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Fractional Reduced Differential Transform Method for Solving Mutualism Model with Fractional Diffusion

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ABSTRACT. This study presents the fractional reduced differential transform method for a nonlinear mutualism model with fractional diffusion. The fractional derivatives are described by Caputo's fractional operator. In this method, the solution is considered as the sum of an infinite series. Which converges rapidly to the exact solution. The method eliminates the need to use Adomian's polynomials to calculate the nonlinear terms. To show the efficiency and accuracy of this method, we compared the results of the fractional derivatives orders with the ordinary derivative order index $\alpha=1$ for the nonlinear mutualism model with fractional diffusion. Approximate solutions for different values of the fractional derivatives together with non-fractional derivatives and absolute errors are represented graphically in two and three dimensions. From all numerical results, we can conclude the efficiency of the proposed method for solving different types of nonlinear fractional systems of partial differential equations over existing methods.

1. Introduction

Recently, it has turned out that many phenomena in engineering and other sciences can be described by models using mathematical tools from fractional calculus [1], fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order could be extended to still be valid when n is not an integer. Diffusion phenomena is one the most important topic in heat transfer, especially in mechanics engineering and biological population. In the earlier literature most of the

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discussions are devoted to coupled systems of two equations. In the recent years, attention has been given to reaction-diffusion systems with three population species, the densities of three populations u, v, w are governed by the following coupled equations: ([2], [3])

$$\begin{aligned} D_t^\alpha u - \mathrm{d}_1\nabla^2 \mathrm{u} &= u(a_1(t,x) - b_1(t,x)u + c_1(t,x)v) \\ D_t^\alpha v - \mathrm{d}_2\nabla^2 v &= v(a_2(t,x) - b_2(t,x)v + c_2(t,x)u + e(t,x)w) \\ D_t^\alpha w - \mathrm{d}_3\nabla^2 w &= w(a_3(t,x) - b_3(t,x)w + c_3(t,x)v) \end{aligned} \tag{1}$$

with initial conditions

$$u(x,0) = u_0,\ \ v(x,0) = v_0,\ \ w(x,0) = w_0$$

where $n-1 < \alpha \leq n$, for each $i = 1,2,3$, d_i is constant and $a_i, b_i, c_i,\ e$ are smooth functions [2], ∇^2 denotes Laplacian with respect to the variables $x = (x_1, x_2, x_3)$ and $u(x,t), v(x,t), w(x,t)$ is solution of Eq. (1). If $d_i = 0$ for each $i = 1,2,3$ in Eq. (1) we obtain a model of Lotka Volterra for prey-predator.

2. Preliminaries and Fractional Calculus

In this section, gives some important definitions, such as the gamma function and basic definitions of the fractional derivatives.

2.1. Gamma Function

Gamma function $\Gamma(n)$ is simply the generalization of factorial to complex and real arguments. The gamma function can be defined as ([5], [6])

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt = (n-1)!,\ n \in IN \tag{2}$$

which is convergent for $n > 0$. A recurrence formula for gamma function is ([5], [6])

$$\Gamma(n+1) = n\Gamma(n)\ for\ n \in IR^+ \tag{3}$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{\mathrm{n}}\ for\ n \in IR^- \tag{4}$$

2.2. Fractional Derivatives

Definition (1): Riemann-Liouville Fractional Integral Operator

Suppose that $\alpha > 0$, $n-1 < \alpha \leq n$, the Riemann-Lioville fractional integral define as [5]

$${}_{a}^{RL}D_t^{-\alpha}\big(f(t)\big) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}f(u)du \tag{5}$$

Note: Riemann-Liouville fractional differential operator define as

$${}^{Rl}D^\alpha f(t) = D^n D^{\alpha-n} f(t), \qquad \alpha < n \tag{6}$$

Definition (2): Caputo Fractional Differential Operator

Suppose that $\alpha > 0$, $n-1 < \alpha \leq n$, the Caputo fractional differential define as [5]

$$ {}_{a}^{C}D_{t}^{\alpha}\big(f(t)\big) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{a}^{t} \dfrac{f^{n}(u)}{(t-u)^{\alpha-n+1}}\,du, & n-1<\alpha<n \\ \dfrac{d^{n}}{dt^{n}} f(t) & \alpha = n \in N \end{cases} \quad (7) $$

