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Fractional Vector Cross Product in Euclidean 3-Space

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Abstract. In this research a new definition of fractional vector cross product of two vectors in Euclidean 3-space is presented. Using this new definition the formulae for Euclidean norm, fractional triple vector cross product, curl and divergence of fractional vector cross product of an electromagnetic vector field is discussed. In particular cases, all the properties satisfy definition of standard vector cross product for standard orthogonal basis of R^3. The new definition has application in radiation characteristic in micro-strip antenna and can be applied in other branches of mathematics and physics.

1. Introduction

In 1967, Crowe [1] presented vector analysis. In 2013, Das [2] defined fractional vector cross product and derived fractional curl. He further defined the formulations that were useful in applications in electrodynamics, elastodynamics, fluid flow etc. Mishra et. al. [4] gave a simulation approach and applied fractional vector cross product in radiation characteristic in micro-strip antenna. Fractional vector cross product was defined by Tripathi and Kim [5] in 2022. In this paper we present new definition of fractional vector cross product in Euclidean 3-space different than in [5] and discussed its properties further with respect to the new definition.

2. Fractional Vector Cross Product in Euclidean 3 - Space

Further to the study of fractional vector cross product in [5], we defined an alternative definition below:

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Definition: Let R^3 be the Euclidean 3-space equipped with standard inner product $\langle .,. \rangle$. Let (e_1, e_2, e_3) be standard orthonormal basis of R^3 and $\beta \in [0,1]$ a real number. Then, for vectors $a = a_1e_1 + a_2e_2 + a_3e_3$, $b = b_1e_1 + b_2e_2 + b_3e_3$ in R^3, the $\beta - fractional\ vector\ cross\ product$ is defined by

$$\begin{aligned} a \times^{\beta} b = &\left\{(a_2b_3 - a_3b_2)\sin\left(\frac{\beta\pi}{2}\right) + (b_2 + b_3)a_1\cos\left(\frac{\beta\pi}{2}\right)\right\}e_1 \\ &+\left\{(a_3b_1 - a_1b_3)\sin\left(\frac{\beta\pi}{2}\right) + (b_3 + b_1)a_2\cos\left(\frac{\beta\pi}{2}\right)\right\}e_2 \\ &+\left\{(a_1b_2 - a_2b_1)\sin\left(\frac{\beta\pi}{2}\right) + (b_1 + b_2)a_3\cos\left(\frac{\beta\pi}{2}\right)\right\}e_3 \end{aligned} \tag{2.1}$$

From eqn (2.1) we have,

$$e_i \times^{\beta} e_j = \cos\left(\frac{\beta\pi}{2}\right)e_i + \sin\left(\frac{\beta\pi}{2}\right)e_k \tag{2.2}$$

$$e_j \times^{\beta} e_i = \cos\left(\frac{\beta\pi}{2}\right)e_j - \sin\left(\frac{\beta\pi}{2}\right)e_k \tag{2.3}$$

$$e_l \times^{\beta} e_l = 0 \ for \ l = \{1,2,3\} \tag{2.4}$$

where (i, j, k) is a cyclic permutation of (1, 2, 3). The equations (2.2), (2.3) and (2.4) are similar to that in [4] and [5].

Matrix representation

By (2.2), (2.3), (2.4) and linearity we can write equation (2.1) as

$$\begin{aligned} a \times^{\beta} b = &\sin\left(\frac{\beta\pi}{2}\right)\left\{(a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3\right\} \\ &+\cos\left(\frac{\beta\pi}{2}\right)\left\{(b_2 + b_3)a_1e_1 + (b_3 + b_1)a_2e_2 + (b_1 + b_2)a_3e_3\right\} \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} a \times^{\beta} b = &\sin\left(\frac{\beta\pi}{2}\right)\left\{(a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3\right\} \\ &+\cos\left(\frac{\beta\pi}{2}\right)(b_1 + b_2 + b_3)a - \cos\left(\frac{\beta\pi}{2}\right)(a_1b_1e_1 + a_2b_2e_2 + a_3b_3e_3) \end{aligned} \tag{2.6}$$

