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Asymptotic Behavior of Solution for Coupled Reaction Diffusion System by Order m

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Abstract. The aim of this paper is to prove that asymptotic behavior in the time of solutions for the weakly coupled reaction diffusion system:

$$\begin{cases} \frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i\left(u_1, u_2, \ldots, u_m\right) & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u_i}{\partial \eta} = 0 & \text{in } \partial\Omega \times \mathbb{R}^+, \\ u_i\left(., 0\right) = u_i^0\left(.\right) & \text{in } \Omega, \end{cases} \tag{0.1}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, $u_i(t, x)$, $i = \overline{1, m}$, $t \geq 0$, $x \in \Omega$ are real valued functions. We treat the system (0.1) as a dynamical system in $C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) \times \ldots \times C\left(\overline{\Omega}\right)$ and apply Lyapunov type stability techniques. A key ingredient in this analysis is a result which establishes that the orbits of the dynamical system are precompact in $C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) \times \ldots \times C\left(\overline{\Omega}\right)$. As a consequence of Arzela-Ascoli theorem, this will be satisfied if the orbits are, for example, uniformly bounded in $C^1\left(\overline{\Omega}\right) \times C^1\left(\overline{\Omega}\right) \times \ldots \times C^1\left(\overline{\Omega}\right)$ for $t > 0$.

1. Introduction

The existence, uniqueness, and asymptotic behavior of the solution of a balanced two component reaction diffusion system have been investigated. It was shown that a global and unique solution existed and it's second component can be estimated using the Lyapunov Functional see [1, 14, 15].

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It was, also, demonstrated that each component of the solution converged, at infinity, to a constant which can be used in terms of the reacting function and the initial data.

The results of the current research can be used in several areas of applied mathematics, especially when the system equations originate from mathematical models of real systems such as in Biology, Chemistry, Population Dynamics, and other disciplines. We know that the problem (0.1) has a unique global solution see [5, 7, 17]. The main question we want to address is asymptotic behavior the solutions for system (0.1). In fact the subject of the asymptotic behavior of reaction diffusion systems has received a lot of attention in the last decades and several outstanding results have been proved by some of the major experts in the field.

This question has been investigated by many authors by considering special forms of the nonlinear terms f_i.

In the case where $i = \overline{1,2}$:

$$\begin{cases} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = f_1(u_1, u_2) & \text{in } \Omega \times \mathbb{R}^+, \\ \\ \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = f_2(u_1, u_2) & \text{in } \Omega \times \mathbb{R}^+, \\ \\ \lambda_i \frac{\partial u_i}{\partial \eta} + (1 - \lambda_i) u_i = 0, & \text{in } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

when $d_1 \neq d_2$, and nonnegative initial data arise, for example, as models for the diffusion of substances which at the same time react with each other chemically (cf. [8, 16]). Also (1.1) is related to the Rosenzweig-Mac Arthur equation in ecology (cf. [2]).

In the case where $f_1(u_1, u_2) = -f_2(u_1, u_2) = -u_1 u_2^{\sigma}$, Note that, the behavior of non-negative total solutions (1.1) is treated in the paper of Alikakos [2] obtained L^{∞}-bounds of solutions global existence when $1 < \sigma < \dfrac{n+2}{n}$, and Masuda [14] who showed that solutions exist globally for every $\sigma \geq 1$ and, in addition, showed that the solutions converge as t goes to $+\infty$. Recently, Haraux and Youkana [13] established a global existence result of a system (1.1) for a large class of the function f_1 and f_2. More precisely, they showed that for

$$f_1(u_1, u_2) = f_2(u_1, u_2) = -u_1 \Psi(u_2),$$

the problem (1.1) admits a global solution provided that the following condition holds:

$$\lim_{u_2 \to +\infty} \frac{[\log(1 + \Psi(u_2))]}{u_2} = 0.$$

In the case where $d_2 \geq 0$, systems of the type (1.1) occur in many applications (cf. [8]).