Riemann-Liouville and Caputo fractional integral operator for polynomial is [5]

$$ {}_{0}^{RL}D_{t}^{-\alpha}(t^{n}) = {}_{0}^{C}D_{t}^{-\alpha}\big(f(t)\big) = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} \quad (8) $$

Definition (3): The Mittag- Leffler Function

Suppose $\alpha > 0, \beta > 0$, then the Mittag-Leffler function define by [5]

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k+\beta)} \quad (9) $$

3. Fractional Reduced Differential Transform Method

Fractional Reduced Differential Transform Method (FRDTM) is iteration method, suppose $u(t, x_1, x_2, \dots, x_n)$ be analytical and continuously differentiable with respect to $n+1$ variables $t, x_1, x_2, .., x_n$ in the domain of interest; then FRDTM in n dimensions for the following differential equation

$$ D_{t}^{\alpha} u + Lu + N(u) = 0 \quad (10) $$

where D_{t}^{α} is differential operator with respect time, L differential operator with respect variables $x_1, x_2, .., x_n$ and $N(u)$ is nonlinear term ([7]-[10]).

$$ u_k(x_1, x_2, \dots, x_n) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)} \left[-L(u_k) - \sum_{r=0}^{k} N(u_r) N(u_{k-r})\right] \quad (11) $$

The approximate solution is given by ([7], [8]).

$$ u(t, x_1, x_2, \dots, x_n) = \sum_{k=0}^{\infty} u_k t^{\alpha k} = u_0 + u_1 t^{\alpha} + u_2 t^{2\alpha} + \cdots \quad (12) $$

4. Numerical Results

In this section, we assume $d_i = 1, a_i = 1 \; for \; i = 1,2,3,\; b_1 = b_3 = 1,\; b_2 = 3,\; c_1 = c_2 = c_3 = 0.5,$ $e = 0.5$ in Eq. (1)

$$ D_{t}^{\alpha} u = \mathrm{u}_{\mathrm{xx}} + u - u^{2} + 0.5uv $$
$$ D_{t}^{\alpha} v = \mathrm{v}_{\mathrm{xx}} + v - 3v^{2} + 0.5uv + 0.5wv $$
$$ D_{t}^{\alpha} w = \mathrm{w}_{\mathrm{xx}} + w - w^{2} + 0.5vw $$

with initial conditions

$$u(x,0)=e^{x},\ \ v(x,0)=x,\ \ \ w(x,0)=x-\pi,\ \ 0\le x\le 10$$

Applied FRDTM

$$u_{k+1}=\frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}((u_k)_{xx}+u_k-\sum_{r=0}^{k}u_r u_{k-r}+\frac{1}{2}\sum_{r=0}^{k}u_r v_{k-r})$$

$$v_{k+1}=\frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}((v_k)_{xx}+v_k-3\sum_{r=0}^{k}v_r v_{k-r}+\frac{1}{2}\sum_{r=0}^{k}v_r u_{k-r}+\frac{1}{2}\sum_{r=0}^{k}v_r w_{k-r})$$

$$w_{k+1}=\frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}((w_k)_{xx}+w_k-\sum_{r=0}^{k}w_r w_{k-r}+\frac{1}{2}\sum_{r=0}^{k}w_r v_{k-r})$$

Given $$u_0=e^{x},\ \ \ v_0=x,\ \ \ w_0=x-\pi$$

when $k=0$

$$u_1=\frac{1}{\Gamma(\alpha+1)}((u_0)_{xx}+u_0-u_0^2+\frac{1}{2}u_0 v_0)$$

$$u_1=\frac{(2-e^{x}+0.5x)e^{x}}{\Gamma(\alpha+1)}$$

$$v_1=\frac{1}{\Gamma(\alpha+1)}((v_0)_{xx}+v_0-3v_0^2+\frac{1}{2}v_0 u_0+\frac{1}{2}v_0 w_0)$$

