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FIXED POINT AND COMMON FIXED POINT THEOREMS FOR α–PROPERTY IN CONE BALL-METRIC SPACES

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ABSTRACT. In this paper, we define a new cone ball-metric and get fixed points and common fixed points for the $\alpha - property$ in cone ball-metric spaces.

1. Introduction and preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers.

Let $\Diamond : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a binary operation satisfying the following conditions :

(i) $\Diamond$ is associative and commutative,
(ii) $\Diamond$ is continuous.

Five typical examples of $\Diamond$ are :

(i) $a\Diamond b = \max\{a, b\}$,
(ii) $a\Diamond b = a + b$,
(iii) $a\Diamond b = ab$,
(iv) $a\Diamond b = ab + a + b$,
(v) $a\Diamond b = \frac{ab}{\max\{a,b,1\}}$.

Definition 1.1. The binary operation $\Diamond$ is said to satisfy $\alpha - property$ if there exists a positive real number α such that

$$a\Diamond b \leq \alpha \max\{a, b\}$$

for all $a, b \in \mathbb{R}^+$.

Following five examples are stand for $\alpha - property$.

Example 1.2. If $a\Diamond b = a + b$, for each $a, b \in \mathbb{R}^+$, then for $\alpha \geq 2$, we have $a\Diamond b \leq \alpha \max\{a, b\}$.

Example 1.3. If $a\Diamond b = \frac{ab}{\max\{a,b,1\}}$, for each $a, b \in \mathbb{R}^+$, then for $\alpha \geq 1$, we have $a\Diamond b \leq \alpha \max\{a, b\}$.

Example 1.4. If $a\Diamond b = ab$, for each $a, b \in \mathbb{R}^+$, then for $\alpha \geq a$, we have $a\Diamond b \leq \alpha \max\{a, b\}$.

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Example 1.5. If $a\Diamond b = ab + a + b$, for each $a, b \in \mathbb{R}^+$, then for $\alpha \geq a + 2$, we have $a\Diamond b \leq \alpha \max\{a, b\}$.

Example 1.6. If $a\Diamond b = \max\{a, b\}$, for each $a, b \in \mathbb{R}^+$, then for $\alpha > 1$, we have $a\Diamond b \leq \alpha \max\{a, b\}$.

Huang and Zhang [3] have introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. In 2006, Mustafa and Sims [5] introduced a more appropriate generalization of metric spaces, G-metric spaces. Recently, Beg et al. [2] introduced the notion of generalized cone metric spaces, and proved some fixed point results for mappings satisfying certain contractive conditions. In this paper, we define a new cone ball-metric and get fixed point and common fixed results for the $\alpha - property$ in cone ball-metric spaces.

We recall some definitions of the cone metric spaces and some of the properties [3], as follow:

Definition 1.7. [3] Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is nonempty, closed, and $P \neq \{\theta\}$,
(ii) $a, b \in \mathbb{R}^+$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

For given a cone $P \subset E$, we can define a partial ordering with respect to P by $x \preccurlyeq y$ or $x \succcurlyeq y$ if and only if $y - x \in P$ for all $x, y \in E$. The real Banach space E equipped with the partial ordered induced by P is denoted by $(E, \preccurlyeq)$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$\theta \preccurlyeq x \preccurlyeq y \quad \Rightarrow \quad \|x\| \leq K\|y\|.$$

The least positive number K satisfying above is called the normal constant of P.

The cone P is called regular if every non-decreasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq \cdots \preccurlyeq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every non-increasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Moreover, P is called stronger minihedral if every subset of E which is bounded above has a supremum [1].

In the following we always suppose that E is a real Banach space with a stronger minihedral regular cone P and $intP \neq \phi$, and $\preccurlyeq$ is a partial ordering with respect to P.

Metric spaces are playing an important role in mathematics and the applied sciences. In 2003, Mustafa and Sims [5] introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 1.8. [5] Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(2) $G(x, x, y) > 0$, for all $x \neq y$;
(3) $G(x, y, z) \geq G(x, x, y)$, for all $x, y, z \in X$;
(4) $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables);
(5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$, for all $x, y, z, w \in X$.

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

This research subject is interesting and widespread. But is too abstract makes the human difficulty with to understand. So we introduce the concept of cone ball-metric spaces and we prove fixed point results on such spaces for functions satisfying the contractions involving the $\alpha - property$.