For triangular diffusion matrix, global bounds were proved by Kirane in [11] if $u_2^0(x) \geq \frac{d_2}{d_1 - d_3} u_1^0(x) \geq 0$, $x \in \Omega$,. The author proved also that the solution (u_1, u_2) converges to a constant vector $k = (k_1, k_2)$ as $t \to \infty$, uniformly in $\bar{\Omega}$. Furthermore, $k_1 \geq 0$, $k_2 \geq 0$ and $k_1 \Psi(k_2) = 0$.

In this paper we shall generalize the results obtained in [11]. We prove the asymptotic behavior of solutions of m-component reaction-diffusion systems with diagonal matrix and homogeneous Neumann conditions. The reaction terms are assumed to be of polynomial growth.
We consider the following m-equations of reaction-diffusion system, with $m \geq 2$:

$$\frac{\partial U}{\partial t} - A_m \Delta U = F(U) \quad \text{in } \Omega \times (0, +\infty), \tag{1.2}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, the vectors U and F and the matrix A_m are defined as:

$$U = (u_1, ..., u_m)^T, \ F = (f_1, ..., f_m)^T,$$

$$A_m = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \ddots & \vdots \\ 0 & 0 & d_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & d_m \end{pmatrix}.$$

The constants $(d_i)_{i=1}^m$, are supposed to be strictly positive which reflects the parabolicity of the system and implies at the same time that the diffusion matrix A_m is positive defnite. The boundary conditions and initial data (respectively) for the proposed system are assumed to satisfy:

$$\partial_\eta U = 0 \quad \text{on} \quad \partial\Omega \times (0, +\infty),$$

and

$$U(0, x) = U_0(x) = (u_1^0, ..., u_m^0)^T \quad \text{on} \quad \Omega,$$

where $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on the boundary $\partial\Omega$, the vectors U_0 are defined as:

$$U_0 = (u_1^0, ..., u_m^0)^T.$$

2. Notations and Preliminary

In the following we denote by

$\|.\|_P$ the norm in $L^P(\Omega)$ for $1 \leq P \leq +\infty$,

$\|.\|_\infty$ the norm in $C(\Omega)$, and

$\|.\|_{1,\infty}$ the norm in $C^1(\Omega)$.

For $1 < P < \infty$, set

$$\begin{cases} D(A) = \left\{ u : u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega \right\}, \\ Au = \Delta u \text{ for } u \in D(A). \end{cases}$$

It is well known (cf. for example, [6]) that A is m-dissipative in $L^P(\Omega)$ for $1 < P < \infty$. Moreover, the restriction of A to $C\left(\bar{\Omega}\right)$ is m-dissipative.
Let us now recall an overview of the asymptotic behavior of the solution for coupled reaction diffusion systems. This will pave the way to introduce our findings later on.

Consider the initial value problem

$$\begin{cases} u_t(t) = Lu(t) + f(u(t)) \\ u(0) = u_0, \end{cases} \tag{IVP}$$

where L is the infinitesimal generator of a C_0-semigroup $S(t)$ on a real Banach space X with norm $\|.\|$, $f : X \to \mathbb{R}$ is a given function, and $u_0 \in X$ is a given initial datum.

Theorem 2.1. *[19] Let $T > 0$. A function $u : [0, T] \to X$ is a weak solution of (IVP) on $[0, T]$ if and only if $f(u(t)) \in L^1(0, T, X)$ and u satisfies the variation of constants formula*

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(u(s))\, ds, \quad \text{for all } s \in [0, T].$$

Definition 2.1. *A function $u : [0, T] \to X$ is called a strong solution of (IVP) if $u(t)$ is strongly continuously differentiable in the interval $0 < t < T$, $u(t) \in D(L)$ for $0 < t < T$, that equation (IVP) is satisfied for $0 < t < T$ and $u(t) \to u_0$ as $t \to 0$.*

Theorem 2.2. *[18] Let $f : X \to X$ be locally Lipschitz continuous. Then for $u_0 \in X$, (IVP) has a unique weak solution defined in a maximal interval of existence $[0, T_{\max})$, $T_{\max} > 0$, $u \in C([0, T_{\max}), X)$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t \to T_{\max}} \|u(t)\| = +\infty.$$

Now, let us recall the following definition.

