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Order of Approximation by a New Univariate Kantorovich Type Operator

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Abstract. In order to approximate Lebesgue integrable functions on $[0, 1]$, a sequence of linear positive integral operators of Kantorovich type $L_\sigma^{<s_\sigma>} f(x)$ with a parameter s_σ is introduced. The estimates for rates of approximation for functions with a specific smoothness are proved using the appropriate modulus of continuity.

1. Introduction

As per the Weierstrass approximation theorem a continuous function on a compact domain is the limit of a uniform convergent sequences of polynomials. A number of proofs of Weierstrass theorem have been supplied by researchers over the time (see pp.10 [21], [22]). S. Bernstein provided a constructive proof of Weierstrass theorem by introducing a certain sequence of polynomials, which is given by

$$B_\sigma f(x) := \sum_{r=0}^{\sigma} f\left(\frac{r}{\sigma}\right) p_{\sigma,r}(x), \quad 0 \le x \le 1,$$

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where $f \in C[0,1]$ and

$$p_{\sigma,r}(x) = \begin{cases} 0 \quad \text{if} & r < 0 \\ \binom{\sigma}{r} x^r (1-x)^{\sigma-r} \quad \text{if} & 0 \le r \le \sigma \\ 0 \quad \text{if} & r > \sigma. \end{cases}$$

It is known that(pp.5–6, [22]) on $[0,1]$, the sequence $B_\sigma f(x)$ converges to $f(x)$ whenever f is continuous at x. Moreover, the convergence of $B_\sigma f(x)$ to $f(x)$ is uniform if f is continuous on $[0,1]$. Although, the operators $B_\sigma f(x)$ demonstrate interesting approximation and shape preservation properties (see, [16]), they are not suitable for the approximation of integrable functions. In [10], Kantorovich devised a mechanism to approximate Lebesgue integrable function on $[0,1]$ by introducing an integral variant $P_n^f(x)$, of the Bernstein polynomials $B_\sigma f(x)$. These operators are defined below.

Let $C[0,1]$ be the space of continuous functions normed by $\|f\| := \max\limits_{0\leqslant x \leqslant 1} |f(x)|$ and let $H_\sigma[0,1]$ be the subspace of all polynomials of degree not exceeding σ. Then, the Kantorovich operators P_σ^f : $C[0,1] \to H_\sigma[0,1], \sigma \in \mathbb{N}$ are given by,

$$P_\sigma^f(x) = \sum_{k=0}^{\sigma} \binom{\sigma}{k} x^k (1-x)^{\sigma-k} (\sigma+1) \int_{k/(\sigma+1)}^{(k+1)/(\sigma+1)} f(t)\, dt.$$

It was shown by G.G. Lorentz (see [12]) that if $f \in L_1[0,1]$ then

$$\int_0^1 |P_\sigma^f(x) - f(x)|\, dx \to 0 \ (\sigma \to \infty).$$

The operator $P_\sigma^f(x)$ have been studied by several authors for approximation properties. In [23], Özarslan and Duman studied a modified Kantorovich operator and explored certain approximation properties. Recently, similar modification of Kantorovich operator was introduced by Dhamija and Deo [3] wherein they proved that the error estimation of these operators is better than the classical ones. Gupta et al. [9] in their paper worked on inequalities for the Kantorovich operator in terms of moduli of continuity. In order to extend the results of Arab et al. [11]for Bernstein polynomials $B_\sigma f(x)$, a Kantorovich-Bernstein operator was introduced and studied by Gupta et al. in [9]. In this paper we introduce a Kantorovich type operator $L_\sigma^{<s_\sigma>} : L_B[0,1] \to H_\sigma[0,1]$ on the space of Lebesgue integrable function $L_B[0,1]$ as

$$L_\sigma^{<s_\sigma>} f(x) = \frac{\sigma + s_\sigma - 1}{s_\sigma - 1} \sum_{r=0}^{\sigma} \left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}} f(y)\, dy \right) p_{\sigma,r}(x), \tag{1.1}$$