In [6] Chen and Tsai introduce the following notion of the cone ball-metric $\mathcal{B}$.

Definition 1.9. [6] Let (X, d) be a cone metric space, $\mathcal{B} : X \times X \times X \to E$, $x, y, z \in X$ and we denote

$$\delta(B) = \sup\{d(a, b) : a, b \in B\},$$

and

$$\mathcal{B}(x, y, z) = \delta(B),$$

where $B = \cap\{F \subset X | F \text{ is a closed ball and } \{x, y, z\} \subset F\}$. Then we call $\mathcal{B}$ a ball-metric with respect to the cone metric d, and $(X, \mathcal{B})$ a cone ball-metric space. It is clear that $\mathcal{B}(x, x, y) = d(x, y)$.

Remark 1.10. It is clear that the cone ball-metric $\mathcal{B}$ has the following properties:

(1) $\mathcal{B}(x, y, z) = \theta$ if and only if $x = y = z$;
(2) $\mathcal{B}(x, x, y) \succ \theta$, for all $x \neq y$;
(3) $\mathcal{B}(x, x, y) \preccurlyeq \mathcal{B}(x, y, z)$, for all $x, y, z \in X$;
(4) $\mathcal{B}(x, y, z) = \mathcal{B}(x, z, y) = \mathcal{B}(z, y, x) = \cdots$ (symmetric in all three variables);
(5) $\mathcal{B}(x, y, z) \preccurlyeq \mathcal{B}(x, w, w) + \mathcal{B}(w, y, z)$, for all $x, y, z, w \in X$;
(6) $\mathcal{B}(x, y, z) \preccurlyeq \mathcal{B}(x, w, w) + \mathcal{B}(y, w, w) + \mathcal{B}(z, w, w)$, for all $x, y, z, w \in X$.

Definition 1.11. [6] Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(1) Cauchy sequence if for every $\varepsilon \in E$ with $\theta \ll \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m, l > n_0$, $\mathcal{B}(x_n, x_m, x_l) \ll \varepsilon$.
(2) Convergent sequence if for every $\varepsilon \in E$ with $\theta \ll \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $\mathcal{B}(x_n, x_m, x) \ll \varepsilon$ for some $x \in X$. Here x is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.12. [6] Let $(X, \mathcal{B})$ be a cone ball-metric space. Then X is said to be complete if every Cauchy sequence is convergent in X.

Proposition 1.13. [6] *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(i) *$\{x_n\}$ converges to x;*
(ii) *$\mathcal{B}(x_n, x_n, x) \to \theta$ as $n \to \infty$;*
(iii) *$\mathcal{B}(x_n, x, x) \to \theta$ as $n \to \infty$;*
(iv) *$\mathcal{B}(x_n, x_m, x) \to \theta$ as $n, m \to \infty$.*

Proposition 1.14. [6] *Let $(X,\mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X, $x,y \in X$. If $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$.*

Proof. Let $\varepsilon \in E$ with $\theta \ll \varepsilon$ be given. Since $x_n \to x$ and $x_n \to y$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $m,n > n_0$,

$$\mathcal{B}(x_n,x_m,x) \ll \frac{\varepsilon}{3} \text{ and } \mathcal{B}(x_n,x_m,y) \ll \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned}\mathcal{B}(x,x,y) &\preccurlyeq \mathcal{B}(x,x_n,x_n) + \mathcal{B}(x_n,x,y)\\ &= \mathcal{B}(x,x_n,x_n) + \mathcal{B}(y,x_n,x)\\ &\preccurlyeq \mathcal{B}(x,x_n,x_n) + \mathcal{B}(y,x_m,x_m) + \mathcal{B}(x_m,x_n,x)\\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence, $\mathcal{B}(x,x,y) \ll \frac{\varepsilon}{\alpha}$ for all $\alpha \geq 1$, and so $\frac{\varepsilon}{\alpha} - \mathcal{B}(x,x,y) \in P$ for all $\alpha \geq 1$. Since $\frac{\varepsilon}{\alpha} \to \theta$ as $\alpha \to \infty$ and P is closed, we have that $-\mathcal{B}(x,x,y) \in P$. This implies that $\mathcal{B}(x,x,y) = \theta$, since $\mathcal{B}(x,x,y) \in P$. So $x = y$. □