$$v_1=\frac{x-2.5x^2+0.5xe^{x}+0.5\pi x}{\Gamma(\alpha+1)}$$

$$w_1=\frac{1}{\Gamma(\alpha+1)}((w_0)_{xx}+w_0-w_0^2+\frac{1}{2}w_0 v_0)$$

$$w_1=\frac{(\mathrm{x}-\pi)(1+\pi-0.5x)}{\Gamma(\alpha+1)}$$

when $k=1$

$$u_2=\frac{1}{\Gamma(2\alpha+1)}((u_1)_{xx}+u_1-2u_0u_1+\frac{1}{2}u_0v_1+\frac{1}{2}u_1v_0)$$

$$u_2=\frac{(5-9e^{x}+2e^{2x}+0.25(7-\pi)x-0.5xe^{x}-x^2)e^{x}}{\Gamma(2\alpha+1)}$$

$$v_2=\frac{1}{\Gamma(2\alpha+1)}((v_1)_{xx}+v_1-6v_0v_1+\frac{1}{2}v_0u_1+\frac{1}{2}v_1u_0+\frac{1}{2}v_0w_1+\frac{1}{2}v_1w_0)$$

$$v_2=\frac{(1-0.5\pi-5.5x+0.5e^{x})(x-2.5x^2+0.5xe^{x}+0.5\pi x)-x^3}{\Gamma(2\alpha+1)}$$
$$+\frac{(0.25e^{x}-0.5\pi^2+0.25\pi+0.5)x^2+(1.5e^{x}-0.5e^{2x})x+e^{x}-5}{\Gamma(2\alpha+1)}$$

$$w_2=\frac{1}{\Gamma(2\alpha+1)}((w_1)_{xx}+w_1-2w_0w_1+\frac{1}{2}w_0v_1+\frac{1}{2}w_1v_0)$$

$$w_2=\frac{(x-\pi)(\pi+1-0.5x)(2\pi+1-1.5x)+0.5x(x-\pi)(x-0.5\pi x-2.5x^2+0.5xe^x)-1}{\Gamma(2\alpha+1)}$$

when $k=2$

$$u_3=\frac{1}{\Gamma(3\alpha+1)}((u_2)_{xx}+u_2-2u_0u_2-u_1^2+\frac{1}{2}u_0v_2+\frac{1}{2}u_1v_1+\frac{1}{2}u_2v_0)$$

$$v_3=\frac{1}{\Gamma(3\alpha+1)}((v_2)_{xx}+v_2-6v_0v_2-3v_1^2+\frac{1}{2}u_0v_2+\frac{1}{2}u_1v_1+\frac{1}{2}u_2v_0+\frac{1}{2}v_0w_2+\frac{1}{2}v_1w_1+\frac{1}{2}v_2w_0)$$

$$w_3=\frac{1}{\Gamma(3\alpha+1)}((w_2)_{xx}+w_2-2w_0w_2-w_1^2+\frac{1}{2}w_0v_2+\frac{1}{2}w_1v_1+\frac{1}{2}w_2v_0)$$

$$\vdots$$

$$u(x,t)=\sum_{k=0}^{\infty}u_kt^{\alpha k}=u_0+u_1t^{\alpha}+u_2t^{2\alpha}+\cdots$$

$$u(x,t)\cong e^x+\frac{(2-e^x+0.5x)e^x}{\Gamma(\alpha+1)}t^{\alpha}+\frac{(5-9e^x+2e^{2x}+0.25(7-\pi)x-0.5xe^x-x^2)e^x}{\Gamma(2\alpha+1)}t^{2\alpha}$$

$$v(x,t)=\sum_{k=0}^{\infty}v_kt^{\alpha k}=v_0+v_1t^{\alpha}+v_2t^{2\alpha}+\cdots$$

$$v(x,t)\cong x+\frac{x-2.5x^2+0.5xe^x+0.5\pi x}{\Gamma(\alpha+1)}t^{\alpha}+\frac{(1-0.5\pi-5.5x+0.5e^x)(x-2.5x^2+0.5xe^x+0.5\pi x)-x^3}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{(0.25e^x-0.5\pi^2+0.25\pi+0.5)x^2+(1.5e^x-0.5e^{2x})x+e^x-5}{\Gamma(2\alpha+1)}t^{2\alpha}$$

$$w(x,t)=\sum_{k=0}^{\infty}w_kt^{\alpha k}=w_0+w_1t^{\alpha}+w_2t^{2\alpha}+\cdots$$

$$w(x,t)\cong(x-\pi)+\frac{(x-\pi)(1+\pi-0.5x)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{(x-\pi)(\pi+1-0.5x)(2\pi+1-1.5x)+0.5x(x-\pi)(x-0.5\pi x-2.5x^2+0.5xe^x)-1}{\Gamma(2\alpha+1)}t^{2\alpha}$$