where $x \in [0,1]$, and the sequence (s_σ) is such that $s_\sigma > 1$, $\lim\limits_{n\to\infty} \frac{1}{s_\sigma} = 0$ and $\frac{s_\sigma}{\sigma} \to 0$. Since the sequence (s_σ) can be defined arbitrarily, we are able to obtain different rates of approximation for different values of (s_σ). The results are supported by numerical examples graphically as well as by tabular values. For some other interesting results in this direction it is worth to mention the monographs [18] and [19]. A bivariate version of Bernstin-Durrmeyer operator of type (1.1) has been discussed in [7]. Recently, a modification of Kantorovich operator and Durrmeyer operator in [9] and [8] has been introduced by Gupta et al. with the aim to find better approximation rates. Some new operators of type and their approximation degrees have been established in [20] by Gupta and Tachev. It is worth mentioning the recent work in this direction in [1], [2], [6], [13] and [14]. Recently, Kantorovich variants of a number of operators written in terms of the Choquet integral with respect to a monotone and submodular set function, Gal in [5] proved quantitative approximation estimates.

It is customary to employ the notations $L(f;x)$ rather than the better one $Lf(x)$ to denote the value of the function Lf at x in approximation theory. By the notation $A \sim B$ we mean there exist a positive absolute constant C such that $CA \leq B$ or $A \leq C^{-1}B$. The relationship between Peetre $K-$functional and corresponding modulus of smoothness will be used to prove direct theorems. These quantities are defined as follows.

Let $f \in C[0,1]$ and $\varphi^2(y) = y(1-y), 0 \leq y \leq 1, \lambda \in [0,1]$. Then, the second order modulus of smoothness by Ditzian and Totik (see [17], pp.5-6) is defined by

$$\omega^2_{\varphi^\lambda}(f,\rho) := \sup_{0<h\leq\rho} \left| f(y + h\varphi^\lambda(y)) - 2f(y) + f(y - h\varphi^\lambda(y)) \right|,$$

whenever $y \pm h\varphi^\lambda(y) \in [0,1]$. The $K-$ functional corresponding to $\omega^2_{\varphi^\lambda}(f,\rho)$ is

$$K_{2,\varphi^\lambda}(f,\rho^2) := \inf_{\phi\in\Omega_2} \{\|f-\phi\| + \rho^2\|\varphi^{2\lambda}\phi''\|\},$$

$$\overline{K}_{2,\varphi^\lambda}(f,\rho^2) := \inf_{\phi\in\Omega_2} \{\|f-\phi\| + \rho\|\varphi^{2\lambda}\phi''\| + \rho^4\|\phi''\|\},$$

where

$$\Omega_2 = \{\phi \in C[0,1] : \phi' \in \mathrm{AC}_{loc}(0,1), \|\varphi^{2\lambda}\phi''\| < \infty\}.$$

Here, $\mathrm{AC}_{loc}(0,1)$ is the class of differentiable functions ϕ such that ϕ' is absolutely continuous in every compact interval $[a,b] \subset (0,1)$. It is known that (see [17, p. 11, 24]) $\omega^2_{\varphi^\lambda}(f,\rho) \sim K_{2,\varphi^\lambda}(f,\rho^2)$ and Furthermore, $\overline{K}_{2,\varphi^\lambda}(f,\rho^2) \sim K_{2,\varphi^\lambda}(f,\rho^2)$. Consequently, we have that

$$\omega^2_{\varphi^\lambda}(f,\sqrt{\rho}) \sim K_{2,\varphi^\lambda}(f,\rho) \sim \overline{K}_{2,\varphi^\lambda}(f,\rho). \tag{1.2}$$