Proposition 1.15. [6] *Let $(X,\mathcal{B})$ be a cone ball-metric space and $\{x_n\},\{y_m\},\{z_l\}$ be three sequences in X. If $x_n \to x$, $y_m \to y$, $z_l \to z$ as $n \to \infty$, then $\mathcal{B}(x_n,y_m,z_l) \to \mathcal{B}(x,y,z)$ as $n \to \infty$.*

Proof. Let $\varepsilon \in E$ with $\theta \ll \varepsilon$ be given. Since $x_n \to x$, $y_m \to y$, $z_l \to z$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m,l > n_0$,

$$\mathcal{B}(x_n,x,x) \ll \frac{\varepsilon}{3},\ \ \mathcal{B}(y_m,y,y) \ll \frac{\varepsilon}{3},\ \ \mathcal{B}(z_l,z,z) \ll \frac{\varepsilon}{3},$$

Therefore,

$$\begin{aligned}\mathcal{B}(x_n,y_m,z_l) &\preccurlyeq \mathcal{B}(x_n,x,x) + \mathcal{B}(x,y_m,z_l)\\ &\preccurlyeq \not\succcurlyeq \mathcal{B}(x_n,x,x) + \mathcal{B}(y_m,y,y) + \mathcal{B}(y,x,z_l)\\ &\preccurlyeq \mathcal{B}(x_n,x,x) + \mathcal{B}(y_m,y,y) + \mathcal{B}(z_l,z,z) + \mathcal{B}(z,x,y)\\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \mathcal{B}(x,y,z),\end{aligned}$$

that is,

$$\mathcal{B}(x_n,y_m,z_l) - \mathcal{B}(x,y,z) \ll \varepsilon.$$

Similarly,

$$\mathcal{B}(x,y,z) - \mathcal{B}(x_n,y_m,z_l) \ll \varepsilon.$$

Therefore, for all $\alpha \geq 1$, we have

$$\mathcal{B}(x_n,y_m,z_l) - \mathcal{B}(x,y,z) \ll \frac{\varepsilon}{\alpha},$$

and

$$\mathcal{B}(x,y,z) - \mathcal{B}(x_n,y_m,z_l) \ll \frac{\varepsilon}{\alpha}.$$

These imply that

$$\begin{aligned}&\frac{\varepsilon}{\alpha} - \mathcal{B}(x_n,y_m,z_l) + \mathcal{B}(x,y,z) \in P,\\ &\frac{\varepsilon}{\alpha} + \mathcal{B}(x_n,y_m,z_l) - \mathcal{B}(x,y,z) \in P.\end{aligned}$$

Since P is closed and $\frac{\varepsilon}{\alpha} \to \theta$ as $\alpha \to \infty$, we have that

$$\lim_{n,m,l\to\infty} [-\mathcal{B}(x_n,y_m,z_l) + \mathcal{B}(x,y,z)] \in P,$$

$$\lim_{n,m,l\to\infty}[\mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z)] \in P.$$

These show that

$$\lim_{n,m,l\to\infty}\mathcal{B}(x_n, y_m, z_l) = \mathcal{B}(x, y, z).$$

So we complete the proof. □

2. Main results

Let $(X, \mathcal{B})$ be a cone ball-metric space with P, and let $\Diamond : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$.

We now state our main common fixed point result for the $\alpha - property$ in a cone ball-metric space $(X, \mathcal{B})$, as follows:

Theorem 2.1. *Let $(X, \mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and f, g be two self mappings of X such that $f(X) \subset g(X)$. Suppose that $\Diamond$ satisfies $\alpha - property$ with $\alpha > 0$ such that*

$$\begin{aligned} \mathcal{B}(fx, fy, fz) &\preccurlyeq k_1[\mathcal{B}(gx, gy, gz)\Diamond\mathcal{B}(gx, fx, fx)] \\ &+ k_2[\mathcal{B}(gx, gy, gz)\Diamond\mathcal{B}(gy, fy, fy)] \\ &+ k_3[\mathcal{B}(gx, gy, gz)\Diamond\mathcal{B}(gz, fz, fz)], \end{aligned} \tag{2.1}$$

where $k_1, k_2, k_3 > 0$ and $0 < \alpha(k_1 + k_2 + k_3) < 1$. If $g(X)$ is closed, then f and g have a coincidence point in X.