Definition 2.2. *Let $\{S(t)\}_{t \geq 0}$ be a nonlinear semigroup on a compact metric space X. If $\left(u_1^0, u_2^0, \ldots, u_m^0\right) \in X$, $O\left(u_1^0, u_2^0, \ldots, u_m^0\right) = \left\{S(t)\left(u_1^0, u_2^0, \ldots, u_m^0\right)\right\}_{t \geq 0}$ is the orbit through $\left(u_1^0, u_2^0, \ldots, u_m^0\right)$, then the w-limite set for $\left(u_1^0, u_2^0, \ldots, u_m^0\right)$ is defined by*

$$w\left(u_1^0, u_2^0, \ldots, u_m^0\right) = \{(u_1, u_2, \ldots, u_m) \in X : \exists t_n \to \infty :$$
$$S(t_n)\left(u_1^0, u_2^0, \ldots, u_m^0\right) \to (u_1, u_2, \ldots, u_m)\}.$$

3. The Main Result

In this section, we state the main result.

Theorem 3.1. *The solution $w = (u_1, u_2, \ldots, u_m)$ of the system (1.2) converges a constant vector of the form $\xi = (\xi_1, \xi_2, \ldots, \xi_i)$ as $t \to \infty$, uniformly in $\overline{\Omega}$ i.e $\left(u_i \underset{t\to\infty}{\to} \xi_i\right)$ for $i = \overline{1, m}$.*
Furthermore, we have:

$$\xi_i \geq 0, \ i = \overline{1, m}, \ f_i(\xi_1, \xi_2, \ldots, \xi_m) = 0,$$

and

$$\sum_{i=1}^m \xi_i = \frac{1}{\overline{\Omega}} \int_\Omega \sum_{i=1}^m u_i^0(x)\, dx.$$

The following lemma is a useful tool in the proof of the Theorem 3.1.

Lemma 3.1. *Let* $(u_1, u_2, \ldots, u_m)$ *be a solution of (1.2). We have*

$$\int_{Q_T} |\nabla u_i|^2 \, dxdt < \infty \text{ for } i = \overline{1, m}, \text{ here } Q_T = \Omega \times [0, T] \text{ and } 0 < T < \infty.$$

Proof. We have For $i = \overline{1, m}$.

$$\frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i \left(u_1, u_2, \ldots, u_m\right). \tag{3.1}$$

By integrating over $(0, T)$ is obtained

$$\int_0^T \frac{\partial u_i}{\partial t}(x, t) \, dt = d_i \int_0^T \Delta u_i dt + \int_0^T f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dt,$$

$$u_i(x, T) - u_i(x, 0) = d_i \int_0^T \Delta u_i dt + \int_0^T f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dt,$$

and integrating a second time is collected over Ω

$$\int_\Omega u_i(x, T) \, dx - \int_\Omega u_i(x, 0) \, dx = d_i \int_\Omega \int_0^T \Delta u_i dtdx$$

$$+ \int_\Omega \int_0^T f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dtdx.$$

Green's formula is applied to $\int_\Omega \Delta u_i dx$, we gain

$$\int_\Omega \Delta u_i dx = \int_{\partial\Omega} \frac{\partial u_i}{\partial \eta} d\sigma - \int_\Omega \nabla u_i \nabla 1 dx, \text{ therefore } \int_\Omega \Delta u_i dx = 0,$$

thus

$$\int_\Omega \int_0^T f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dtdx = \int_\Omega u_i(x, T) \, dx - \int_\Omega u_i^0(x) \, dx < \infty,$$

as a result of $u_i(T) \in C\left(\overline{\Omega}\right)$ we have

$$\int_{Q_T} f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dtdx < \infty, \text{ for } i = \overline{1, m}.$$

