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INEQUALITIES FOR CO-ORDINATED $m-$CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we prove some new inequalities of Hadamard-type for $m-$convex functions on the co-ordinates via Riemann-Liouville fractional integrals.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a < b$. The following double inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hadamard's inequality. Both inequalities hold in the reversed direction if f is concave.

In [7], Dragomir defined convex functions on the co-ordinates as following:

Definition 1. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

In [7], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$
\begin{aligned}
(1.1)\qquad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx \right.\\
& \left. + \frac{1}{(d-c)}\int_c^d f(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

Similar results can be found in [7]-[12].

In [17], Toader defined $m-$convex functions as following:

Definition 2. *The function $f : [0,b] \to \mathbb{R}$, $b > 0$ is said to be $m-$convex, where $m \in [0,1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m(b)$ the class of all $m-$convex functions on $[0,b]$ for which $f(0) \leq 0$. Obviously, if we choose $m = 1$, we have ordinary convex functions on $[0,b]$.

In [10], Özdemir *et al.* defined co-ordinated $m-$convex functions as following:

Definition 3. *Let us consider the bidimensional interval $\Delta = [0,b] \times [0,d]$ in $[0,\infty)^2$. The mapping $f : \Delta \to \mathbb{R}$ is $m-$convex on Δ if*

$$f(tx + (1-t)z, ty + m(1-t)w) \leq tf(x,y) + m(1-t)f(z,w)$$

holds for all $(x,y),(z,w) \in \Delta$ and $t \in [0,1]$, $b,d > 0$ and for some fixed $m \in [0,1]$.

In [16], Sarıkaya *et al.* proved some Hadamard's type inequalities for co-ordinated convex functions as followings:

Theorem 2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$
\begin{aligned}
(1.2)\qquad & \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A\right| \\
\leq\ & \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|(a,c) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|(a,d) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|(b,c) + \left|\frac{\partial^2 f}{\partial t\partial s}\right|(b,d)}{4}\right)
\end{aligned}
$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right].$$

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

(1.3)

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|$$

$$\leq\frac{(b-a)(d-c)}{4\left(p+1\right)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right]$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

(1.4)

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy-A\right|$$

$$\leq\frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right)^{\frac{1}{q}}$$

where

$$A=\frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx+\frac{1}{(d-c)}\int_c^d\left[f(a,y)dy+f(b,y)\right]dy\right].$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 4. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a\geq 0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\qquad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\qquad x<b$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-t}u^{\alpha-1}du$, *here is* $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties of this operator can be found in the references [3]-[?].

Throughout of this paper, we will use the following notation:

$$\begin{aligned} B &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right] \\ &\quad -\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{d^-}f(a,c)+J^{\beta}_{d^-}f(b,c)+J^{\beta}_{c^+}f(b,d)+J^{\beta}_{c^+}f(a,d)\right] \\ &\quad -\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{b^-}f(a,d)+J^{\alpha}_{b^-}f(a,c)+J^{\alpha}_{a^+}f(b,d)+J^{\alpha}_{a^+}f(b,c)\right] \end{aligned}$$

where

$$\begin{aligned} J^{\alpha,\beta}_{b^-,d^-}f(a,c) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx \\ J^{\alpha,\beta}_{a^+,d^-}f(b,c) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx \\ J^{\alpha,\beta}_{b^-,c^+}f(a,d) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx \\ J^{\alpha,\beta}_{a^+,c^+}f(b,d) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx. \end{aligned}$$

The main purpose of this paper is to establish inequalities of Hadamard-type inequalities for $m-$convex functions on the co-ordinates via Riemann-Liouville fractional integrals by using a new Lemma and fairly elemantery analysis.