Moreover, if f and g commute at their coincidence points, then f and g have a unique common fixed point in X

Proof. Given $x_0 \in X$. Since $f(X) \subset g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Continuing this process, we define the sequence $\{x_n\}$ in X recursively as follows:

$$fx_n = gx_{n+1} \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

In what follows we will suppose that $fx_{n+1} \neq fx_n$ for all $n \in \mathbb{N}$, since if $fx_{n+1} = fx_n$ for some n, then $fx_{n+1} = gx_{n+1}$, that is , f, g have a coincidence point x_{n+1}, and so we complete the proof.

By (2.1), we have

$$\begin{aligned} \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) &\preccurlyeq k_1[\mathcal{B}(gx_n, gx_{n+1}, gx_{n+1})\Diamond\mathcal{B}(gx_n, fx_n, fx_n)] \\ &+ k_2[\mathcal{B}(gx_n, gx_{n+1}, gx_{n+1})\Diamond\mathcal{B}(gx_{n+1}, fx_{n+1}, fx_{n+1})] \\ &+ k_3[\mathcal{B}(gx_n, gx_{n+1}, gx_{n+1})\Diamond\mathcal{B}(gx_{n+1}, fx_{n+1}, fx_{n+1})]. \end{aligned}$$

Therefore, by the condition of $\alpha - property$, we conclude that for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) &\preccurlyeq k_1\alpha \max\{\mathcal{B}(fx_{n-1}, fx_n, fx_n), \mathcal{B}(fx_{n-1}, fx_n, fx_n)\} \\ &+ k_2\alpha \max\{\mathcal{B}(fx_{n-1}, fx_n, fx_n), \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1})\} \\ &+ k_3\alpha \max\{\mathcal{B}(fx_{n-1}, fx_n, fx_n), \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1})\}. \end{aligned}$$

If $\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) > \mathcal{B}(fx_{n-1}, fx_n, fx_n)$, we obtain

$$\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) \ll \alpha(k_1 + k_2 + k_3)\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}),$$

which contradiction. Hence $\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) \leq \mathcal{B}(fx_{n-1}, fx_n, fx_n)$. similarly it is easy to see that $\mathcal{B}(fx_{n+1}, fx_{n+2}, fx_{n+2}) \leq \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1})$

$$\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) \ll \alpha(k_1+k_2+k_3)\mathcal{B}(fx_{n-1}, fx_n, fx_n),$$

and

$$\begin{aligned}\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) &\preccurlyeq \delta\mathcal{B}(fx_{n-1}, fx_n, fx_n)\\ &\preccurlyeq \cdots\\ &\preccurlyeq \delta^n\mathcal{B}(fx_0, fx_1, fx_1)\end{aligned}$$

where $\alpha(k_1+k_2+k_3)=\delta<1$. So we have

$$\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) \preccurlyeq \delta^n\mathcal{B}(fx_0, fx_1, fx_1) \to \theta \ \ as \ \ n \to \infty.$$

Next, we claim that the sequence $\{fx_n\}$ is a Cauchy sequence. Suppose that $\{fx_n\}$ is not a Cauchy sequence. Then there exists $\gamma \in E$ with $\theta \ll \gamma$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ satisfying:

(1) m_k is even and n_k is odd,
(2) $\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \succcurlyeq \gamma$, and
(3) m_k is the smallest even number such that the conditions (1), (2) hold.

Since $\lim_{n\to\infty}\mathcal{B}(fx_n, fx_{n+1}, fx_{n+1})=\theta$ and by (2), (3), we have that

$$\begin{aligned}\gamma &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k})\\ &\preccurlyeq \mathcal{B}(fx_{n_k}, f_{m_k-1}, f_{m_k-1}) + \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k})\\ &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{m_k-2}, fx_{m_k-2}) + \mathcal{B}(fx_{m_k-2}, fx_{m_k-1}, fx_{m_k-1})\\ &\quad +\mathcal{B}(fx_{m_k-1}, f_{m_k}, f_{m_k})\\ &\preccurlyeq \gamma + \mathcal{B}(fx_{m_k-2}, fx_{m_k-1}, fx_{m_k-1}) + \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}).\end{aligned}$$

Taking $\lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty}\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k})=\gamma.$$

Since

$$\begin{aligned}\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) &\preccurlyeq \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}) + \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k})\\ &\quad +\mathcal{B}(fx_{m_k}, fx_{m_k-1}, fx_{m_k-1}).\end{aligned}$$