Multiply now the i^{th} equation of (1.2) by u_i, for $i = \overline{1, m}$, and integrating over Q_T, we attain

$$\int_\Omega \int_0^T u_i \frac{\partial u_i}{\partial t}(x, t) \, dtdx = d_i \int_\Omega \int_0^T u_i \Delta u_i dtdx$$

$$+ \int_\Omega \int_0^T u_i f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dtdx,$$

by using the Green formula

$$\int_\Omega u_i \Delta u_i dx = \int_{\partial\Omega} u_i \frac{\partial u_i}{\partial \eta} d\sigma - \int_\Omega |\nabla u_i|^2 \, dx, \text{ therefore } \int_\Omega u_i \Delta u_i dx = -\int_\Omega |\nabla u_i|^2 \, dx,$$

and a simple calculation, it becomes

$$\frac{1}{2} \int_\Omega \left[u_i^2(x, t)\right]\Big|_0^T dx = -d_i \int_0^T \int_\Omega |\nabla u_i|^2 \, dxdt$$

$$+ \int_0^T \int_\Omega u_i(x, t) \, f_i \left(u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\right) dxdt,$$

then

$$\int_{\Omega} u_i^2(x,T) + 2d_i \int_{Q_T} |\nabla u_i|^2 \, dxdt = \int_{\Omega} \left(u_i^0(x)\right)^2 dx$$

$$+2\int_{Q_T} u_i(x,t) f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dxdt,$$

consequently

$$2d_i \int_{Q_T} |\nabla u_i|^2 \, dxdt \leq \int_{\Omega} \left(u_i^0(x)\right)^2 dx$$

$$+2\int_{Q_T} u_i(x,t) f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dxdt, \tag{3.2}$$

since

$$\int_{\Omega} \left(u_i^0(x)\right)^2 dx < \infty, \text{ for } i = \overline{1,m}.$$

and

$$\int_{Q_T} u_i(x,t) f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dxdt$$

$$\leq \|u_i\|_{L^\infty(Q_T)} \int_{Q_T} f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dxdt < \infty, \text{ for } i = \overline{1,m}.$$

hence

$$2d_i \int_{Q_T} |\nabla u_i|^2 \, dxdt < \infty, \text{ for } i = \overline{1,m}.$$

consequently

$$\int_{Q_T} |\nabla u_i|^2 \, dxdt < \infty, \text{ for } i = \overline{1,m}. \; \forall T > 0.$$

□

4. Proof of the Main Result (Theorem 3.1)

We are now ready to prove the main result of this work:

Proof of Theorem 3.1. First, if we integrate the i^{th} equation of (1.2) over Ω we have

$$\int_{\Omega} \frac{\partial u_i}{\partial t}(x,t) \, dx = d_i \int_{\Omega} \Delta u_i dx + \int_{\Omega} f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dx,$$

use Green theorem to transform the terms Δu_i in the light of boundary conditions we observe that

$$\int_{\Omega} \frac{\partial u_i}{\partial t}(x,t) \, dx = \int_{\Omega} f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dx,$$

if we add this equations imliying

$$\int_{\Omega} \sum_{i=1}^{m} \frac{\partial u_i}{\partial t}(x,t) \, dx = \int_{\Omega} \sum_{i=1}^{m} f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dx, \text{ for } i = \overline{1,m},$$

if we assume that

$$\sum_{i=1}^{m} \int_{\Omega} f_i(u_1(x,t), u_2(x,t), \dots, u_m(x,t)) \, dx = 0,$$

we get

$$\int_{\Omega}\sum_{i=1}^{m}\frac{\partial u_i}{\partial t}(x,t)\,dx=0,\ \text{for } i=\overline{1,m},$$

as

$$\begin{aligned}\int_0^t\int_{\Omega}\sum_{i=1}^{m}\frac{\partial u_i}{\partial t}(x,t)\,dxdt &= \int_{\Omega}\int_0^t\sum_{i=1}^{m}\frac{\partial u_i}{\partial t}(x,t)\,dtdx\\ &= \int_{\Omega}\sum_{i=1}^{m}u_i(x,t)\Big|_0^t\,dx\end{aligned}$$

$$=\int_{\Omega}\sum_{i=1}^{m}u_i(x,t)\,dx-\int_{\Omega}\sum_{i=1}^{m}u_i^0(x)\,dx=0,$$

we deduce that

$$\int_{\Omega}\sum_{i=1}^{m}u_i(x,t)\,dx=\int_{\Omega}\sum_{i=1}^{m}u_i^0(x)\,dx. \tag{4.1}$$