Table 1. Numerical results of variable $u(x,t)$

x	$\alpha = 1$	$\alpha = 0.8$	$\alpha = 0.5$	$\alpha = 0.2$
1.0e+014 *				
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0002	0.0002	0.0001	0.0001
0.0000	0.0042	0.0034	0.0021	0.0010
0.0000	0.0851	0.0683	0.0425	0.0209
0.0000	1.7093	1.3734	0.8546	0.4193

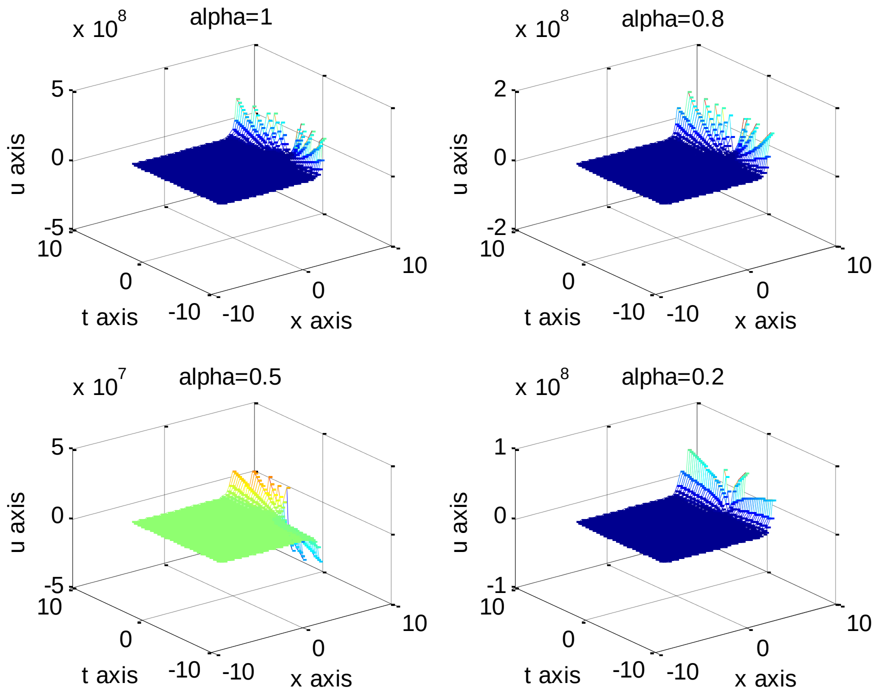

Figure 1. Graphical presentation of variable $u(x,t)$

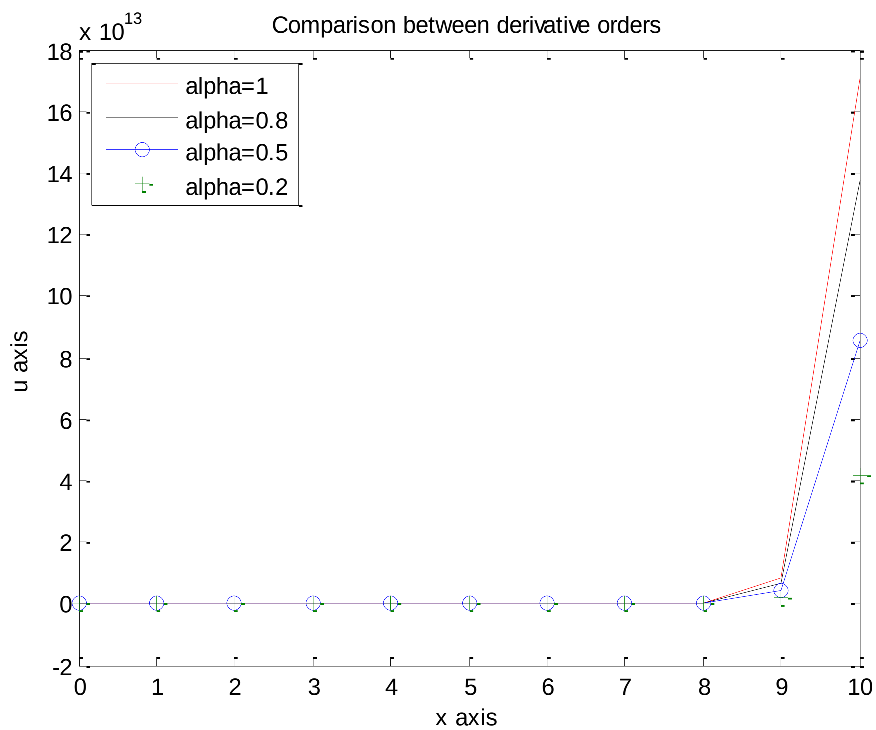

Figure 2. Compression derivatives order between non-fractional order with fractional orders of variable $u(x,t)$.

Table 1 shows the approximate solution of fractional diffusion of variable $u(x,t)$, it is noted that only the Second order of the FRDTM. Figure 1: The surface of diffusion variable $u(x,t)$ is convergence between fractional order and ordinary order, in Figure 2: we get small difference between ordinary order with multiple fractional orders.

Table 2. Numerical results of variable $v(x,t)$

$\boldsymbol{x}$	$\boldsymbol{\alpha = 1}$	$\boldsymbol{\alpha = 0.8}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.2}$
1.0e+009 *				
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0003	-0.0002	-0.0002	-0.0001
0.0000	-0.0023	-0.0019	-0.0012	-0.0006
0.0000	-0.0182	-0.0147	-0.0091	-0.0045
0.0000	-0.1473	-0.1184	-0.0737	-0.0361
0.0000	-1.1998	-0.9641	-0.5999	-0.2943
0.0000	-9.7643	-7.8456	-4.8821	-2.3950