2. MAIN RESULTS

To prove our main result, we need the following Lemma:

Lemma 1. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a twice partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$ and $\alpha,\beta>0$, $a,c\geq 0$, then the following equality holds:*

$$\begin{aligned} &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B \\ &= \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[(1-s)^{\beta}-s^{\beta}\right]\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\,dsdt. \end{aligned} \tag{2.1}$$

Proof. Integration by parts, we can write

$$
\begin{aligned}
K &= \int_0^1\int_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right]\left[(1-s)^{\beta}-s^{\beta}\right]\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sc+(1-s)d\right)dsdt \\
&= \int_0^1 \left[(1-s)^{\beta}-s^{\beta}\right]\left[\int_0^1 (1-t)^{\alpha}\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sc+(1-s)d\right)dt\right. \\
&\qquad\qquad \left. -\int_0^1 t^{\alpha}\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sc+(1-s)d\right)dt\right]ds \\
&= \frac{1}{b-a}\left\{\int_0^1 \left[(1-s)^{\beta}-s^{\beta}\right]\left[\frac{\partial f}{\partial s}\left(b, sc+(1-s)d\right)+\frac{\partial f}{\partial s}\left(a, sc+(1-s)d\right)\right.\right. \\
&\qquad\qquad -\alpha\int_0^1 (1-t)^{\alpha-1}\frac{\partial f}{\partial s}\left(ta+(1-t)b, sc+(1-s)d\right)dt \\
&\qquad\qquad \left.\left. -\alpha\int_0^1 t^{\alpha-1}\frac{\partial f}{\partial s}\left(ta+(1-t)b, sc+(1-s)d\right)dt\right]ds\right\}.
\end{aligned}
$$

By integrating again, we get

$$
\begin{aligned}
K &= \frac{1}{(b-a)(d-c)}\left\{f(a,c)+f(a,d)+f(b,c)+f(b,d)\right. \\
&-\beta\int_0^1 (1-s)^{\beta-1} f\left(b, sc+(1-s)d\right)ds-\beta\int_0^1 s^{\beta-1} f\left(a, sc+(1-s)d\right)ds \\
&-\beta\int_0^1 (1-s)^{\beta-1} f\left(a, sc+(1-s)d\right)ds-\beta\int_0^1 s^{\beta-1} f\left(b, sc+(1-s)d\right)ds \\
&-\alpha\int_0^1 (1-t)^{\alpha-1} f\left(ta+(1-t)b, d\right)dt-\alpha\int_0^1 t^{\alpha-1} f\left(ta+(1-t)b, d\right)dt \\
&-\alpha\int_0^1 (1-t)^{\alpha-1} f\left(ta+(1-t)b, c\right)dt-\alpha\int_0^1 t^{\alpha-1} f\left(ta+(1-t)b, c\right)dt \\
&+\alpha\beta\int_0^1\int_0^1 (1-t)^{\alpha-1}(1-s)^{\beta-1} f\left(ta+(1-t)b, sc+(1-s)d\right)dsdt \\
&+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}(1-s)^{\beta-1} f\left(ta+(1-t)b, sc+(1-s)d\right)dsdt
\end{aligned}
$$

$$+\alpha\beta\int_0^1\int_0^1(1-t)^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sc+(1-s)d)\,dsdt$$
$$\left.+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sc+(1-s)d)\,dsdt\right\}.$$

By using the change of the variables, we can get

$$x=ta+(1-t)b \text{ and } y=sc+(1-s)d,$$

that is

$$t=\frac{x-b}{a-b} \text{ and } s=\frac{y-d}{c-d}.$$

Taking into account these equalities, we obtain

$$
\begin{aligned}
K &= \frac{1}{(b-a)(d-c)}\{f(a,c)+f(a,d)+f(b,c)+f(b,d) \\
&\quad -\frac{\beta}{(d-c)^{\beta-1}}\left[\int_c^d (y-c)^{\beta-1}f(a,y)\,dy+\int_c^d (d-y)^{\beta-1}f(a,y)\,dy\right. \\
&\quad \left.+\int_c^d (y-c)^{\beta-1}f(b,y)\,dy+\int_c^d (d-y)^{\beta-1}f(b,y)\,dy\right] \\
&\quad -\frac{\alpha}{(b-a)^{\alpha-1}}\left[\int_a^b (x-a)^{\alpha-1}f(x,d)\,dx+\int_a^b (x-a)^{\alpha-1}f(x,c)\,dx\right. \\
&\quad \left.+\int_a^b (b-x)^{\alpha-1}f(x,d)\,dx+\int_a^b (b-x)^{\alpha-1}f(x,c)\,dx\right]+\frac{\alpha\beta}{(b-a)^{\alpha-1}(d-c)^{\beta-1}} \\
&\quad \times\left[\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx+\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx\right. \\
&\quad \left.\left.+\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx+\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx\right]\right\}.
\end{aligned}
\tag{2.2}
$$

