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# Fixed Point Theorems in Cone Metric Spaces via $c-$Distance Over Topological Module

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Abstract. In 2011, Wang and Guo introduced c-distance in cone metric spaces. The idea of cone metric spaces over topological modules was presented by Branga and Olaru in 2020. Combining these two ideas, we introduce cone metric spaces with $c-$distance over topological module and establish a fixed point theorem.

## 1. Introduction

Cone metric spaces were first introduced by Huang and Zhang [9]. For detailed study of cone metric spaces, refer [5, 6, 12, 14–16]. Other authors have also established fixed point theorems in cone metric spaces (for instance, [1–3, 10, 11]). Wang and Guo [18] presented cone metric spaces with $c$-distance and proved some fixed point theorems. Cone metric spaces over topological module were introduced by Branga and Olaru [4]. In this paper, we present a new concept namely "cone metric spaces with $c$-distance over topological module" and prove a fixed point theorem.

## 2. Preliminaries

**Definition 2.1.** *[8] Let $(G, +)$ be a group with partial order relation $\leq$ . Then $G$ is said to be a* **partially ordered group** *if translation in $G$ is order preserving:*

$$x \leq y \Rightarrow z + x + w \leq z + y + w, \forall\, x, y, w, z \in G$$

**Definition 2.2.** *[17] Consider a ring $(R, +, .)$ and $1$ be an identity of $(R, +, .)$ such that $1 \neq 0$ and $\leq$ is a partial order on $R$. Then $R$ is called a partially ordered ring if:*

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(1) $(R, +, .)$ *is a partially ordered group;*

(2) $z \geq 0$ *and* $w \geq 0$ *implies* $z.w \geq 0$ *for all* $z, w \in R$.

**Remark 2.1.** *Throughout the paper,* $R^+ = \{r \in R : r \geq 0\}$, $\mathcal{U}(R)$ *and* $\mathcal{U}_+(R)$ *are the notations for the positive cone of* $R$*, the set of invertible elements of* $R$ *and* $\mathcal{U}(R) \cap R^+$*, respectively.*

**Definition 2.3.** *[20] Consider an abelian group* $(G, +)$*. Then* $G$ *is called a topological group if* $G$ *is endowed with a topology* $\mathcal{G}$ *such that the conditions mentioned below are satisfied:*

(1) *For* $(g_1, g_2) \in G \times G$*, the map* $(g_1, g_2) \to g_1 + g_2$ *is continuous where* $g_1 + g_2 \in G$ *and* $G \times G$ *is endowed with the product topology;*

(2) *For* $g \in G$*, the map* $g \to -g$ *is continuous, where* $-g \in G$.

$(G, +, \mathcal{G})$ *or* $(G, \mathcal{G})$ *is the notation for the topological group.*

**Definition 2.4.** *[20] A ring* $(R, +, .)$ *is called a topological ring if* $R$ *considered with the topology* $\mathcal{R}$ *such that* $(R, +, \mathcal{R})$ *is a topological group and for* $(r_1, r_2) \in R \times R$*, the map* $(r_1, r_2) \to r_1.r_2$ *is continuous, where* $r_1.r_2 \in R$ *and* $R \times R$ *is endowed with the product topology.*

$(R, +, ., \mathcal{R})$ is called a Hausdorff topological ring [20] if the topology $\mathcal{R}$ is Hausdorff. Also, $(R, +, ., \mathcal{R})$ or $(R, \mathcal{R})$ is the notation for the topological ring.

**Definition 2.5.** *[20] Consider a topological ring* $(R, \mathcal{R})$*. A left* $R-$*module* $(E, +, .)$ *is called a topological* $R-$*module if a topology* $\mathcal{E}$ *is defined on* $E$ *such that* $(E, +)$ *is a topological abelian group and the condition mentioned below is satisfied:*
*For* $(r, x) \in R \times E$*, the map* $(r, x) \to r.x$ *is continuous, where* $r.x \in E$.
*Further,* $(E, +, ., \mathcal{E})$ *or* $(E, \mathcal{E})$ *is the notation used for the topological left* $R-$*module.*

**Definition 2.6.** *[4] A set* $P \subset E$*, where* $(E, +, ., \mathcal{E})$ *is a topological module, is called a cone if :*

(1) $P$ *is closed, nonempty and* $P \neq \{0_E\}$;

(2) $s.x + t.y \in P$*, whenever* $x, y \in P$ *and* $s, t \in R^+$;

(3) $x \in P$ *and* $-x \in P$ *implies* $x = 0_E$.