Taking $\lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty}\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) \preccurlyeq \gamma. \tag{2.2}$$

On the other hand,

$$\begin{aligned}\gamma &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k})\\ &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{n_k-1}, fx_{n_k-1}) + \mathcal{B}(fx_{n_k-1}, fx_{m_k}, fx_{m_k})\\ &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{n_k-1}, fx_{n_k-1}) + \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1})\\ &\quad +\mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}).\end{aligned}$$

Taking $\lim_{k\to\infty}$, we also deduce

$$\gamma \preccurlyeq \lim_{k\to\infty}\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}). \tag{2.3}$$

By (2.2) and (2.3), we get

$$\lim_{k\to\infty}\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1})=\gamma.$$

And, by (2.1), we have that

$$\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \preccurlyeq \psi(L(x_{n_k}, x_{m_k}, x_{m_k}))$$

where

$$\begin{aligned}\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) &\preccurlyeq k_1[\mathcal{B}(gx_{n_k}, gx_{m_k}, gx_{m_k})\lozenge\mathcal{B}(gx_{n_k}, fx_{n_k}, fx_{n_k})]\\ &+ k_2[\mathcal{B}(gx_{n_k}, gx_{m_k}, gx_{m_k})\lozenge\mathcal{B}(gx_{m_k}, fx_{m_k}, fx_{m_k})]\\ &+ k_3[\mathcal{B}(gx_{n_k}, gx_{m_k}, gx_{m_k})\lozenge\mathcal{B}(gx_{m_k}, fx_{m_k}, fx_{m_k})].\end{aligned}$$

$$\begin{aligned}\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) &\preccurlyeq k_1[\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1})\lozenge\mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k})]\\ &+ k_2[\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1})\lozenge\mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k})]\\ &+ k_3[\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1})\lozenge\mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k})].\end{aligned}$$

$$\begin{aligned}\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) &\preccurlyeq \alpha k_1 \max\{\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}), \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k})\}\\ &+ \alpha k_2 \max\{\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}), \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k})\}\\ &+ \alpha k_3 \max\{\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}), \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k})\}.\end{aligned}$$

(I) If

$$\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}),$$

is maximum then taking $lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) = \gamma,$$

and

$$\gamma \preccurlyeq \lim_{k\to\infty} \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \ll \gamma,$$

a contradiction.

(II) If

$$\mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}),$$

or

$$\mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}),$$

is maximum then taking $lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}) = \theta,$$

$$\lim_{k\to\infty} \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}) = \theta,$$

and

$$\gamma \preccurlyeq \lim_{k\to\infty} \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \preccurlyeq \theta,$$

a contradiction.

Follow (I) and (II), we get the sequence $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and $g(X)$ is closed, there exist $\nu, \mu \in X$ such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = g(\mu) = \nu.$$

We shall show that μ is a coincidence point of f and g, that is, we claim that

$$\mathcal{B}(g\mu, f\mu, f\mu) = \theta.$$

If not, assume that $\mathcal{B}(g\mu, f\mu, f\mu) \neq \theta$, then by (2.1), we have

$$\begin{aligned}\mathcal{B}(g\mu, f\mu, f\mu) &\preccurlyeq \mathcal{B}(g\mu, fx_n, fx_n) + \mathcal{B}(fx_n, f\mu, f\mu) \\ &\preccurlyeq \mathcal{B}(g\mu, fx_n, fx_n) + k_1[\mathcal{B}(gx_n, g\mu, g\mu)\Diamond\mathcal{B}(gx_n, fx_n, fx_n)] \\ &\quad +k_2[\mathcal{B}(gx_n, g\mu, g\mu)\Diamond\mathcal{B}(g\mu, f\mu, f\mu)] \\ &\quad +k_3[\mathcal{B}(gx_n, g\mu, g\mu)\Diamond\mathcal{B}(g\mu, f\mu, f\mu)], \\ &\preccurlyeq \mathcal{B}(g\mu, fx_n, fx_n) \\ &\quad +\alpha(k_1+k_2+k_3)\max\{\mathcal{B}(gx_n, g\mu, g\mu), \mathcal{B}(gx_n, fx_n, fx_n), \mathcal{B}(g\mu, f\mu, f\mu)\},\end{aligned}$$