Integrating the i^{th} equation of the system (1.2) in Ω, for $i=\overline{1,m}$, we have:

$$\int_{\Omega}\frac{\partial u_i}{\partial t}(x,t)\,dx=\int_{\Omega}f_i\left(u_1(x,t),u_2(x,t),\ldots,u_m(x,t)\right)dx>0,$$

as a means that $\frac{d}{dt}\int_{\Omega}u_i(x,t)\,dx>0$. then the fonction $t\rightarrow\int_{\Omega}u_i(x,t)\,dx$ is increasing and Ω is bounded. Then $t\rightarrow\frac{1}{|\Omega|}\int_{\Omega}u_i(x,t)\,dx$ is increasing and according to the positivity of u_i was $\frac{1}{|\Omega|}\int_{\Omega}u_i(x,t)\,dx\geq 0$. Therefore $\frac{1}{|\Omega|}\int_{\Omega}u_i(x,t)\,dx$ is bounded below and increasing.
Formerly $\exists\lim\limits_{t\rightarrow\infty}\frac{1}{|\Omega|}\int_{\Omega}u_i(x,t)\,dx=l_i$, for $i=\overline{1,m}$.
On the other hand, since sets $\underset{t\geq 0}{\cup}\{u_i(t)\}_{t\geq 0}$, for $i=\overline{1,m}$ are precompacts in $C\left(\overline{\Omega}\right)$. There exists a sequence $(t_n)_{n\geq 0}$, $t_n\rightarrow\infty$ such that

$$\lim_{n\rightarrow\infty}u_i(t_n)=u_i^s,\ \text{for } i=\overline{1,m}\quad\text{in } C\left(\overline{\Omega}\right),$$

Now, denote by $w\left(u_1^0,u_2^0,\ldots,u_m^0\right)$ the w-limite set for $\left(u_1^0,u_2^0,\ldots,u_m^0\right)$ and Φ the set of the solution of the elliptic system

$$\begin{cases}-d_i\Delta u_i^s=f_i\left(u_1^s(x,t),u_2^s(x,t),\ldots,u_m^s(x,t)\right) & \text{in } \Omega,\\ \frac{\partial u_i^s}{\partial\eta}=0 & \text{in } \partial\Omega,\end{cases} \tag{4.2}$$

and prove $\Phi=\{(\xi_1,\xi_2,\ldots,\xi_m)\}$ where $\xi_1,\xi_2,\ldots,\xi_m$ are constants, in fact, multipliying the i^{th} equation of the problem (4.2) by u_i^s for $i=\overline{1,m}$ and integrating over Ω yields:

$$-d_i\int_{\Omega}u_i^s\Delta u_i^s dx=\int_{\Omega}u_i^s f_i\left(u_1^s(x,t),u_2^s(x,t),\ldots,u_m^s(x,t)\right)dx.$$

Apply Green formular:

$$d_i\int_{\Omega}|\nabla u_i^s|^2\,dx=\int_{\Omega}u_i^s f_i\left(u_1^s(x,t),u_2^s(x,t),\ldots,u_m^s(x,t)\right)dx.$$

Adding the i^{th} equations yields

$$\sum_{i=i}^{m} d_i \int_{\Omega} |\nabla u_i^s|^2 \, dx = \sum_{i=i}^{m} \int_{\Omega} u_i^s f_i \left(u_1^s(x,t), u_2^s(x,t), \ldots, u_m^s(x,t)\right) dx.$$