**Remark 2.2.** *Throughout the paper,* $P^\circ$ *and* $\bar{P}$ *are be the notations for the interior and closure of* $P$*, respectively. Furthermore, the cone* $P$ *is said to be solid if* $P^\circ$ *is non-empty. Also, a partial ordering* $\leq_P$ *with respect to* $P$ *is defined by* $y - x \in P$ *if and only if* $x \leq_P y$ *and* $x <_P y$ *indicates* $x \leq_P y, x \neq y$*. Moreover, the partial ordering* $x \ll y$ *indicates* $y - x \in P^\circ$.

Next, we shall consider the following hypotheses:

(Hyp I) [4] Consider a Hausdorff topological ring $(R, +, \odot, \mathcal{R})$ such that:

(1) $\mathcal{U}_+(R)$ is non-empty.

(2) $0_R$ is a limit point of $\mathcal{U}_+(R)$.

(3) $\leq_R$ indicates the partial ordering on $R$.

(Hyp II) [4] $(E, +, ., \mathcal{E})$ is a topological left $R-$module.

(Hyp III) [4] $P$ is a solid cone contained in $E$.

**Proposition 2.1.** *[4] Consider a topological left $R-$module $(E, +, ., \mathcal{E})$ and $P$ be a cone contained in $E$ such that the hypothesis $HypI, HypII$ and $HypIII$ are satisfied. Then:*

1. *$P^\circ + P^\circ \subseteq P^\circ$;*
2. *$\beta \odot P^\circ \subseteq P^\circ$, where $\beta$ is an element of $\mathcal{U}_+(R)$;*
3. *If $v \leq_P u$ and $\beta \in R^+$, then $\beta \odot v \leq_P \beta \odot u$;*
4. *If $y \leq_P x$ and $x \ll z$, then $y \ll z$;*
5. *If $y \ll x$ and $x \leq_P z$, then $y \ll z$;*
6. *If $y \ll x$ and $x \ll w$, then $y \ll w$;*
7. *If $0_E \leq_P y \ll v, \forall v \in P^\circ$, then $v = 0$;*
8. *If $0_E \ll v$ and $\{b_n\}$ is a sequence in $E$ such that $b_n \to 0_E$, then there is a natural number $m_0$ such that $b_n \ll u$ for $m \geq m_0$.*

From now onwards, $(R, +, ., \mathcal{R})$ denotes a Hausdorff topological ring.

**Definition 2.7.** *[4] A directed set is a partially ordered set $(\Lambda, \leq)$ such that the condition mentioned below is fulfilled:*
*If $\lambda_1, \lambda_2$ in $\Lambda$ there is $\lambda_3$ in $\Lambda$ so that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.*

**Definition 2.8.** *[4] A sequence $\{x_\lambda\}_{\lambda\in\Lambda}$ in $R$ is a family of elements in $R$ that is indexed by a directed set.*

**Definition 2.9.** *[4] A family $\{x_\lambda\}_{\lambda\in\Lambda}$ contained in $R$ is convergent to an element $x \in R$ if for each neighborhood $W$ of $x$ there exists $\lambda_0 \in \Lambda$ such that $x_\lambda$ is an element of $W$ for every $\lambda \in \Lambda$, where $\lambda \geq \lambda_0$.*

**Definition 2.10.** *[4] A sequence $\{x_\lambda\}_{\lambda\in\Lambda}$ is called a Cauchy sequence if for each neighborhood $W$ of $0_R$ there exists $\lambda_0 \in \Lambda$ such that $x_{\lambda_1} - x_{\lambda_2} \in W$ for each $\lambda_0 \leq \lambda_1$ and $\lambda_0 \leq \lambda_2$.*

**Theorem 2.1.** *[4] Every convergent sequence $\{x_\lambda\}_{\lambda\in\Lambda}$ in $R$ is a Cauchy sequence.*

To exemplify the summability of the family of elements of a topological ring, consider $\mathcal{H}(\Lambda)$ to be the set of all finite sets contained in $\Lambda$ directed by the inclusion $\subseteq$ .