From [3] we have

$$a \times^{\beta} b = \begin{pmatrix} 0 & a_1\cos\left(\frac{\beta\pi}{2}\right) - a_3\sin\left(\frac{\beta\pi}{2}\right) & a_1\cos\left(\frac{\beta\pi}{2}\right) + a_2\sin\left(\frac{\beta\pi}{2}\right) \\ a_2\cos\left(\frac{\beta\pi}{2}\right) + a_3\sin\left(\frac{\beta\pi}{2}\right) & 0 & a_2\cos\left(\frac{\beta\pi}{2}\right) - a_1\sin\left(\frac{\beta\pi}{2}\right) \\ a_3\cos\left(\frac{\beta\pi}{2}\right) - a_2\sin\left(\frac{\beta\pi}{2}\right) & a_3\cos\left(\frac{\beta\pi}{2}\right) + a_1\sin\left(\frac{\beta\pi}{2}\right) & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2.7}$$

or

$$a \times^{\beta} b = \begin{pmatrix} (b_2+b_3)\cos\left(\frac{\beta\pi}{2}\right) & b_3\sin\left(\frac{\beta\pi}{2}\right) & -b_2\sin\left(\frac{\beta\pi}{2}\right) \\ -b_3\sin\left(\frac{\beta\pi}{2}\right) & (b_3+b_1)\cos\left(\frac{\beta\pi}{2}\right) & b_1\sin\left(\frac{\beta\pi}{2}\right) \\ b_2\sin\left(\frac{\beta\pi}{2}\right) & -b_1\sin\left(\frac{\beta\pi}{2}\right) & (b_1+b_2)\cos\left(\frac{\beta\pi}{2}\right) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{2.8}$$

Thus considering $a \in R^3$ as a fixed column vector in R^3, we see that $a \times^{\beta} b : R^3 \to R^3$ is a linear function as given in eqn (2.7) and (2.8).

Particular case:

Case 1: The 0-fractional vector cross product is defined by:

$$a \times^0 b = (b_2+b_3)a_1e_1 + (b_3+b_1)a_2e_2 + (b_1+b_2)a_3e_3$$

By eqn (2.7) we have, $a \times^0 b = \begin{pmatrix} 0 & a_1 & a_1 \\ a_2 & 0 & a_2 \\ a_3 & a_3 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Example 1: Using eqn (2.2) we have

$$e_i \times^0 e_j = e_i, e_j \times^0 e_i = e_j, e_l \times^0 e_l = 0.$$

Case 2: The 1-fractional vector cross product is defined by:

$$a \times b = (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3.$$

By eqn (2.7) we have

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Example 2: Using eqn (2.2) we have

$$e_i \times e_j = e_k, e_j \times e_i = -e_k, e_l \times e_l = 0.$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Thus, 1-fractional vector cross product is simply the standard vector cross product.

3. Properties of Fractional Vector Cross product

Theorem 3.1. *The β - fractional vector cross product satisfies*

$$a \times^{\beta} (b+c) = a \times^{\beta} b + a \times^{\beta} c \tag{3.1}$$

$$(a+b) \times^{\beta} c = a \times^{\beta} c + b \times^{\beta} c \tag{3.2}$$

$$(\lambda a + \mu b) \times^{\beta} c = \lambda a \times^{\beta} c + \mu b \times^{\beta} c \tag{3.3}$$

Proof. From eqn (2.5), equations (3.1), (3.2) and (3.3) can be proved.

□

Particular case:

Case 1: The 0-fractional vector cross product satisfies following properties:

$a \times^{0} (b+c) = a \times^{0} b + a \times^{0} c$

$(a+b) \times^{0} c = a \times^{0} c + b \times^{0} c$

$(\lambda a + \mu b) \times^{0} c = \lambda a \times^{0} c + \mu b \times^{0} c$

Case 2: The 1-fractional vector cross product behaves as standard cross product.

$a \times (b+c) = a \times b + a \times c$

$(a+b) \times c = a \times c + b \times c$

$(\lambda a + \mu b) \times c = \lambda a \times c + \mu b \times c$

Theorem 3.2. *The β - fractional vector cross product satisfies*

$$\langle a, a \times^{\beta} b \rangle = \cos\left(\frac{\beta\pi}{2}\right)\left\{a_1^2(b_2+b_3) + a_2^2(b_1+b_3) + a_3^2(b_1+b_2)\right\} \tag{3.4}$$

$$\langle a, a \times^{\beta} b \rangle = \cos\left(\frac{\beta\pi}{2}\right)(b_1+b_2+b_3)\|a\|^2 - \cos\left(\frac{\beta\pi}{2}\right)(a_1^2 b_1 + a_2^2 b_2 + a_3^2 b_3) \tag{3.5}$$

Proof. From eqns (2.5) and (2.6), equations (3.4) and (3.5) can be proved.