(III) If

$$\max\{\mathcal{B}(gx_n, g\mu, g\mu), \mathcal{B}(gx_n, fx_n, fx_n), \mathcal{B}(g\mu, f\mu, f\mu)\} = \mathcal{B}(gx_n, g\mu, g\mu),$$

then taking $\lim_{n\to\infty}$, we deduce

$$\lim_{n\to\infty}\mathcal{B}(gx_n, g\mu, g\mu) = \mathcal{B}(g\mu, g\mu, g\mu) = \theta,$$

and

$$\begin{aligned}\mathcal{B}(g\mu, f\mu, f\mu) &= \lim_{n\to\infty}\mathcal{B}(g\mu, fx_n, fx_n) + \alpha(k_1+k_2+k_3)\lim_{n\to\infty}\mathcal{B}(gx_n, g\mu, g\mu) \\ &\preccurlyeq \theta,\end{aligned}$$

a contradiction.

(IV) If

$$\max\{\mathcal{B}(gx_n, g\mu, g\mu), \mathcal{B}(gx_n, fx_n, fx_n), \mathcal{B}(g\mu, f\mu, f\mu)\} = \mathcal{B}(gx_n, fx_n, fx_n),$$

then taking $\lim_{n\to\infty}$, we deduce

$$\lim_{n\to\infty}\mathcal{B}(gx_n, fx_n, fx_n) = \mathcal{B}(g\mu, g\mu, g\mu) = \theta,$$

and

$$\mathcal{B}(g\mu, f\mu, f\mu) = \lim_{n\to\infty}\mathcal{B}(g\mu, fx_n, fx_n) + \alpha(k_1+k_2+k_3)\lim_{n\to\infty}\mathcal{B}(gx_n, fx_n, fx_n) \preccurlyeq \theta,$$

a contradiction.

(V) If

$$\max\{\mathcal{B}(gx_n, g\mu, g\mu), \mathcal{B}(gx_n, fx_n, fx_n), \mathcal{B}(g\mu, f\mu, f\mu)\} = \mathcal{B}(g\mu, f\mu, f\mu),$$

then

$$\mathcal{B}(g\mu, f\mu, f\mu) = \alpha(k_1+k_2+k_3)\mathcal{B}(g\mu, f\mu, f\mu) \ll \mathcal{B}(g\mu, f\mu, f\mu),$$

a contradiction.

Follow (III)-(V), we obtain that $\mathcal{B}(g\mu, f\mu, f\mu) = \theta$, that is, $g\mu = f\mu = \nu$, and so μ is a coincidence point of f and g.

Suppose that f and g commute at μ. Then

$$f\nu = fg\mu = gf\mu = g\nu.$$

Later, we claim that $\mathcal{B}(f\mu, f\nu, f\nu) = \theta$. By (2.1), we have

$$\begin{aligned}
\mathcal{B}(f\mu, f\nu, f\nu) &\preccurlyeq k_1[\mathcal{B}(g\mu, g\nu, g\nu)\Diamond\mathcal{B}(g\mu, f\mu, f\mu)] \\
&+ k_2[\mathcal{B}(g\mu, g\nu, g\nu)\Diamond\mathcal{B}(g\nu, f\nu, f\nu)] \\
&+ k_3[\mathcal{B}(g\mu, g\nu, g\nu)\Diamond\mathcal{B}(g\nu, f\nu, f\nu)] \\
&= \alpha(k_1 + k_2 + k_3)\max\{\mathcal{B}(f\mu, f\nu, f\nu), \mathcal{B}(f\mu, f\mu, f\mu), \mathcal{B}(f\nu, f\nu, f\nu)\} \\
&= \alpha(k_1 + k_2 + k_3)\max\{\mathcal{B}(f\mu, f\nu, f\nu), \theta\}.
\end{aligned}$$

Therefore, if

$$\mathcal{B}(f\mu, f\nu, f\nu) \preccurlyeq \alpha(k_1 + k_2 + k_3)\mathcal{B}(f\mu, f\nu, f\nu) \ll \mathcal{B}(f\mu, f\nu, f\nu),$$

then we get a contradition, which implies that $\mathcal{B}(f\mu, f\nu, f\nu) = \theta$, $\mathcal{B}(\nu, f\nu, f\nu) = \theta$, that is, $\nu = f\nu = g\nu$. So ν is a common fixed point of f and g.