**Definition 2.11.** *[4] An element $t$ of $R$ is sum of family $\{x_\lambda\}_{\lambda\in\Lambda}$ contained in $R$ if the sequence $\{t_I\}_{I\in\mathcal{H}(\Lambda)}$ is convergent to $t$, where for each $I \in \mathcal{H}(\Lambda)$,*

$$t_I = \sum_{\lambda\in I} x_\lambda.$$

*The family $\{x_\lambda\}_{\lambda\in I}$ is said to be summable if $\{x_\lambda\}_{\lambda\in I}$ has a sum $t$ in $R$.*

**Definition 2.12.** *[4] A family $\{x_\lambda\}_{\lambda\in I}$ contained in $R$ is said to satisfy Cauchy condition if for every neighborhood $W$ of $0_R$ there exists $I_W$ in $\mathcal{H}(\Lambda)$ such that $\sum_{\lambda\in J} x_\lambda \in W$, for each $J \in \mathcal{H}(\Lambda)$ disjoint with $I_W$.*

**Theorem 2.2.** *[4] A sequence $\{t_I\}_{I\in\mathcal{H}(\Lambda)}$ is a Cauchy sequence if and only if the family $\{x_\lambda\}_{\lambda\in\Lambda}$ contained in $R$ satisfies Cauchy condition.*

**Theorem 2.3.** *[4] Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a summable family in $R$. Then for each neighborhood $W$ of $0_R$, there exists $J$ in $\mathcal{H}(\Lambda)$ so that $x_\lambda \in W$ for each $\lambda \in \Lambda \setminus J$.*

**Definition 2.13.** *[4] Let $(R, +, ., \mathcal{R})$ be a topological ring. Then $R$ is said to be complete if the topological additive group of ring $(R, +, \mathcal{R})$ is complete.*

**Definition 2.14.** *[4] Let $X$ be a non-empty set, $(E, +, ., \mathcal{E})$ be a topological left $R-$module and the map $d : X \times X \to E$ satisfies:*

(1) $d(x, y) \geq_P 0_E \ \forall x, y \in X$.
(2) $d(x, y) = 0_E$ *if and only if* $x = y \ \forall x, y \in X$.
(3) $d(x, y) = d(y, x) \ \forall x, y \in X$.
(4) $d(x, y) \leq_P d(x, z) + d(z, y) \ \forall x, y \in X$.

*Then $d$ is said to be a cone metric on $X$ and the pair $(X, d)$ is said to be a cone metric space over topological left $R-$module $E$.*

**Definition 2.15.** *[4] Let $(X, d)$ be a cone metric space over the topological left $R-$module $E$, $x$ be an element of $X$ and $\{x_n\}$ be a sequence in $X$. Then*

(1) *$\{x_n\}$ is said to be convergent to $x$ if for each $u \gg 0$, there is a natural number $N$ such that $d(x_n, x) \ll u, \ \forall \ n > N$.*
(2) *$\{x_n\}$ contained in $X$ is said to be a Cauchy sequence if for each $u \gg 0$, there is a natural natural number $N$ such that $d(x_n, x_m) \ll u, \ \forall \ m, n > N$.*

## 3. Fixed point theorem via c-distance

In this section we first introduce a new notion namely $c-$distance in cone metric spaces over topological module. Next we discuss some results regarding the same. Further, a fixed point theorem in cone metric spaces via $c-$distance over topological module has been established.