Supposing $\sum\limits_{i=i}^{m} u_i^s f_i \left(u_1^s(x,t), u_2^s(x,t), \ldots, u_m^s(x,t)\right) \leq 0$ for $i = \overline{1,m}$, then

$$0 \leq \sum_{i=i}^{m} d_i \int_{\Omega} |\nabla u_i^s|^2 \, dx \leq 0,$$

therefore

$$\sum_{i=i}^{m} d_i \int_{\Omega} |\nabla u_i^s|^2 \, dx = 0.$$

We deduce that

$$\int_{\Omega} |\nabla u_i^s|^2 \, dx = 0 \Rightarrow \nabla u_i^s = 0 \Rightarrow u_i^s = \xi_i. \tag{4.3}$$

Replacing $u_i^s = \xi_i$, for $i = \overline{1,m}$ in the i^{th} equation (4.2).
It is clear that $f_i(\xi_1, \xi_2, \ldots, \xi_m) = 0$.
Hereafter, we are going to show that $w\left(u_1^0, u_2^0, \ldots, u_m^0\right) \neq \phi$. Now, $\forall x \in \Omega$, $\sigma \in \,]-1,1[$ and let

$$p_i^n(x,\sigma) = u_i(x, t_n + \sigma), \text{ for } i = \overline{1,m},$$

multiply the i^{th} equation of the poblem (1.2) by $\frac{\partial u_i}{\partial t}$

$$\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - d_i \frac{\partial u}{\partial t}\Delta u_i = \frac{\partial u_i}{\partial t} f_i(u_1, u_2, \ldots, u_m),$$

and integrate over Ω we get:

$$\int_{\Omega} \left(\frac{\partial u_i}{\partial t}\right)^2 dx - d_i \int_{\Omega} \frac{\partial u_i}{\partial t}\Delta u_i dx = \int_{\Omega} \frac{\partial u_i}{\partial t} f_i(u_1, u_2, \ldots, u_m)\, dx,$$

as

$$\left\| \frac{\partial u_i}{\partial t} \right\|_{L^2(\Omega)}^2 = d_i \int_{\Omega} \frac{\partial u_i}{\partial t}\Delta u dx + \int_{\Omega} \frac{\partial u_i}{\partial t} f_i(u_1, u_2, \ldots, u_m)\, dx,$$

intégrating result over $(t_0, +\infty)$, we have:

$$\int_{t_0}^{+\infty} \left\| \frac{\partial u_i}{\partial t} \right\|_{L^2(\Omega)}^2 dt = d_i \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u_i}{\partial t}\Delta u_i dx dt + \int_{t_0}^{+\infty} \int_{\Omega} \frac{\partial u_i}{\partial t} f_i(u_1, u_2, \ldots, u_m)\, dx dt < \infty,$$

thus $\frac{\partial u_i}{\partial t} \in L^2\left(t_0, +\infty, L^2(\Omega)\right)$, $\forall \sigma \in \left]-1, 1\right[$ we get:

$$\begin{aligned}
p_i^n(x,\sigma) - u_i(x,t_n) &= u_i(x, t_n+\sigma) - u_i(x,t_n) \\
&= \int_{t_n}^{t_n+\sigma} \frac{\partial u_i}{\partial t}(x,t)\,dt \\
&\leq \int_{t_n-1}^{t_n+1} \frac{\partial u_i}{\partial t}(x,t)\,dt \text{ by reason of } (t_n - 1 < t_n,\ \sigma < 1,\ t_n+\sigma < t_n+1) \\
&\leq \left(\int_{t_n-1}^{t_n+1} (1)^2\,dt\right)^{\frac{1}{2}} \left(\int_{t_n-1}^{t_n+1} \left(\frac{\partial u_i}{\partial t}(x,t)\right)^2 dt\right)^{\frac{1}{2}}, \\
&\leq \sqrt{2}\left(\int_{t_n-1}^{t_n+1} \left(\frac{\partial u_i}{\partial t}(x,t)\right)^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

as follows

$$|p_i^n(x,\sigma) - u_i(x,t_n)|^2 = 2\int_{t_n-1}^{t_n+1} \left(\frac{\partial u_i}{\partial t}(x,t)\right)^2 dt,$$