Multiplying both sides of (2.2) by $\frac{(b-a)(d-c)}{4}$ and using the Riemann-Liouville integrals, we obtain equality (2.1). This completes the proof. ∎

Theorem 5. *Let $f:\Delta=[0,b]\times[0,d]\to\mathbb{R}$ be a partial differentiable mapping on Δ and $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, $\alpha,\beta>0$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is $m-$convex function on the co-ordinates* *on* Δ *where* $0 \leq a < b < \infty$ *and* $0 \leq c < d < \infty$, *then the following inequality holds;*

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + B \right| \\ \leq & \frac{(b-a)(d-c)}{4} M_\alpha M_\beta \\ & \times \left(\left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + m \left| \frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{d}{m}\right) \right| + m \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{d}{m}\right) \right| \right) \end{aligned}$$

where

$$\begin{aligned} M_\alpha &= \left[\frac{1}{\alpha+1} - \frac{\left(\frac{1}{2}\right)^\alpha}{\alpha+1} \right] \\ M_\beta &= \left[\frac{1}{\beta+1} - \frac{\left(\frac{1}{2}\right)^\beta}{\beta+1} \right]. \end{aligned}$$

Proof. From Lemma 1 and using the property of modulus, we have

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + B \right| \\ \leq & \frac{(b-a)(d-c)}{4} \int_0^1 \int_0^1 \left| (1-t)^\alpha - t^\alpha \right| \left| (1-s)^\beta - s^\beta \right| \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right| dsdt. \end{aligned}$$

Since $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is co-ordinated $m-$convex, we can write

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + B \right| \\ \leq & \frac{(b-a)(d-c)}{4} \int_0^1 \int_0^1 \left| (1-s)^\beta - s^\beta \right| \left| (1-t)^\alpha - t^\alpha \right| \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + mt(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{d}{m}\right) \right| \right. \\ & \left. +(1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + m(1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{d}{m}\right) \right| \right\} dtds \end{aligned}$$

By computing these integrals, we obtain

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + B \right| \\ \leq & \frac{(b-a)(d-c)}{4} \left[\frac{1}{\alpha+1} - \frac{\left(\frac{1}{2}\right)^\alpha}{\alpha+1} \right] \\ & \times \int_0^1 \left| (1-s)^\beta - s^\beta \right| \left(s \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + m(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{d}{m}\right) \right| \right. \\ & \left. +s \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| + m(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{d}{m}\right) \right| \right) ds. \end{aligned}$$

Using co-ordinated $m-$convexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ again, we get

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right| \\
\leq\ & \frac{(b-a)(d-c)}{4}\left[\frac{1}{\alpha+1}-\frac{\left(\frac{1}{2}\right)^{\alpha}}{\alpha+1}\right]\left[\frac{1}{\beta+1}-\frac{\left(\frac{1}{2}\right)^{\beta}}{\beta+1}\right] \\
&\times\left(\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|+m\left|\frac{\partial^2 f}{\partial t \partial s}\left(a,\frac{d}{m}\right)\right|+m\left|\frac{\partial^2 f}{\partial t \partial s}\left(b,\frac{d}{m}\right)\right|\right)
\end{aligned}$$

Thus, the proof is completed. ▮

Remark 1. *Suppose that all the assumptions of Theorem 5 are satisfied. If we choose $\alpha=\beta=m=1$, we obtain the inequality* (1.2).