2. Preliminaries

Since, $L^{<s_\sigma>}_\sigma f_0(x) = 1$ and $L^{<s_\sigma>}_\sigma$ are positive linear operators, $\|L^{<s_\sigma>}_\sigma(f,\cdot)\| \leqslant \|f\|$. Further, it will be shown that $\lim\limits_{\sigma\to\infty} L^{<s_\sigma>}_\sigma f(x) = f(x)$ at a point x of continuity of f on $[0,1]$.
By straight forward computations we have

Lemma 2.1. *Let $f_i(y) = y^i$, where $i \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Then*

(1) $L^{<s_\sigma>}_\sigma f_0(x) = 1$,
(2) $L^{<s_\sigma>}_\sigma f_1(x) = \frac{-1+s_\sigma+2\sigma x}{2(\sigma+s_\sigma-1)}$,
(3) $L^{<s_\sigma>}_\sigma f_2(x) = \frac{1-2s_\sigma+s_\sigma^2+3\sigma s_\sigma x-3\sigma x^2+3\sigma^2x^2}{3(\sigma+s_\sigma-1)^2}$.
(4) $L^{<s_\sigma>}_\sigma f_3(x) = \frac{6(\sigma-1)\sigma(s_\sigma+1)x^2+2\sigma(s_\sigma(2s_\sigma-1)+1)x+4(\sigma-2)(\sigma-1)\sigma x^3+(s_\sigma-1)^3}{4(\sigma+s_\sigma-1)^3}$.

(5) $L_{\sigma}^{<s_\sigma>} f_4(x) = \frac{(1-4s_\sigma+6s_\sigma^2-4s_\sigma^3+s_\sigma^4+5\sigma s_\sigma x-5\sigma s_\sigma^2 x+5\sigma s_\sigma^3 x-15\sigma x^2}{5(-1+\sigma+s_\sigma)^4}$
$+\frac{15\sigma^2x^2-10\sigma s_\sigma x^2+10\sigma^2 s_\sigma x^2-10\sigma s_\sigma^2 x^2+10\sigma^2 s_\sigma^2 x^2+40\sigma x^3-60\sigma^2 x^3+20\sigma^3 x^3}{5(-1+\sigma+s_\sigma)^4}$
$+\frac{20\sigma s_\sigma x^3-30\sigma^2 s_\sigma x^3+10\sigma^3 s_\sigma x^3-30\sigma x^4+55\sigma^2 x^4-30\sigma^3 x^4+5\sigma^4 x^4)}{5(-1+\sigma+s_\sigma)^4}$.

Corollary 2.1. *Let* $A_{m,\sigma}^{<s_\sigma>}(x) = L_{\sigma}^{<s_\sigma>}(f_1 - xf_0)^m(x)$, *where* $m \in \mathbb{N}_0$. *Then*

(1) $A_{0,\sigma}^{<s_\sigma>}(x) = 1$,
(2) $A_{1,\sigma}^{<s_\sigma>}(x) = -\frac{(s_\sigma-1)(2x-1)}{2(\sigma+s_\sigma-1)}$,
(3) $A_{2,\sigma}^{<s_\sigma>}(x) = \frac{-3\sigma x^2+3\sigma x+(s_\sigma-1)^2+3(s_\sigma-1)^2x^2-3(s_\sigma-1)^2x}{3(\sigma+s_\sigma-1)^2}$.
(4) $A_{3,\sigma}^{<s_\sigma>}(x) = -\frac{(2x-1)\left(-6\sigma(s_\sigma-1)x^2-4\sigma x^2+6\sigma(s_\sigma-1)x+4\sigma x+(s_\sigma-1)^3+2(s_\sigma-1)^3x^2-2(s_\sigma-1)^3x\right)}{4(\sigma+s_\sigma-1)^3}$,
(5) $A_{4,\sigma}^{<s_\sigma>}(x) = \frac{15\sigma^2(x-1)^2x^2-5\sigma(x-1)x\left(2(s_\sigma-1)+2(s_\sigma-1)^2(3(x-1)x+1)+8(s_\sigma-1)(x-1)x+6(x-1)x+1\right)}{5(\sigma+s_\sigma-1)^4}$
$+\frac{(s_\sigma-1)^4(5(x-1)x((x-1)x+1)+1)}{5(\sigma+s_\sigma-1)^4}$.