□

Particular case:

Case 1: The 0-fractional vector cross product satisfies following properties:

$\langle a, a \times^0 b\rangle = \{a_1^2(b_2 + b_3) + a_2^2(b_1 + b_3) + a_3^2(b_1 + b_2)\}$

$\langle a, a \times^0 b\rangle = (b_1 + b_2 + b_3)\|a\|^2 - (a_1^2 b_1 + a_2^2 b_2 + a_3^2 b_3)$

Case 2: The 1-fractional vector cross product satisfies following properties:

$\langle a, a \times b\rangle = 0$

Example 3: Using eqn (2.2) we have

$\langle e_i, e_i \times^\beta e_j\rangle = \cos\left(\frac{\beta\pi}{2}\right)$

For 0-fractional vector cross product $\langle e_i, e_i \times^0 e_j\rangle = 1$

For 1-fractional vector product $\langle e_i, e_i \times e_j\rangle = 0$.

Theorem 3.3. *The β - fractional vector cross product satisfies*

$$\langle b, a \times^\beta b\rangle = \cos\left(\frac{\beta\pi}{2}\right)\{a_1 b_1(b_2 + b_3) + a_2 b_2(b_1 + b_3) + a_3 b_3(b_1 + b_2)\} \tag{3.6}$$

$$\langle b, a \times^\beta b\rangle = \cos\left(\frac{\beta\pi}{2}\right)(b_1 + b_2 + b_3)\langle a, b\rangle - \cos\left(\frac{\beta\pi}{2}\right)(a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2) \tag{3.7}$$

Proof. From eqns (2.5) and (2.6), equations (3.6) and (3.7) can be proved. □

Particular case:

Case 1: The 0-fractional vector cross product satisfies following properties:

$\langle b, a \times^0 b\rangle = \{a_1 b_1(b_2 + b_3) + a_2 b_2(b_1 + b_3) + a_3 b_3(b_1 + b_2)\}$

$\langle a, a \times^0 b\rangle = (b_1 + b_2 + b_3)\langle a, b\rangle - (a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2)$

Case 2: The 1-fractional vector cross product satisfies following properties:

$$\langle a, a \times b \rangle = 0$$

Example 4: Using eqn (2.2) we have

$$\langle e_j, e_i \times^\beta e_j \rangle = 0$$

Theorem 3.4. *The β - fractional vector cross product satisfies*

$$\begin{aligned}\langle a+b, a \times^\beta b \rangle &= \cos\big(\frac{\beta\pi}{2}\big)\big\{(a_1+b_1)(b_2+b_3)a_1 \\ &+(a_2+b_2)(b_1+b_3)a_2+(a_3+b_3)(b_1+b_2)a_3\big\}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\langle a+b, b \times^\beta a \rangle &= \cos\big(\frac{\beta\pi}{2}\big)\big\{(a_1+b_1)(a_2+a_3)b_1 \\ &+(a_2+b_2)(a_1+a_3)b_2+(a_3+b_3)(a_1+a_2)b_3\big\}\end{aligned} \tag{3.9}$$

$$\begin{aligned}\langle a+b, a \times^\beta b \rangle - \langle a+b, b \times^\beta a \rangle = \cos\big(\frac{\beta\pi}{2}\big)\big\{ &\begin{vmatrix} a_2+b_2 & a_3+b_3 & a_1+b_1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ - &\begin{vmatrix} a_3+b_3 & a_1+b_1 & a_2+b_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\big\}\end{aligned} \tag{3.10}$$

Proof. This theorem can be proved from eqn (3.6).

□

Particular case:

Case 1: The 0-fractional vector cross product satisfies following properties:

$$\langle a+b, a \times^0 b \rangle = \big\{(a_1+b_1)(b_2+b_3)+(a_2+b_2)(b_1+b_3)+(a_3+b_3)(b_1+b_2)\big\}$$

$$\langle a+b, b \times^0 a \rangle = \big\{(a_1+b_1)(a_2+a_3)b_1+(a_2+b_2)(a_1+a_3)b_2+(a_3+b_3)(a_1+a_2)b_3\big\}$$

$$\langle a+b, a \times^0 b \rangle - \langle a+b, b \times^0 a \rangle$$
$$= \big\{ \begin{vmatrix} a_2+b_2 & a_3+b_3 & a_1+b_1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_3+b_3 & a_1+b_1 & a_2+b_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \big\}$$

Case 2: The 1-fractional vector cross product satisfies following properties:

$\langle a+b, a\times b\rangle = 0, \langle a+b, b\times a\rangle = 0$

$\langle a+b, a\times b\rangle - \langle a+b, b\times a\rangle = 0$

Example 5: Using eqn (2) we have

$\langle e_i+e_j, e_i \times^{\beta} e_j\rangle - \langle e_i+e_j, e_j \times^{\beta} e_i\rangle = (e_i - e_j)\cos\left(\frac{\beta\pi}{2}\right)$

For $\beta = 0, \langle e_i+e_j, e_i \times^{0} e_j\rangle - \langle e_i+e_j, e_j \times^{0} e_i\rangle = e_i - e_j$

For $\beta = 1, \langle e_i+e_j, e_i \times e_j\rangle - \langle e_i+e_j, e_j \times e_i\rangle = 0$

Theorem 3.5. *The β - fractional vector cross product satisfies*

$$\begin{aligned}(a\times^{\beta} b) + (b\times^{\beta} a) &= \cos\left(\frac{\beta\pi}{2}\right)\Big\{(b_1+b_2+b_3)(a_1e_1+a_2e_2+a_3e_3)\\ +(a_1+a_2+a_3)(b_1e_1+b_2e_2+b_3e_3)\Big\} &- 2\cos\left(\frac{\beta\pi}{2}\right)(a_1b_1e_1+a_2b_2e_2+a_3b_3e_3)\end{aligned} \tag{3.11}$$

Proof. Using eqn (2.6) we can prove the theorem.

□

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$$\begin{aligned}(a\times^{0} b) + (b\times^{0} a) &= \Big\{(b_1+b_2+b_3)(a_1e_1+a_2e_2+a_3e_3)\\ +(a_1+a_2+a_3)(b_1e_1+b_2e_2+b_3e_3)\Big\} &- 2(a_1b_1e_1+a_2b_2e_2+a_3b_3e_3)\end{aligned} \tag{3.12}$$

Case 2: The 1-fractional vector cross product satisfies following property:

$$(a\times b) + (b\times a) = 0 \tag{3.13}$$

Example 6: Using eqn (2.2) and (2.3) we have

$(e_i \times^{\beta} e_j) + (e_j \times^{\beta} e_i) = \cos\left(\frac{\beta\pi}{2}\right)(e_i + e_j)$

For $\beta = 0,\ (e_i \times^{0} e_j) + (e_j \times^{0} e_i) = e_i + e_j$

For $\beta = 1$, $(e_i \times e_j) + (e_j \times e_i) = 0$

Theorem 3.6. *The β - fractional vector cross product satisfies*

$$\begin{aligned} \|a \times^{\beta} b\|^2 &= \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\} \\ &+\left\{2b_1b_2(a_1a_2 + a_3^2) + 2b_1b_3(a_1a_3 + a_2^2) + 2b_2b_3(a_2a_3 + a_1^2)\right\}\cos\left(\frac{\beta\pi}{2}\right)^2 \\ &+ \begin{vmatrix} a_1(b_3 + b_3) & a_2(b_1 + b_3) & a_3(b_1 + b_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \sin\left(\beta\pi\right) \end{aligned} \tag{3.14}$$

$$\begin{aligned} \|b \times^{\beta} a\|^2 &= \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\} \\ &+\left\{2a_1a_2(b_1b_2 + b_3^2) + 2a_1a_3(b_1b_3 + b_2^2) + 2a_2a_3(b_2b_3 + b_1^2)\right\}cos\left(\frac{\beta\pi}{2}\right)^2 \\ &+ \begin{vmatrix} b_1(a_3 + a_3) & b_2(a_1 + a_3) & b_3(a_1 + a_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \sin\left(\beta\pi\right) \end{aligned} \tag{3.15}$$

Proof. Using theorem 3.2 we can prove the theorem. □

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$$\|a \times^0 b\|^2 = \left\{\left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\} + 2b_1b_2(a_1a_2 + a_3^2) + 2b_1b_3(a_1a_3 + a_2^2) + 2b_2b_3(a_2a_3 + a_1^2)\right\}$$

$$\|b \times^0 a\|^2 = \left\{\left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\} + 2a_1a_2(b_1b_2 + b_3^2) + 2a_1a_3(b_1b_3 + b_2^2) + 2a_2a_3(b_2b_3 + b_1^2)\right\}$$

Case 2: The 1-fractional vector cross product satisfies following property:

$$\|a \times b\|^2 = \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\}, \|b \times a\|^2 = \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\}$$