integrating the latter inequality Ω yields

$$\int_\Omega |p_i^n(x,\sigma) - u_i(x,t_n)|^2\,dx \leq 2\int_\Omega \int_{t_n-1}^{t_n+1} \left(\frac{\partial u_i}{\partial t}(x,t)\right)^2 dt dx,$$

we pass to the limit as $n \to \infty$, we have:

$$\|p_i^n(x,\sigma) - u_i^s\|_{L^2(\Omega)}^2 \leq 2\lim_{n\to\infty} \int_\Omega \int_{t_n-1}^{t_n+1} \left(\frac{\partial u_i}{\partial t}(x,t)\right)^2 dt dx = 0,$$

so

$$\|p_i^n(x,\sigma) - u_i^s\|_{L^2(\Omega)}^2 \underset{n\to\infty}{\rightarrow} 0\,.$$

As a result, we will all $\sigma \in \left]-1, 1\right[$,

$$\|p_i^n(x,\sigma) - u_i^s\|_{L^2(\Omega)}^2 \underset{n\to\infty}{\rightarrow} 0, \text{ hence } \sup_{-1<\sigma<1} \|p_i^n(x,\sigma) - u_i^s\|_{L^2(\Omega)}^2 \underset{n\to\infty}{\rightarrow} 0,$$

and by the same mode are obtained:

$$\sup_{-1<\sigma<1} \|p_i^n(x,\sigma) - u_i^s\|_{L^2(\Omega)}^2 \underset{n\to\infty}{\rightarrow} 0, \text{ for } i = \overline{1,m}.$$

Also, we can have:

$$\sup_{-1<\sigma<1} \|\nabla p_i^n(x,\sigma) - \nabla u_i^s\|_{L^2(\Omega)}^2 \underset{n\to\infty}{\rightarrow} 0 \text{ for } i = \overline{1,m},$$

through positivity and boundedness of the solution was:

$$0 \leq u_i(x, t_n+\sigma) \leq M_i,$$

remarkably $f_i \in C^\infty(\mathbb{R}^n)$, we can conclude, using Lebesgue's theorem, that

$$f_i(p_1(x,\sigma), p_2(x,\sigma), \ldots, p_m(x,\sigma)) \to f_i(u_1^s, u_2^s, \ldots, u_m^s) \text{ in } L^2(\Omega\times(-1,1)) \text{ weak.}$$

Now, let $\varrho_i \in C^1\left(\overline{\Omega}\right)$ such that $\varrho_i = 0$ on $\partial\Omega$ where $i = \overline{1,m}$, and let $\gamma \in C^1\left(\overline{\Omega}\right)$ such that $supp\gamma \subset [-1,1]$,

$$\int_{-1}^{1} \gamma(s)\, ds = 1 \text{ and } \gamma(-1) = \gamma(1).$$

We multiply the i^{th} equation of problem (1.2) by $\gamma(t-t_n)\varrho_i$ and integrate over $\Omega \times (t_n - 1, t_n + 1)$, we obtain

$$\begin{array}{c} \int_{t_n-1}^{t_n+1} \int_\Omega \gamma(t-t_n)\varrho_i \frac{\partial u_i}{\partial t} dxdt - d_i \int_{t_n-1}^{t_n+1} \int_\Omega \gamma(t-t_n)\varrho_i \Delta u_i dxdt \\ = \int_{t_n-1}^{t_n+1} \int_\Omega \gamma(t-t_n)\varrho_i f_i(u_1, u_2, \ldots, u_m)\, dxdt. \end{array} \tag{4.4}$$

Forecast the integral $\int_{t_n-1}^{t_n+1} \gamma(t-t_n)\varrho_i \frac{\partial u_i}{\partial t} dt$ by part, we find

$$\int_{t_n-1}^{t_n+1} \gamma(t-t_n)\varrho_i \frac{\partial u_i}{\partial t} dt = -\int_{t_n-1}^{t_n+1} \gamma'(t-t_n)\varrho_i u_i(x,t)\, dt, \tag{4.5}$$

to appraise $\int_\Omega \gamma(t-t_n)\varrho_i \Delta u_i dx$ applying Green's formula