Remark 2.1. *By simple calculations, we have*

$$A_{2,\sigma}^{<s_\sigma>}(x) \sim \frac{s_\sigma}{\sigma + s_\sigma - 1}\varphi^2(x) \sim \frac{s_\sigma}{\sigma}\varphi^2(x),\ x \in [0,1]$$

where $\varphi^2(x) = x(1-x)$.

Lemma 2.2. *Let* $0 \le \lambda \le 1$, $0 < x < 1$ *and* $g \in AC_{loc}(0,1)$. *Then*

$$\left|\int_x^y g'(\xi)\, d\xi\right| \le 2\sqrt{2}(\varphi^{-\lambda}(x))|y-x|\|\varphi^{\lambda}g'\|.$$

Proof. In view of Hölder's inequality, we have

$$\begin{aligned}\int_z^y g'(\xi)\, d\xi &\le \|\varphi^{\lambda}g'\| \left|\int_z^y \frac{d\xi}{\varphi^{\lambda}(\xi)}\right| \\ &\le \|\varphi^{\lambda}g'\||y-z|^{1-\lambda}\left|\int_z^y \frac{d\xi}{\varphi(\xi)}\right|^{\lambda}.\end{aligned}$$

Now,

$$\begin{aligned}\left|\int_z^y \frac{d\xi}{\varphi(\xi)}\right| &= \left|\int_z^y \frac{1}{\sqrt{\xi(\xi-1)}}\right| \\ &\le \left|\int_z^y \left(\frac{1}{\sqrt{\xi}} + \frac{1}{\sqrt{1-\xi}}\right)\, d\xi\right| \\ &\le 2(|\sqrt{y}-\sqrt{z}| + |\sqrt{1-y}-\sqrt{1-z}|) \\ &\le 2|y-z|\left(\frac{1}{\sqrt{y}+\sqrt{z}} + \frac{1}{\sqrt{1-y}+\sqrt{1-z}}\right) \\ &\le 2|y-z|\left(\frac{1}{\sqrt{z}} + \frac{1}{\sqrt{1-z}}\right) \\ &\le \frac{2\sqrt{2}|y-z|}{\sqrt{z(1-z)}}.\end{aligned}$$

By an application of the inequality $|a+b|^p \leq |a|^p + |b|^p$, $0 \leq p \leq 1$, we obtain

$$\begin{aligned}\left|\int_z^y g'(\xi)d\xi\right| &\leq \|\varphi^\lambda g'\||y-z|\frac{(2\sqrt{2})^\lambda}{z^{\frac{\lambda}{2}}}\left|\frac{1}{\sqrt{1-z}}\right|^\lambda \\ &\leq \|\varphi^\lambda g'\||y-z|\frac{(2\sqrt{2})^\lambda}{z^{\frac{\lambda}{2}}}\left|\frac{1}{\sqrt{1-z}}\right|^\lambda \\ &\leq \|\varphi^\lambda g'\||y-z|(2\sqrt{2})^\lambda(z^{\frac{-\lambda}{2}}(1-z)^{\frac{-\lambda}{2}}) \\ &\leq 2\sqrt{2}(\varphi^{-\lambda}(z))|y-z|\|\varphi^\lambda g'\|.\end{aligned}$$

This completes the proof. □

3. Main Results

Theorem 3.1. *If $f \in C[0,1]$, then $L_\sigma^{<s_\sigma>}f(x)$ converges uniformly to $f(x)$ on the interval [0,1].*

Proof. By lemma2.1,

$$\lim_{\sigma\to\infty} L_\sigma^{<s_\sigma>}f_j(x) = x^j, \quad j = 0,1,2.$$

Further, the convergence is uniform. Therefore, by the Korovkin theorem

$$\lim_{\sigma\to\infty} \|L_\sigma^{<s_\sigma>}f - f\| = 0.$$

□

For a twice differentiable function in $[0,1]$, we have following asymptotic formula of Voronoskaja(see [15]) type.