**Definition 3.1.** *Let $(X, d)$ be a cone metric space over a topological left $R-$module $E$. Then a map $p : X \times X \to E$ is said to be a $c-$distance on $X$ if it satisfies the following conditions:*

(p1) $p(x, y) \geq_P 0_E \ \forall \ x, y \in X$;
(p2) $p(x, z) \leq_P p(x, y) + p(y, z) \ \forall \ x, y, z \in X$;
(p3) *For every $y \in X$ and $n \geq 1$, $p(y, x_n) \leq_P u$ for some $u = u_y \in P$, then $p(y, x) \leq_P u$ whenever $\{x_n\}$ in $X$ is convergent to a point $x \in X$;*

(p4) *For each* $u \gg 0$, *there exists* $v \gg 0$ *such that* $p(z,x) \ll v$ *and* $p(z,y) \ll v$ *implies that* $d(x,y) \ll u$ *where* $u, v \in E$.

**Lemma 3.1.** *Let* $(X,d)$ *be a cone metric space over a topological left* $R$*-module* $E$, $p$ *be a* $c-$*distance on* $X$. *Consider the sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* $X$. *Next, let* $\{a_n\}$ *be a sequence in* $P$ *convergent to* $0_E$ *and* $x, y, z \in X$. *Then the following hold:*

(i) *If* $p(x_n, y_n) \leq_P a_n$ *and* $p(x_n, z) \leq_P a_n$ *for every natural number* $n$, *then* $\{y_n\}$ *is convergent to* $z$.
(ii) *If* $p(x_n, y) \leq_P a_n$ *and* $p(x_n, z) \leq_P a_n$ *for every natural number* $n$, *then* $y = z$.
(iii) *If* $p(x_n, x_m) \leq_P a_n$ *for all* $m > n$, *where* $m, n$ *are natural numbers, then* $\{x_n\}$ *is a Cauchy sequence in* $X$.
(iv) *If* $p(y, x_n) \leq_P a_n$, *then* $\{x_n\}$ *is a Cauchy sequence in* $X$.

*Proof.* (i) Let $u \gg 0$. Then there is $v \gg 0$ such that $p(u', v') \ll v$ and $p(u', z) \ll v$ implies that $d(v', z) \ll u$. Let $m_0$ be any natural number such that $a_n \ll v$ and $b_n \ll v$ for each $n \geq m_0$. Next for each $n \geq m_0, p(x_n, y_n) \leq_P a_n \ll v$ and $p(x_n, z) \leq_P b_n \ll v$. Therefore, $d(y_n, z) \ll u$. This proves that $\{y_n\}$ is convergent to $z$.
(ii) Proof clearly follows from $(i)$.
(iii) Let $u \gg 0$. Proceeding as proof of $(i)$ let $v \gg 0$ and $m_0$ be any natural number. Next, for $n, m \geq m_0 + 1, p(x_{m_0}, x_n) \leq u_{m_0} \ll v$ and $p(x_{m_0}, x_m) \leq u_{m_0} \ll v$. Therefore, $d(x_n, x_m) \ll v$. This proves that $\{x_n\}$ is a Cauchy sequence.
(iv) Proof clearly follows from $(iii)$.

□

**Theorem 3.1.** *Let* $(X,d)$ *be a cone metric space over a topological left* $R-$*module* $E$ *and* $p$ *be a* $c-$*distance on* $X$. *Suppose that the hypothesis* $HypI, HypII$ *and* $HypIII$ *are satisfied. Define*

$$\mathcal{S} = \{r \in R^+ | \{r^n\} \text{ is a summable family}\}.$$

*and the map* $T : X \to X$ *satisfies:*

$$p(Tx, Ty) \leq_P rp(x,y), \ \forall \ x, y \in X.$$

*Then* $T$ *has a fixed point* $x^*$ *in* $X$ *and for each* $x \in X, \{T^n x\}$ *is convergent to the fixed point. If* $\zeta = T\zeta$, *then* $p(\zeta, \zeta) = 0$. *Also,* $T$ *has a unique fixed point.*

*Proof.* Fix $x_0 \in X$. Let $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0$. Then
$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq_P rp(x_{n-1}, x_n) \leq_P r^2 p(x_{n-2}, x_{n-1}) \leq_P \ldots \leq_P r^n p(x_0, x_1)$.