Example 7: Using eqns (2.2) and (2.3) we have

$\|e_i \times^{\beta} e_j\|^2 = \cos\left(\frac{\beta\pi}{2}\right)^2, \|e_j \times^{\beta} e_i\|^2 = cos\left(\frac{\beta\pi}{2}\right)^2$

For $\beta = 0$, $\|e_i \times^0 e_j\|^2 = 1$, $\|e_j \times^0 e_i\|^2 = 1$

For $\beta = 1$, $\|e_i \times e_j\|^2 = 0$, $\|e_j \times e_i\|^2 = 0$

Theorem 3.7. *The β - fractional vector cross product satisfies*

$$\begin{aligned} \langle a \times^{\beta} b, c\rangle = \sin\left(\frac{\beta\pi}{2}\right)\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \cos\left(\frac{\beta\pi}{2}\right)(b_1 + b_2 + b_3)\langle a, c\rangle \\ - \cos\left(\frac{\beta\pi}{2}\right)(a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3) \end{aligned} \tag{3.16}$$

Proof. Using eqn (3.4) we can prove the theorem. □

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$\langle a \times^0 b, c\rangle = (b_1 + b_2 + b_3)\langle a, c\rangle - (a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3)$

Case 2: The 1-fractional vector cross product satisfies following property:

$\langle a \times b, c\rangle = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example 8: Using eqn (2.2) and (2.3) we have

$\langle e_i \times^{\beta} e_j, e_k\rangle = \sin\left(\frac{\beta\pi}{2}\right)$

For $\beta = 0$, $\langle e_i \times^0 e_j, e_k\rangle = 0$

For $\beta = 1$, $\langle e_i \times e_j, e_k\rangle = 1$

Theorem 3.8. *The β - fractional vector cross product satisfies*

$$
\begin{aligned}
(a \times^{\beta} b) \times^{\beta} c &= \sin\Big(\frac{\beta\pi}{2}\Big)^2 (\langle a, c\rangle b - \langle b, c\rangle a) \\
&+ \cos\Big(\frac{\beta\pi}{2}\Big)\Big\{ \sin\Big(\frac{\beta\pi}{2}(b_1 + b_2 + b_3)(a \times c) \\
- \sin\Big(\frac{\beta\pi}{2}\Big)(a_1 b_1 e_1 + a_2 b_2 e_2 + a_3 b_3 e_3) \times c &+ \sin\Big(\frac{\beta\pi}{2}\Big) \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} c \\
+ \cos\Big(\frac{\beta\pi}{2}\Big)\big((b_2 + b_3)a_1 + (b_1 + b_3)a_2 + (b_1 + b_2)a_3\big)c &- \sin\Big(\frac{\beta\pi}{2}\Big) \begin{vmatrix} c_1 e_1 & c_2 e_2 & c_3 e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
- \cos\Big(\frac{\beta\pi}{2}\Big)\big((b_2 + b_3)a_1 c_1 e_1 &+ (b_1 + b_3)a_2 c_2 e_2 + (b_1 + b_2)a_3 c_3 e_3\big)\Big\}
\end{aligned} \tag{3.17}
$$

Proof. For vectors $c = c_1 e_1 + c_2 e_2 + c_3 e_3$ and $d = d_1 e_1 + d_2 e_2 + d_3 e_3$ in R^3 in view of eqn (2.6), we have

$$
\begin{aligned}
d \times^{\beta} c &= \sin\Big(\frac{\beta\pi}{2}\Big)\big\{(d_2 c_3 - d_3 c_2)e_1 + (d_3 c_1 - d_1 c_3)e_2 + (d_1 c_2 - d_2 c_1)e_3\big\} \\
&+ \cos\Big(\frac{\beta\pi}{2}\Big)\big\{(c_2 + c_3)d_1 e_1 + (c_3 + c_1)d_2 e_2 + (c_1 + c_2)d_3 e_3\big\} \\
&= \sin\Big(\frac{\beta\pi}{2}\Big)\big\{(d_2 c_3 - d_3 c_2)e_1 + (d_3 c_1 - d_1 c_3)e_2 + (d_1 c_2 - d_2 c_1)e_3\big\} \\
&+ \cos\Big(\frac{\beta\pi}{2}\Big)(c_1 + c_2 + b_3)d - \cos\Big(\frac{\beta\pi}{2}\Big)(d_1 c_1 e_1 + d_2 c_2 e_2 + d_3 c_3 e_3)
\end{aligned} \tag{3.18}
$$