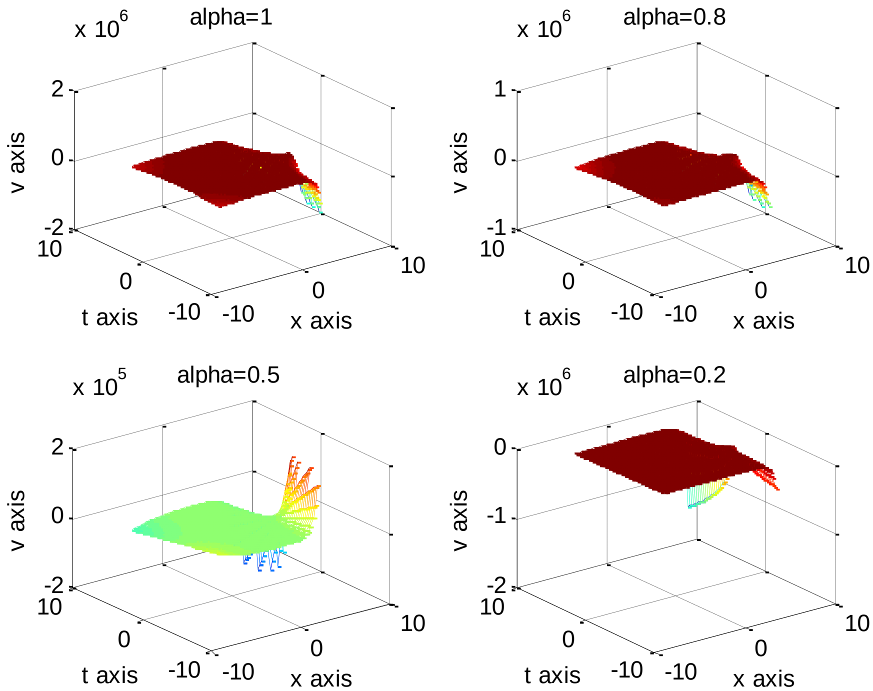

Figure 3. Graphical presentation of variable $v(x,t)$

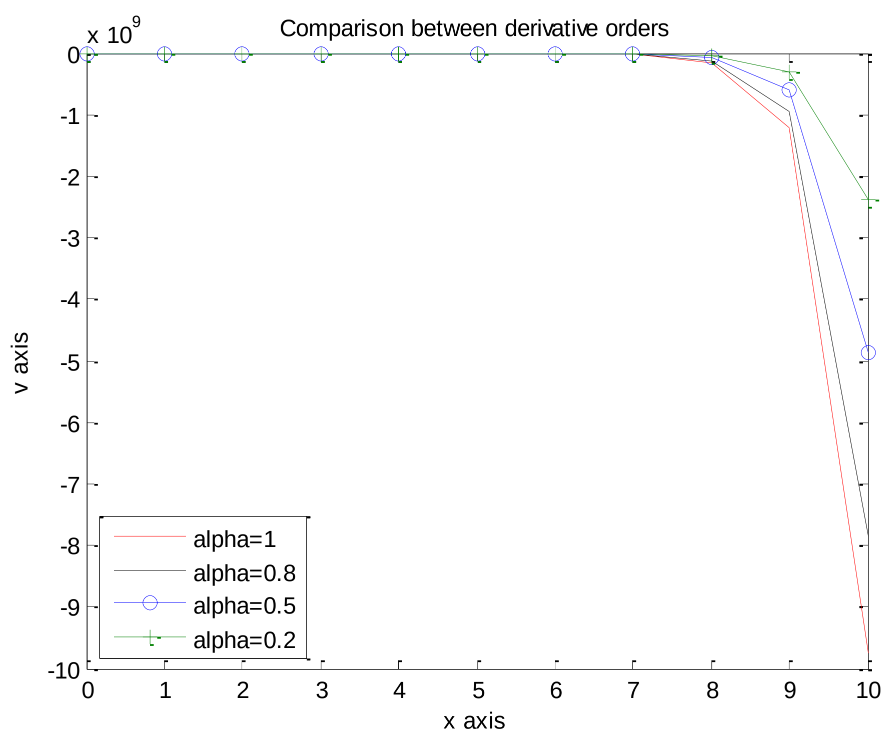

Figure 4. Compression derivatives order between non-fractional order with fractional orders of variable $v(x,t)$.

Table 2 shows the approximate solution of fractional diffusion of variable $v(x,t)$, it is noted that only the Second order of the FRDTM. Figure 3: The surface of diffusion of variable $v(x,t)$ is convergence between fractional order and ordinary order, in Figure 4: we get small difference between ordinary order with multiple fractional orders.

Table 3. Numerical results of variable $w(x,t)$

$\boldsymbol{x}$	$\boldsymbol{\alpha = 1}$	$\boldsymbol{\alpha = 0.8}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.2}$
1.0e+007 *				
0.0000	-0.0001	-0.0001	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	0.0001	0.0001	0.0000	0.0000
0.0000	0.0011	0.0009	0.0006	0.0003
0.0000	0.0077	0.0062	0.0038	0.0019
0.0000	0.0401	0.0322	0.0201	0.0098
0.0000	0.1828	0.1469	0.0914	0.0448
0.0000	0.7647	0.6144	0.3823	0.1876
0.0000	3.0144	2.4220	1.5072	0.7394