On the basis of above inequality, for all $q \geq 1$, we have

$$\begin{aligned} p(x_n, x_{n+q}) &\leq_P p(x_n, x_{n+1}) + \ldots + p(x_{n+q-1}, x_{n+q}) \\ &\leq_P r^n p(x_0, x_1) + \ldots + r^{n+q-1} p(x_0, x_1) \\ &\leq_P r^n (1_R + r + \ldots + r^{q-1}) p(x_0, x_1) \\ &\leq_P r^n (\sum_{i=0}^{+\infty} r^i) p(x_0, x_1). \end{aligned}$$

Using Theorem 2.3, we have $r^n \overset{R}{\to} 0_R$ as $n \to \infty$. Also $\{r^n\}$ is a summable family, right multiplication is continuous and using Proposition 2.1 (8) we see that for each $u \gg 0$, there is a natural number $N$ such that $p(x_n, x_{n+q}) \ll u \ \forall \ n \geq N$ and $q \geq 1$. Then $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $X$, there is $x^*$ in $X$ such that $x_n$ is convergent to $x^*$ as $n$ tends to $\infty$. By $(p3)$ we see that

$$p(x_n, x^*) \ll u. \ (0.3.1)$$

Also,

$$\begin{aligned} p(x_n, Tx^*) &= p(Tx_{n-1}, Tx^*) \\ &\leq_P rp(x_{n-1}, x^*) \\ &\leq_P r^n (\sum_{i=0}^{+\infty} r^i) p(x_0, x_1) \\ &\ll u. \ (0.3.2) \end{aligned}$$

Now, $p(x_n, x^*) \ll u$ and $p(x_n, Tx^*) \ll u$. Let $v \gg 0$. Then by $(p4)$, we have $d(x^*, T(x^*)) \ll v$. Next using Proposition 2.1 (7) we have

$$x^* = Tx^*.$$

Hence $x^*$ is a fixed point of $T$. Next suppose that $y^*$ is a fixed point of $T$. Then for $u \in P^\circ$, we have $q(x^*, y^*) = q(Tx^*, Ty^*) \leq_P rq(x^*, y^*) \leq_P r^2 q(x^*, y^*) \leq_p \ldots \leq_P r^n q(x^*, y^*) \ll u$. Hence $q(x^*, y^*) = 0$. Also, $q(x^*, x^*) = 0$. Using Lemma 3.1 $(ii)$, we have $x^* = y^*$. Hence $T$ has a unique fixed point. □

**Corollary 3.1.** *Let $(X, d)$ be a cone metric space over a topological left $R-$module $E$ and $p$ be a $c-$distance on $X$. Define*

$$\mathcal{S} = \{r \in R^+ | \{r^n\} \text{ is a summable family}\}.$$

*and the map $T : X \to X$ satisfies:*

$$p(T^n x, T^n y) \leq_P rp(x, y), \ \forall \ x, y \in X.$$

*Then $T$ has a fixed point $x^*$ in $X$. If $\zeta = T\zeta$, then $p(\zeta, \zeta) = 0$.*

*Proof.* Using Theorem 3.1, we see that $x^*$ is a unique fixed point of $T^n$. Also, $T^n(Tx^*) = T(T^n x^*) = T(x^*)$. Therefore, $T(x^*)$ is a fixed point of $T^n$. So, $x^* = Tx^*$. This shows that $x^*$ is a fixed point of $T$. Also, the fixed point of $T$ is also a fixed point of $T^n$ shows that $T$ has a unique fixed point.

Next suppose that $\zeta = T\zeta$. We see that the fixed point of $T$ is also a fixed point of $T^n$. From this, for $u \gg 0$, we have

$p(\zeta,\zeta) = p(T\zeta, T\zeta) = p(T^n\zeta, T^n\zeta) \leq_P rp(\zeta,\zeta) \leq_P r^2q(\zeta,\zeta) \leq_P \ldots \leq_P r^n p(\zeta,\zeta) \ll u.$

Hence $p(\zeta,\zeta) = 0$. □

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