\in A\times B$. Then

$$
\begin{aligned}
(\mu\times\delta)((x_2,y_2),q) &= \max\{\mu(x_2,q),\delta(y_2,q)\}\\
&\leq \max\{\max\{\mu(x_1 * x_2,q),\mu(x_1,q)\},\max\{\delta(y_1\star y_2,q),\delta(y_1,q)\}\}\\
&= \max\{\max\{\mu(x_1 * x_2,q),\delta(y_1\star y_2,q)\},\max\{\mu(x_1,q),\delta(y_1,q)\}\}\\
&= \max\{(\mu\times\delta)((x_1 * x_2, y_1\star y_2),q),(\mu\cdot\delta)((x_1,y_1),q)\}\\
&= \max\{(\mu\times\delta)((x_1 * y_1)\diamond(x_2\star y_2),q),(\mu\cdot\delta)((x_1,y_1),q)\}.
\end{aligned}
$$

Hence $\mu\times\delta$ is an anti-q-fuzzy deductive system of $A\times B$. □

Corollary 3.5. *Let $(A,\cdot,1_A)$ and $(B,\star,1_B)$ be Hilbert algebras. If μ is an anti-Q-fuzzy deductive system of A and δ is an anti-Q-fuzzy deductive system of B, then $\mu\times\delta$ is an anti-Q-fuzzy deductive system of $A\times B$.*

Theorem 3.8. *If μ is a Q-fuzzy set of A and δ is a Q-fuzzy set of B such that $\mu\times\delta$ is an anti-q-fuzzy deductive system of $A\times B$, then the following statements hold:*

(1) *either $\mu(1_A,q)\leq\mu(x,q)$ for all $x\in A$ or $\delta(1_B,q)\leq\delta(x,q)$ for all $x\in B$,*
(2) *if $\mu(1_A,q)\leq\mu(x,q)$ for all $x\in A$, then either $\delta(1_B,q)\leq\mu(x,q)$ for all $x\in A$ or $\delta(1_B,q)\leq\delta(x,q)$ for all $x\in B$,*
(3) *if $\delta(1_A,q)\leq\delta(x,q)$ for all $x\in B$, then either $\mu(1_A,q)\leq\mu(x,q)$ for all $x\in A$ or $\mu(1_A,q)\leq\delta(x,q)$ for all $x\in B$.*

Proof. (1). Suppose that there exist $x\in A$ and $y\in B$ such that $\mu(1_A,q)>\mu(x,q)$ and $\delta(1_B,q)>\delta(y,q)$. Then $(\mu\times\delta)((x,y),q)=\max\{\mu(x,q),\delta(y,q)\}<\max\{\mu(1_A,q),\delta(1_B,q)\}\}=(\mu\times\delta)((1_A,1_B),q)$, which is a contradiction. Hence $\mu(1_A,q)\leq\mu(x,q)$ for all $x\in A$ or $\delta(1_B,q)\leq\delta(x,q)$ for all $x\in B$.
(2). Assume that $\mu(1_A,q)\leq\mu(x,q)$ for all $x\in A$. Suppose that there exist $x\in A$ and $y\in B$ such that $\mu(1_A,q)>\mu(x,q)$ and $\delta(1_B,q)>\delta(y,q)$. Then $\mu(1_A,q)\leq\mu(x,q)<\delta(1_B,q)$. Thus

$$
\begin{aligned}
(\mu\times\delta)((x,y),q) &= \max\{\mu(x,q),\delta(y,q)\}\\
&< \max\{\mu(1_A,q),\delta(1_B,q)\}\}\\
&= \delta(1_B,q)\\
&= \max\{\mu(1_A,q),\delta(1_B,q)\}\\
&= (\mu\times\delta)((1_A,1_B),q),
\end{aligned}
$$

which is a contradiction. Hence $\delta(1_A,q)\leq\mu(x,q)$ for all $x\in A$ or $\delta(1_B,q)\leq\delta(x,q)$ for all $x\in B$.
(3). Assume that $\delta(1_A,q)\leq\delta(x,q)$ for all $x\in B$. Suppose that there exist $x\in A$ and $y\in B$ such

that $\mu(1_A, q) > \mu(x, q)$ and $\mu(1_A, q) > \delta(y, q)$. Then $\delta(1_B, q) \leq \delta(x, q) < \mu(1_A, q)$. Thus

$$\begin{aligned}(\mu \times \delta)((x, y), q) &= \max\{\mu(x, q), \delta(y, q)\} \\ &> \max\{\mu(1_A, q), \mu(1_A, q)\}\} \\ &= \mu(1_A, q) \\ &= \max\{\mu(1_A, q), \delta(1_B, q)\} \\ &= (\mu \times \delta)((1_A, 1_B), q),\end{aligned}$$

which is a contradiction. Hence $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$ or $\mu(1_B, q) \leq \delta(x, q)$ for all $x \in B$. □