Let $\overline{\nu}$ be another common fixed point of f and g. By (2.1),

$$\mathcal{B}(\overline{\nu}, \nu, \nu) = \mathcal{B}(f\overline{\nu}, f\nu, f\nu),$$

where

$$\begin{aligned}
\mathcal{B}(f\overline{\nu}, f\nu, f\nu) &\preccurlyeq k_1[\mathcal{B}(g\overline{\nu}, g\nu, g\nu)\Diamond\mathcal{B}(g\overline{\nu}f\overline{\nu}, f\overline{\nu})] \\
&+ k_2[\mathcal{B}(g\overline{\nu}, g\nu, g\nu)\Diamond\mathcal{B}(g\nu, f\nu, f\nu)] \\
&+ k_3[\mathcal{B}(g\overline{\nu}, g\nu, g\nu)\Diamond\mathcal{B}(g\nu, f\nu, f\nu)] \\
&= \alpha(k_1 + k_2 + k_3)\max\{\mathcal{B}(f\overline{\nu}, f\nu, f\nu), \mathcal{B}(f\overline{\nu}, f\overline{\nu}, f\overline{\nu}), \mathcal{B}(f\nu, f\nu, f\nu)\} \\
&= \alpha(k_1 + k_2 + k_3)\max\{\mathcal{B}(f\overline{\nu}, f\nu, f\nu), \theta\} \\
&= \alpha(k_1 + k_2 + k_3)\max\{\mathcal{B}(\overline{\nu}, \nu, \nu), \theta\} \\
&\preccurlyeq \mathcal{B}(\overline{\nu}, \nu, \nu).
\end{aligned}$$

Therefore, we also conclude that $\mathcal{B}(\overline{\nu}, \nu, \nu) = \theta$, that is $\overline{\nu} = \nu$. So we show that ν is the unique common fixed point of g and f. □

Corollary 2.2. *Let $(X, \mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and f, g be two self mappings of X such that $f(X) \subset g(X)$. Suppose that*

$$\begin{aligned}
\mathcal{B}(fx, fy, fz) &\preccurlyeq \alpha\max\{\mathcal{B}(gx, gy, gz), \mathcal{B}(gx, fx, fx), \\
&\qquad \mathcal{B}(gy, fy, fy), \mathcal{B}(gz, fz, fz)\}
\end{aligned} \tag{2.4}$$

where $\alpha \in [0, 1)$. If $g(X)$ is closed, then f and g have a coincidence point in X.

Moreover, if f and g commute at their coincidence points, then f and g have a unique common fixed point in X

Proof. It is sufficient if we take $a\Diamond b = \alpha\max\{a, b\}$ for $\alpha \in [0, 1)$ in Theorem 2.1 then we get the result. □

Corollary 2.3. *Let $(X, \mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and f be a self mapping of X. Suppose that $\Diamond$ satisfies $\alpha-$property with $\alpha > 0$ such that*

$$\begin{aligned}
\mathcal{B}(fx, fy, fz) &\preccurlyeq k_1[\mathcal{B}(x, y, z)\Diamond\mathcal{B}(x, fx, fx)] \\
&+ k_2[\mathcal{B}(x, y, z)\Diamond\mathcal{B}(y, fy, fy)] \\
&+ k_3[\mathcal{B}(x, y, z)\Diamond\mathcal{B}(z, fz, fz)],
\end{aligned} \tag{2.5}$$

where $k_1, k_2, k_3 > 0$ and $0 < \alpha(k_1 + k_2 + k_3) < 1$. Then f has a fixed point in X.

Proof. It is sufficient if we take $g = I$ (identity mapping) in Theorem 2.1 then we get the result. □

References

[1] C.D. Aliprantis and R. Tourky, *Cones and Duality, in: Graduate studies in Mathematics*, Amer. Math. Soc. **84** (2007). 215–240.

[2] I. Beg, M. Abbas and T. Nazir, *Generalized cone metric spaces*, J. Nonlinear Sci. Appl. **3** (2010), no. 1, 23–31.

[3] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **322** (2007), 1468–1476.

[4] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

[5] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear and Convex Anal. **7** (2006), no. 2, 289–297.

[6] C. Chen and P. Tsai, *Fixed point and common fixed point theorems for the Meir-Keeler type functions in cone ball-metric spaces*, Ann. Funct. Anal. **3** (2012), no. 2, 155–169.

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