Theorem 6. *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on Δ and $\frac{\partial^2 f}{\partial t \partial s}\in L(\Delta)$, $\alpha,\beta\in(0,1]$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q>1$, is $m-$convex function on the co-ordinates on Δ where $0\leq a<b<\infty$ and $0\leq c<d<\infty$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right| \\
\leq\ & \frac{(b-a)(d-c)}{4(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q+m\left|\frac{\partial^2 f}{\partial t \partial s}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q+m\left|\frac{\partial^2 f}{\partial t \partial s}\left(b,\frac{d}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

where $p^{-1}+q^{-1}=1$.

Proof. From Lemma 1 and by applying the well-known Hölder inequality for double integrals, then one has

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right| \\
\leq\ & \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left[\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|(1-s)^{\beta}-s^{\beta}\right|\right]^p dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

By using the fact that

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha}$$

for $\alpha\in(0,1]$ and $t_1,t_2\in[0,1]$, we get

$$\begin{aligned}
\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt &\leq \int_0^1|1-2t|^{\alpha p}dt \\
&= \frac{1}{\alpha p+1}
\end{aligned}$$

and

$$\int_0^1 \left|\left|(1-s)^{\beta}-s^{\beta}\right|\right|^p dt \leq \int_0^1 |1-2s|^{\beta p}\, dt = \frac{1}{\beta p+1}.$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is co-ordinated $m-$convex, we can write

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right| \\
\leq\ & \frac{(b-a)(d-c)}{4(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \\
&\times\left(\int_0^1\int_0^1\left[ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + mt(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{d}{m}\right)\right|^q\right]\right. \\
&\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + m(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{d}{m}\right)\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

By computing these integrals, we obtain

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right| \\
\leq\ & \frac{(b-a)(d-c)}{4(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + m\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{d}{m}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + m\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{d}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Which completes the proof. ∎

Remark 2. *Suppose that all the assumptions of Theorem 6 are satisfied. If we choose* $\alpha=\beta=m=1$, *we obtain the inequality* (1.3).

Theorem 7. *Let* $f:\Delta\to\mathbb{R}$ *be a partial differentiable mapping on* Δ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, $\alpha,\beta\in(0,1]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, *is* $m-$*convex function on the co-ordinates on* Δ *where* $0\leq a<b<\infty$ *and* $0\leq c<d<\infty$, *then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right| \\
\leq\ & \frac{(b-a)(d-c)}{4}\left(\left[\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha+1}\right]\left[\frac{1-\left(\frac{1}{2}\right)^{\beta}}{\beta+1}\right]\right)^{1-\frac{1}{q}} M_{\alpha}^{\frac{1}{q}}M_{\beta}^{\frac{1}{q}} \\
&\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + m\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{d}{m}\right)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + m\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{d}{m}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where M_{α}, M_{β} *are defined as in Theorem 5.*

Proof. From Lemma 1 and by applying the well-known Power-mean inequality for double integrals, then one has

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|(1-s)^{\beta}-s^{\beta}\right| dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|(1-s)^{\beta}-s^{\beta}\right|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated $m-$convex, we can write

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|(1-s)^{\beta}-s^{\beta}\right| dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|(1-s)^{\beta}-s^{\beta}\right|\left[ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+mt(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{d}{m}\right)\right|^q\right]\right.$$
$$\left.+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+m(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{d}{m}\right)\right|^q dsdt\right)^{\frac{1}{q}}.$$

By computing these integrals, we obtain

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+B\right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left(\left[\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha+1}\right]\left[\frac{1-\left(\frac{1}{2}\right)^{\beta}}{\beta+1}\right]\right)^{1-\frac{1}{q}} M_{\alpha}^{\frac{1}{q}} M_{\beta}^{\frac{1}{q}}$$
$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|m\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{d}{m}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+m\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{d}{m}\right)\right|^q\right)^{\frac{1}{q}}$$

which completes the proof. ∎

Remark 3. *Suppose that all the assumptions of Theorem 7 are satisfied. If we choose $\alpha=\beta=m=1$, we obtain the inequality* (1.4).

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