Corollary 3.6. *If μ is a Q-fuzzy set of A and δ is a Q-fuzzy set of B such that $\mu \times \delta$ is an anti-Q-fuzzy deductive system of $A \times B$, then the following statements hold:*

(1) *for all $q \in Q$, either $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \leq \delta(x, q)$ for all $x \in B$,*
(2) *for all $q \in Q$, if $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$, then either $\delta(1_B, q) \leq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \leq \delta(x, q)$ for all $x \in B$,*
(3) *for all $q \in Q$, if $\delta(1_A, q) \leq \delta(x, q)$ for all $x \in B$, then either $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$ or $\mu(1_A, q) \leq \delta(x, q)$ for all $x \in B$.*

Theorem 3.9. *Let $(A, \cdot, 1_A)$ and $(B, \star, 1_B)$ be Hilbert algebras and let μ be a Q-fuzzy set in A and δ be a Q-fuzzy set in B. If $\mu \times \delta$ is an anti-q-fuzzy deductive system of $A \times B$, then either μ is an anti-q-fuzzy deductive system of A or δ is an anti-q-fuzzy deductive system of B.*

Proof. Assume that $\mu \times \delta$ is an anti-q-fuzzy deductive system of $A \times B$. Suppose that μ is not an anti-q-fuzzy deductive system of A and δ is not an anti-q-fuzzy deductive system of B. Then we have $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \leq \delta(x, q)$ for all $x \in B$. Suppose that $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$. Then either $\delta(1_B, q) \leq \mu(x, q)$ for all $x \in A$ or $\delta(1_B, q) \leq \delta(x, q)$ for all $x \in B$. If $\delta(1_B, q) \leq \mu(x, q)$ for all $x \in A$, then $(\mu \times \delta)((x, 1_B), q) = \max\{\mu(x, q), \delta(1_B, q)\} = \mu(x, q)$. We consider, for all $x, y \in A$,

$$\begin{aligned}\mu(y, q) &= \max\{\mu(y, q), \delta(1_B, q)\} \\ &= (\mu \times \delta)((y, 1_B), q) \\ &\leq \max\{(\mu \times \delta)((x, 1_B) \diamond (y, 1_B), q), (\mu \times \delta)((x, 1_B), q)\} \\ &= \max\{(\mu \times \delta)((x * y, 1_B \star 1_B), q), (\mu \times \delta)((x, 1_B), q)\} \\ &= \max\{(\mu \times \delta)((x * y, 1_B), q), (\mu \times \delta)((x, 1_B), q)\} \\ &= \max\{\max\{\mu(x * y, q), \delta(1_B, q)\}, \max\{\mu(x, q), \delta(1_B, q)\}\} \\ &= \max\{\mu(x * y, q), \mu(x, q)\}.\end{aligned}$$

Hence μ is an anti-q-fuzzy deductive system of A, which is a contradiction. Suppose that $\delta(1_B, q) \leq \delta(x, q)$ for all $x \in B$. Then either $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in B$ or $\mu(1_A, q) \leq \delta(x, q)$ for all $x \in B$.

If $\mu(1_A, q) \leq \delta(x, q)$ for all $x \in B$, then $(\mu \cdot \delta)((1_A, x), q) = \max\{\mu(1_A, q), \delta(x, q)\} = \delta(x, q)$. We consider, for all $x, y \in B$,

$$\begin{aligned}
\delta(y, q) &= \max\{\mu(1_A, q), \delta(y, q)\} \\
&= (\mu \times \delta)((1_A, y), q) \\
&\leq \max\{(\mu \times \delta)((1_A, x) \diamond (1_A, y), q), (\mu \times \delta)((1_A, x), q)\} \\
&= \max\{(\mu \times \delta)((1_A * 1_A, x * y), q), (\mu \times \delta)((1_A, x), q)\} \\
&= \max\{(\mu \times \delta)((1_A, x * y), q), (\mu \times \delta)((1_A, x), q)\} \\
&= \max\{\min\{\mu(1_A, q), \delta(x * y, q)\}, \max\{\mu(1_A, q), \delta(x, q)\}\} \\
&= \max\{\delta(x * y, q), \delta(x, q)\}.
\end{aligned}$$

Hence δ is an anti-q-fuzzy deductive system of B, which is a contradiction. Since μ is not an anti-q-fuzzy deductive system of A and δ is not an anti-q-fuzzy deductive system of B, we have $\mu(1_A, q) \leq \mu(x, q)$ for all $x \in A$ and $\delta(1_B, q) \leq \delta(x, q)$ for all $x \in B$. Let $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that $\mu(x_2, q) > \max\{\mu(x_1 * x_2, q), \mu(x_1, q)\}$ and $\delta(y_2, q) > \max\{\delta(y_1 \star y_2, q), \delta(y_1, q)\}$, so $\max\{\mu(x_2, q), \delta(y_2, q)\} > \max\{\max\{\mu(x_1 * x_2, q), \mu(x_1, q)\}, \max\{\delta(y_1 \star y_2, q), \delta(y_1, q)\}\}$. Thus $\max\{\mu(x_2, q), \delta(y_2, q)\}$