$$\begin{aligned} \int_\Omega \gamma(t-t_n)\varrho_i \Delta u_i dx &= \int_{\partial\Omega} \gamma(t-t_n)\varrho_i \frac{\partial u_i}{\partial \eta} d\sigma - \int_\Omega \nabla\gamma(t-t_n)\varrho_i \nabla u_i dx \\ &= -\int_\Omega \nabla\gamma(t-t_n)\varrho_i \nabla u_i dx, \end{aligned}$$

extremely

$$\int_\Omega \gamma(t-t_n)\varrho_i \Delta u_i dx = -\int_\Omega \nabla\gamma(t-t_n)\varrho_i \nabla u_i dx, \tag{4.6}$$

injected (4.5) and (4.6) in (4.4) is accessed

$$\begin{aligned} &-\int_{t_n-1}^{t_n+1} \int_\Omega \gamma'(t-t_n)\varrho_i u_i(x,t)\, dxdt + d_i \int_{t_n-1}^{t_n+1} \int_\Omega \nabla\gamma(t-t_n)\varrho_i \nabla u_i dxdt \\ &+ \int_{t_n-1}^{t_n+1} \int_\Omega \gamma(t-t_n)\varrho_i f_i(u_1, u_2, \ldots, u_m)\, dxdt = 0 \text{ for } i = \overline{1,m}. \end{aligned} \tag{4.7}$$

By making the following change of variable $\sigma = t - t_n \to d\sigma = dt$

$$\text{if } \left\{ \begin{array}{l} t = t_n - 1 \\ t = t_n + 1 \end{array} \right. \to \left\{ \begin{array}{c} \sigma = -1 \\ \sigma = 1 \end{array} \right.$$

accordingly the integral (4.7) becomes

$$\int_{-1}^{+1} \int_\Omega \gamma'(\sigma)\varrho_i p_i^n(x,\sigma)\, dxd\sigma - d_i \int_{-1}^{+1} \int_\Omega \nabla\gamma(\sigma)\varrho_i \nabla p_i^n(x,\sigma)\, dxd\sigma$$

$$\int_{-1}^{+1} \int_\Omega \gamma(\sigma)\varrho_i f_i(p_1(x,\sigma), p_2(x,\sigma), \ldots, p_m(x,\sigma))\, dxd\sigma = 0, \text{ for } i = \overline{1,m}. \tag{4.8}$$

Applying Lesbegue's theorem we gain: for $i = \overline{1, m}$

$$\begin{aligned}
\lim_{n\to\infty} \int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma)\, \varrho_i p_i^n(x,\sigma)\, dxdt &= \int_{-1}^{+1} \int_{\Omega} \gamma'(\sigma)\, \varrho_i u_i^s dxd\sigma \\
&= \int_{-1}^{+1} \gamma'(\sigma)\, d\sigma \int_{\Omega} \varrho_i u_i^s dx \\
&= \gamma(\sigma)|_{-1}^{+1} \int_{\Omega} \varrho_i u_i^s dx = 0 \text{ by virtue of } \gamma(1) = \gamma(-1),
\end{aligned}$$

from inequality (4.8), we make: for $i = \overline{1, m}$

$$-d_i \int_{\Omega} \nabla \varrho_i \nabla u_i^s + \int_{\Omega} \varrho_i f_i\left(u_1^s, u_2^s, \dots, u_m^s\right) dx = 0,$$

Which is the variational formulation of (4.2), hence $w = \Phi$.

Finally, combining (4.2) with (4.1) yields

$$\begin{aligned}
\int_{\Omega} \sum_{i=1}^{m} \xi_i dx &= \int_{\Omega} \sum_{i=1}^{m} u_i^0 dx, \\
|\Omega| \sum_{i=1}^{m} \xi_i &= \int_{\Omega} \sum_{i=1}^{m} u_i^0 dx,
\end{aligned}$$

$$\sum_{i=1}^{m} \xi_i = \frac{1}{|\Omega|} \int_{\Omega} \sum_{i=1}^{m} u_i^0 dx,$$

the proof of the theorem is complete. □

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