Using $d = a \times^{\beta} b$, then from eqn (2.1) we have,

$$d_1 = \big\{(a_2 b_3 - a_3 b_2)\sin\big(\tfrac{\beta\pi}{2}\big) + (b_2 + b_3)a_1 \cos\big(\tfrac{\beta\pi}{2}\big)\big\}e_1$$

$$d_2 = \big\{(a_3 b_1 - a_1 b_3)\sin\big(\tfrac{\beta\pi}{2}\big) + (b_3 + b_1)a_2 \cos\big(\tfrac{\beta\pi}{2}\big)\big\}e_2$$

$$d_3 = \big\{(a_1 b_2 - a_2 b_1)\sin\big(\tfrac{\beta\pi}{2}\big) + (b_1 + b_2)a_3 \cos\big(\tfrac{\beta\pi}{2}\big)\big\}e_3$$

Using above details with eqn (3.18), we can prove theorem 3.8. □

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$$
\begin{aligned}
&(a \times^{0} b) \times^{0} c = \big((b_2 + b_3)a_1 + (b_1 + b_3)a_2 + (b_1 + b_2)a_3\big)c \\
&- \big((b_2 + b_3)a_1 c_1 e_1 + (b_1 + b_3)a_2 c_2 e_2 + (b_1 + b_2)a_3 c_3 e_3\big)\big\}
\end{aligned}
$$

Case 2: The 1-fractional vector cross product satisfies following property:

$$(a \times b) \times c = \langle a, c\rangle b - \langle b, c\rangle a$$

Example 9: Using (2.2) we have

$$(e_i \times^{\beta} e_j) \times^{\beta} e_k = \cos\left(\tfrac{\beta\pi}{2}\right)\left(e_i \cos\left(\tfrac{\beta\pi}{2}\right) - \sin\left(\tfrac{\beta\pi}{2}\right) e_k\right)$$

For $\beta = 0$, $(e_i \times^0 e_j) \times^0 e_k = \left(e_i \cos\left(\tfrac{\beta\pi}{2}\right)\right.$

For $\beta = 1$, $(e_i \times e_j) \times e_k = 0$

4. Divergence and Curl of Fractional Vector Cross Product

4.1. **Curl of a Curl of Fractional Vector Cross Product.** Using eqn (2.5) we have,

$$a \times^{\beta} b = \sin\left(\frac{\beta\pi}{2}\right) \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \cos\left(\frac{\beta\pi}{2}\right) \begin{pmatrix} 0 & a_1 & a_1 \\ a_2 & 0 & a_2 \\ a_3 & a_3 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{4.1}$$

Alternatively

$$\begin{aligned} a \times^{\beta} b = \sin\left(\frac{\beta\pi}{2}\right) \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ + \cos\left(\frac{\beta\pi}{2}\right) \begin{pmatrix} (b_2 + b_3) & 0 & 0 \\ 0 & (b_3 + b_1) & 0 \\ 0 & 0 & (b_1 + b_2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \end{aligned} \tag{4.2}$$

From eqn (4.1) and [2] for a vector field $F = F_1 e_1 + F_2 e_2 + F_3 e_3$ the fractional curl is:

$$\nabla \times^{\beta} F = \sin\left(\frac{\beta\pi}{2}\right) \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial y} & \frac{\partial^{\beta}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \cos\left(\frac{\beta\pi}{2}\right) \begin{pmatrix} 0 & \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial x} \\ \frac{\partial^{\beta}}{\partial y} & 0 & \frac{\partial^{\beta}}{\partial y} \\ \frac{\partial^{\beta}}{\partial z} & \frac{\partial^{\beta}}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \tag{4.3}$$

Alternatively, from eqn (4.2) we have

$$a \times^{\beta} b = \sin\big(\frac{\beta\pi}{2}\big)\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial y} & \frac{\partial^{\beta}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \cos\big(\frac{\beta\pi}{2}\big)\begin{pmatrix} (F_2+F_3) & 0 & 0 \\ 0 & (F_3+F_1) & 0 \\ 0 & 0 & (F_1+F_2) \end{pmatrix}\begin{pmatrix} \frac{\partial^{\beta}}{\partial x} \\ \frac{\partial^{\beta}}{\partial y} \\ \frac{\partial^{\beta}}{\partial z} \end{pmatrix} \tag{4.4}$$