$$\begin{aligned}
&= (\mu \cdot \delta)((x_2, y_2), q) \\
&\leq \max\{(\mu \cdot \delta)((x_1, y_1) \diamond (x_2, y_2), q), (\mu \cdot \delta)((x_1, y_1), q)\} \\
&= \max\{(\mu \cdot \delta)((x_1 * x_2, y_1 \star y_2), q), (\mu \cdot \delta)((x_1, y_1), q)\} \\
&= \max\{\max\{\mu(x_1 * x_2, q), \delta(y_1 \star y_2, q)\}, \max\{\mu(x_1, q), \delta(y_1, q)\}\} \\
&= \max\{\max\{\mu(x_1 * x_2, q), \mu(x_1, q)\}, \max\{\delta(y_1 \star y_2, q), \delta(y_1, q)\}\}.
\end{aligned}$$

It follows that $\max\{\mu(x_2, q), \delta(y_2, q)\} \not> \max\{\max\{\mu(x_1, q), \mu(x_2, q)\}, \max\{\delta(y_1, q), \delta(y_2, q)\}\}$, which is a contradiction. Hence μ is an anti-q-fuzzy deductive system of A or δ is an anti-q-fuzzy deductive system of B. □

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References

[1] B. Ahmad, A. Kharal, On Fuzzy Soft Sets, Adv. Fuzzy Syst. 2009 (2009), 586507. https://doi.org/10.1155/2009/586507.

[2] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87-96. https://doi.org/10.1016/s0165-0114(86)80034-3.

[3] A.K. Adak, D.D. Salokolaei, Some Properties of Pythagorean Fuzzy Ideal of Near-Rings, Int. J. Appl. Oper. Res. 9 (2019), 1-9.

[4] M. Atef, M.I. Ali, T.M. Al-Shami, Fuzzy Soft Covering-Based Multi-Granulation Fuzzy Rough Sets and Their Applications, Comput. Appl. Math. 40 (2021), 115. https://doi.org/10.1007/s40314-021-01501-x.

[5] D. Busneag, A note on deductive systems of a Hilbert algebra, Kobe. J. Math. 2 (1985), 29-35. https://cir.nii.ac.jp/crid/1570854175360486400.

[6] D. Busneag, Hilbert algebras of fractions and maximal Hilbert algebras of quotients, Kobe. J. Math. 5 (1988), 161-172. https://cir.nii.ac.jp/crid/1570572702603831808.

[7] N. Cağman, S. Enginoğlu, and F. Citak, Fuzzy soft set theory and its application, Iran. J. Fuzzy Syst. 8 (2011), 137-147.

[8] I. Chajda, R.Halas, Congruences and ideals in Hilbert algebras, Kyungpook Math. J. 39 (1999), 429-429.

[9] A. Diego, Sur les algébres de Hilbert, Collect. Log. Math. Ser. A (Ed. Hermann, Paris). 21 (1966), 1-52.

[10] W.A. Dudek, Y.B. Jun, On fuzzy ideals in Hilbert algebra, Novi Sad J. Math. 29 (1999), 193-207.

[11] W. A. Dudek, On fuzzification in Hilbert algebras, Contrib. Gen. Algebra, 11 (1999), 77-83.

[12] H. Garg, S. Singh, A novel triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators, Iran. J. Fuzzy Syst. 15 (2018), 69-93. https://doi.org/10.22111/ijfs.2018.4159.

[13] H. Garg, K. Kumar, An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making, Soft Comput. 22 (2018), 4959-4970. https://doi.org/10.1007/s00500-018-3202-1.

[14] H. Garg, K. Kumar, Distance measures for connection number sets based on set pair analysis and its applications to decision-making process, Appl. Intell. 48 (2018), 3346-3359. https://doi.org/10.1007/s10489-018-1152-z.

[15] Y.B. Jun, Deductive systems of Hilbert algebras, Math. Japon. 43 (1996), 51-54. https://cir.nii.ac.jp/crid/1571417124616097792.

[16] K.H. Kim, On intuitionistic Q-fuzzy ideals of semigroups, Sci. Math. Japon. e–2006 (2006), 119-126.

[17] P.M. Sithar Selvam, T. Priya, K.T. Nagalakshmi, T. Ramachandran, A note on anti Q-fuzzy KU-subalgebras and homomorphism of KU-algebras, Bull. Math. Stat. Res. 1 (2013), 42-49.

[18] K. Tanamoon, S. Sripaeng, A. Iampan, Q-fuzzy sets in UP-algebras, Songklanakarin J. Sci. Technol. 40 (2018), 9-29.

[19] L.A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965), 338-353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[20] J. Zhan, Z. Tan, Intuitionistic fuzzy deductive systems in Hilbert algebra, Southeast Asian Bull. Math. 29 (2005), 813-826.