Theorem 3.2. *Let f be such that $f''(x)$ exists at $x \in [0,1]$. Then*

$$\lim_{\sigma\to\infty} \frac{(\sigma+s_\sigma-1)}{(s_\sigma-1)}(L_\sigma^{<s_\sigma>}f(x) - f(x)) = \frac{d}{dx}\left(\frac{\varphi^2(x)f'(x)}{2}\right).$$

Proof. We have,

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + E(y;x)(y-x)^2, \tag{3.1}$$

where $E(y,x)$ is continuous on $[0,1]$ and $\lim_{y\to x} E(y,x) = 0$. Therefore, by an application of $L_\sigma^{<s_\sigma>}$ on (3.1) and corollary(2.1), we obtain

$$\begin{aligned}L_\sigma^{<s_\sigma>}f(x) &= L_\sigma^{<s_\sigma>}f_0(x)f(x) + f'(x)L_\sigma^{<s_\sigma>}(f_1 - xf_0)(x) + \frac{f''(x)}{2}L_\sigma^{<s_\sigma>}(f_1 - f_0x)^2(x) \\ &\quad + L_\sigma^{<s_\sigma>}(E(f_1;x)(f_1-x)^2))(x).\end{aligned}$$

So that

$$L_{\sigma}^{<s_\sigma>} f(x) - f(x) = f'(x)\left[-\frac{(s_\sigma-1)(2x-1)}{2(\sigma+s_\sigma-1)}\right] + \\ + \frac{f''(x)}{2}\left[\frac{(-1+s_\sigma)^2+3(n-(-1+s_\sigma)^2)x+3(-n+(-1+s_\sigma)^2)x^2}{3(\sigma+s_\sigma-1)^2}\right] \\ + L_{\sigma}^{<s_\sigma>}(E(y;x)(f_1-x)^2(x). \tag{3.2}$$

Now,

$$L_{\sigma}^{<s_\sigma>}(E(y;x)(f_1-x)^2(x) \leq \sqrt{L_{\sigma}^{<s_\sigma>}(E^2(y;x)(x)}\sqrt{L_{\sigma}^{<s_\sigma>}((f_1-x)^4(x)}$$

by Cauchy-Schwarz inequality. Also $(\sigma+s_\sigma-1)\sqrt{L_{\sigma}^{<s_\sigma>}((f_1-x)^4(x)} \to 0$ as $\sigma \to \infty$ by straight forward calculations and identity (5). Further, by continuity of $E(y,x)$ and $\lim_{y\to x} E(y,x) = 0$, it follows that $\lim_{\sigma\to\infty} L_{\sigma}^{<s_\sigma>}(E^2(y;x)(x) = 0$. Finally, by multiplying (3.2) with $2(\sigma+s_\sigma-1)$

$$\lim_{\sigma\to\infty} 2\frac{(\sigma+s_\sigma-1)}{s_\sigma-1}(L_{\sigma}^{<s_\sigma>} f(x) - f(x)) = \left((1-2x)f'(x) + x(1-x)f''(x)\right).$$

This completes the proof. □

Corollary 3.1. *If $f' \in AC_{loc}(0,1)$ and f'' is bounded, then*

$$\|L_{\sigma}^{<s_\sigma>}(f) - f\| \sim O\left(\frac{s_\sigma}{\sigma}\right) \sim O\left(\frac{s_\sigma}{\sigma}\right)\|f''\|.$$