By theorem (3.8) we have,

$$\begin{aligned} a \times^{\beta} (b \times^{\beta} c) &= \sin\big(\frac{\beta\pi}{2}\big)^2\big(b\langle a, c\rangle - c\langle a, b\rangle\big) + \cos\big(\frac{\beta\pi}{2}\big)\Big\{ \sin\big(\frac{\beta\pi}{2}(c_1+c_2+c_3)(a \times b) \\ &- \sin\big(\frac{\beta\pi}{2}\big)(b_1c_1e_1 + b_2c_2e_2 + b_3c_3e_3) \times c + \sin\big(\frac{\beta\pi}{2}\big)\begin{vmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} c \\ &+ \cos\big(\frac{\beta\pi}{2}\big)a\big((c_2+c_3)b_1 + (c_1+c_3)b_2 + (c_1+c_2)b_3\big) - \sin\big(\frac{\beta\pi}{2}\big)\begin{vmatrix} a_1e_1 & a_2e_2 & a_3e_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &- \cos\big(\frac{\beta\pi}{2}\big)\big((c_2+c_3)a_1b_1e_1 + (c_1+c_3)a_2b_2e_2 + (c_1+c_2)a_3b_3e_3\big)\Big\} \end{aligned} \tag{4.5}$$

From (4.3) and (4.5) we have

$$\begin{aligned} \nabla \times^{\beta} (\nabla \times^{\beta} F) &= \sin\big(\frac{\beta\pi}{2}\big)^2\big(\nabla\langle \nabla, F\rangle - F\langle \nabla, \nabla\rangle\big) \\ &+ \cos\big(\frac{\beta\pi}{2}\big)\Big\{ \sin\big(\frac{\beta\pi}{2}\big)(\nabla \times \nabla)(F_1+F_2+F_3) \\ &- \sin\big(\frac{\beta\pi}{2}\big)\big(\frac{\partial^{\beta}}{\partial x}F_1e_1 + \frac{\partial^{\beta}}{\partial y}F_2e_2 + \frac{\partial^{\beta}}{\partial z}F_3e_3\big) \times F + \sin\big(\frac{\beta\pi}{2}\big)\begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial y} & \frac{\partial^{\beta}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} c \\ &+ \cos\big(\frac{\beta\pi}{2}\big)\nabla\big(\frac{\partial^{\beta}}{\partial x}(F_2+F_3) + \frac{\partial^{\beta}}{\partial y}(F_1+F_3) + \frac{\partial^{\beta}}{\partial z}(F_1+F_2)\big) - \sin\big(\frac{\beta\pi}{2}\big)\begin{vmatrix} \frac{\partial^{\beta}}{\partial x}e_1 & \frac{\partial^{\beta}}{\partial y}e_2 & \frac{\partial^{\beta}}{\partial z}e_3 \\ \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial y} & \frac{\partial^{\beta}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &- \cos\big(\frac{\beta\pi}{2}\big)\big((\frac{\partial^{\beta}}{\partial x})^2(F_2+F_3)e_1 + (\frac{\partial^{\beta}}{\partial y})^2(F_1+F_3)e_2 + (\frac{\partial^{\beta}}{\partial z})^2(F_1+F_2)e_3\big)\Big\} \end{aligned} \tag{4.6}$$

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$$\begin{aligned}\nabla \times^0 (\nabla \times^0 F) = \big\{\nabla\big(\frac{\partial^0}{\partial x}(F_2+F_3)+\frac{\partial^0}{\partial y}(F_1+F_3)+\frac{\partial^0}{\partial z}(F_1+F_2)\big)\\ -\big((\frac{\partial^0}{\partial x})^2(F_2+F_3)e_1+(\frac{\partial^0}{\partial y})^2(F_1+F_3)e_2+(\frac{\partial^0}{\partial z})^2(F_1+F_2)e_3\big)\big\}\end{aligned} \tag{4.7}$$

Case 2: The 1-fractional vector cross product satisfies following property:

$$\nabla \times (\nabla \times F) = \nabla\langle\nabla, F\rangle - F\langle\nabla,\nabla\rangle \tag{4.8}$$