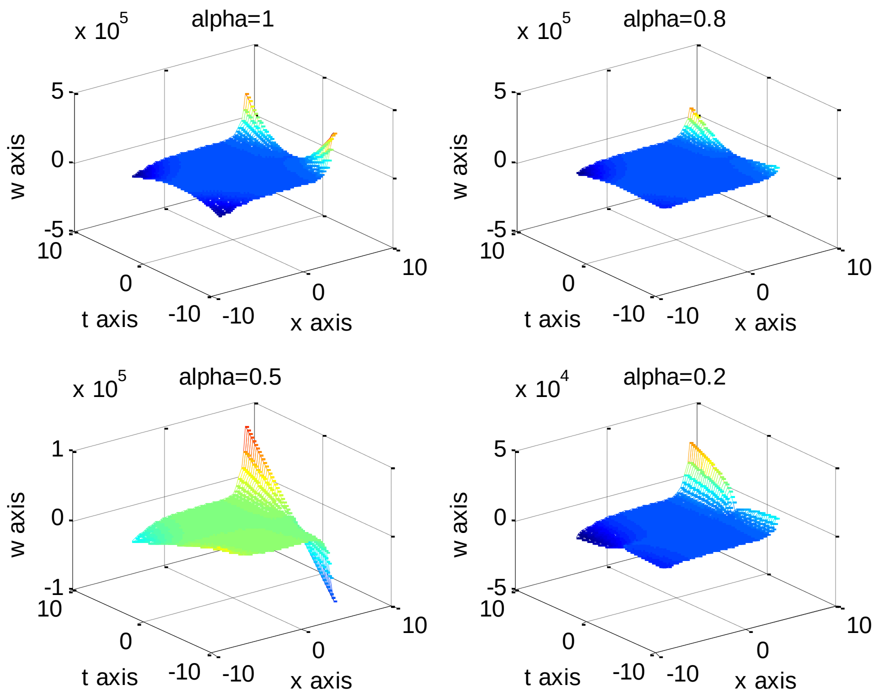

Figure 5. Graphical presentation of variable $w(x,t)$

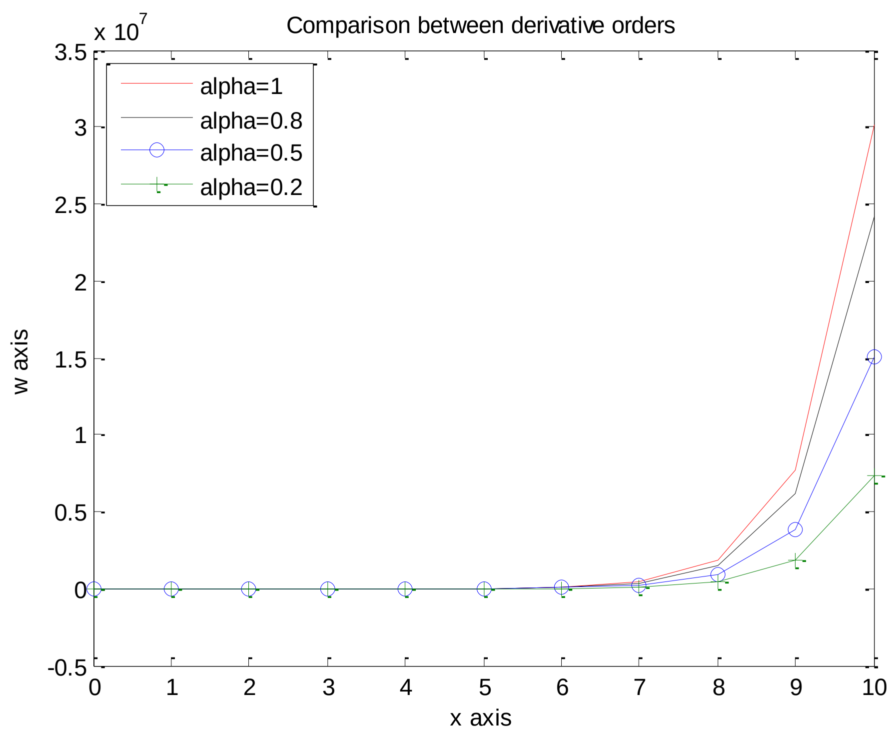

Figure 6. Compression derivatives order between non-fractional order with fractional orders of variable $w(x,t)$.

Table 3 shows the approximate solution of fractional diffusion of variable $w(x,t)$, it is noted that only the Second order of the FRDTM. Figure 5: The surface of diffusion of variable $w(x,t)$ is convergence between fractional order and ordinary order, in Figure 6: we get small difference between ordinary order with multiple fractional orders.

5. Conclusions

The fractional reduced differential transform method has been successfully applied to obtain an analytical approximate solution for the mutualism model with fractional diffusion. It is easy to recognize that FRDTM is powerful mathematical tool for solving different kinds of linear and/or nonlinear fractional partial differential equations the FRDTM is no need to use Adomian's polynomials to calculate the nonlinear terms. We have concluded that the fractional derivative of diffusion mutualism model is more accurate than ordinary derivative order. From all numerical results, we can conclude the efficiency of the proposed method for solving different types of nonlinear fractional partial differential equations so we recommended researchers would use Fractional reduced differential transform method when derivation the mathematical models (biological phenomena) for fractional derivatives.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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