4.2. **Divergence of a Curl of Fractional Vector Cross Product.** Using eqn (2.5) and (4.3) we have,

$$\begin{aligned}\nabla.(\nabla \times^\beta F) = (\frac{\partial^\beta}{\partial x}+\frac{\partial^\beta}{\partial y}+\frac{\partial^\beta}{\partial z})\sin\big(\frac{\beta\pi}{2}\big)\\ \big\{(\frac{\partial^\beta}{\partial y}F_3-\frac{\partial^\beta}{\partial z}F_2)e_1+(\frac{\partial^\beta}{\partial z}F_1-\frac{\partial^\beta}{\partial x}F_3)e_2+(\frac{\partial^\beta}{\partial x}F_2-\frac{\partial^\beta}{\partial y}F_1)e_3\big\}\\ +\cos\big(\frac{\beta\pi}{2}\big)\big\{\frac{\partial^\beta}{\partial x}(F_2+F_3)e_1+\frac{\partial^\beta}{\partial y}(F_3+F_1)e_2+\frac{\partial^\beta}{\partial z}(F_1+F_2)e_3\big\}\end{aligned} \tag{4.9}$$

$$\begin{aligned}\nabla.(\nabla \times^\beta F) = \cos\big(\frac{\beta\pi}{2}\big)\big\{(\frac{\partial^\beta}{\partial x})^2(F_2+F_3)e_1+(\frac{\partial^\beta}{\partial y})^2(F_3+F_1)e_2\\ +(\frac{\partial^\beta}{\partial z})^2(F_1+F_2)e_3\big\}\end{aligned} \tag{4.10}$$

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$$\nabla.(\nabla \times^0 F) = \big\{(\frac{\partial^0}{\partial x})^2(F_2+F_3)e_1+(\frac{\partial^0}{\partial y})^2(F_3+F_1)e_2+(\frac{\partial^0}{\partial z})^2(F_1+F_2)e_3\big\} \tag{4.11}$$

Case 2: The 1-fractional vector cross product satisfies following property:

$$\nabla.(\nabla \times F) = 0 \tag{4.12}$$

4.3. **Curl of a Divergence of Fractional Vector Cross Product.**

$$\nabla.F = \frac{\partial^\beta}{\partial x}F_1+\frac{\partial^\beta}{\partial y}F_2+\frac{\partial^\beta}{\partial z}F_3 \tag{4.13}$$

$$\nabla \times^\beta (\nabla.F) = \frac{\partial^\beta}{\partial x}F_1+\frac{\partial^\beta}{\partial y}F_2+\frac{\partial^\beta}{\partial z}F_3 \tag{4.14}$$

Using eqn (4.3) we have,

$$\nabla \times^{\beta} (\nabla.F) = \sin\left(\frac{\beta\pi}{2}\right)\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial y} & \frac{\partial^{\beta}}{\partial z} \\ \frac{\partial^{\beta}}{\partial x}F_1 & \frac{\partial^{\beta}}{\partial y}F_2 & \frac{\partial^{\beta}}{\partial z}F_3 \end{vmatrix} + \cos\left(\frac{\beta\pi}{2}\right)\begin{pmatrix} 0 & \frac{\partial^{\beta}}{\partial x} & \frac{\partial^{\beta}}{\partial x} \\ \frac{\partial^{\beta}}{\partial y} & 0 & \frac{\partial^{\beta}}{\partial y} \\ \frac{\partial^{\beta}}{\partial z} & \frac{\partial^{\beta}}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial^{\beta}}{\partial x}F_1 \\ \frac{\partial^{\beta}}{\partial y}F_2 \\ \frac{\partial^{\beta}}{\partial z}F_3 \end{pmatrix} \tag{4.15}$$

Particular case:

Case 1: The 0-fractional vector cross product satisfies following property:

$$\nabla \times^{0} (\nabla.F) = \begin{pmatrix} 0 & \frac{\partial^{0}}{\partial x} & \frac{\partial^{0}}{\partial x} \\ \frac{\partial^{0}}{\partial y} & 0 & \frac{\partial^{0}}{\partial y} \\ \frac{\partial^{0}}{\partial z} & \frac{\partial^{0}}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial^{0}}{\partial x}F_1 \\ \frac{\partial^{0}}{\partial y}F_2 \\ \frac{\partial^{0}}{\partial z}F_3 \end{pmatrix} = 0 \tag{4.16}$$

Case 2: The 1-fractional vector cross product satisfies following property:

$$\nabla \times (\nabla.F) = 0 \tag{4.17}$$

5. Conclusion

Fractional vector cross product is one of the important property in the study of fractional calculus. In this research we have defined a new fractional vector cross product named as β fractional vector cross product and further discussed properties of Euclidean norm, fractional triple vector cross product, curl and divergence of fractional vector cross product. This definition is useful for applications in electrodynamics, polarisation, impedance studies etc. where we can see real application of fractional solution.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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