Theorem 3.3. *Let $L_{\sigma}^{<s_\sigma>} f(x)$ be the operator defined by (1.1) and $f \in C[0,1]$. Then,*

$$|L_{\sigma}^{<s_\sigma>} f(x) - f(x)| \leq 2\omega(f, \Delta_{\sigma}^{<s_\sigma>}(x)),$$

where

$$\Delta_{\sigma}^{<s_\sigma>}(x) = \left(\frac{-3\sigma x^2+3\sigma x+(s_\sigma-1)^2+3(s_\sigma-1)^2x^2-3(s_\sigma-1)^2x}{3(\sigma+s_\sigma-1)^2}\right)^{1/2}.$$

Proof. Since $L_{\sigma}^{<s_\sigma>} f_0, x) = 1$, we have that

$$|L_{\sigma}^{<s_\sigma>} f(x) - f(x)| \leq \frac{\sigma+s_\sigma-1}{s_\sigma-1}\sum_{r=0}^{\sigma}\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}} |f(y)-f(x)|\,dy\right) p_{\sigma,r}(x).$$

By using the inequality

$$\omega(f, k\Delta) \leq (k+1)\omega(f,\Delta)$$

for positive real number k, we have that

$$
\begin{aligned}
&|L_\sigma^{<s_\sigma>} f(x) - f(x)| \\
&\leq \omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left[\frac{\sigma+s_\sigma-1}{s_\sigma-1}\sum_{r=0}^{\sigma}\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}(\Delta_\sigma^{<s_\sigma>}(x)|y-x|+1)\,dy\right)p_{\sigma,r}(x)\right] \\
&\leq \omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left(\frac{\sigma+s_\sigma-1}{s_\sigma-1}\sum_{r=0}^{\sigma}\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}(\Delta_\sigma^{<s_\sigma>}(x)|y-x|)\,dy\right)p_{\sigma,r}(x)\right) \\
&+\omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left(\frac{\sigma+s_\sigma-1}{s_\sigma-1}\sum_{r=0}^{\sigma}\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}\,dy\right)p_{\sigma,r}(x)\right).
\end{aligned}
$$

Applying Hölder's inequality, we obtain

$$
\begin{aligned}
&|L_\sigma^{<s_\sigma>} f(x) - f(x)| \\
&\leq \omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left[\frac{\sigma+s_\sigma-1}{s_\sigma-1}\sum_{k=0}^{\sigma}p_{\sigma,r}(x)\Delta_\sigma^{<s_\sigma>}(x)\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}(y-x)^2\,dy\right)^{\frac{1}{2}}\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}\,dy\right)^{\frac{1}{2}}+1\right] \\
&\leq \omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left[\frac{(\sigma+s_\sigma-1)}{s_\sigma-1}\Delta_\sigma^{<s_\sigma>}(x)\sum_{k=0}^{n}p_{\sigma,r}(x)\left(\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}(y-x)^2\,dy\right)^{\frac{1}{2}}\left(\frac{s_\sigma-1}{\sigma+s_\sigma-1}\right)^{\frac{1}{2}}+1\right] \\
&\leq \omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left[\Delta_\sigma^{<s_\sigma>}(x)\left(\frac{\sigma+s_\sigma-1}{s_\sigma-1}\sum_{k=0}^{n}p_{\sigma,r}(x)\int_{\frac{r}{\sigma+s_\sigma-1}}^{\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}}(y-x)^2\,dy\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}p_{\sigma,r}(x)\right)^{\frac{1}{2}}+1\right] \\
&\leq \omega\left(f; \frac{1}{\Delta_\sigma^{<s_\sigma>}(x)}\right)\left[\Delta_\sigma^{<s_\sigma>}(x)\sqrt{A_{2,\sigma}^{<s_\sigma>}(x)}+1\right].
\end{aligned}
$$

Finally, we have

$$|L_\sigma^{<s_\sigma>} f(x) - f(x)| \leq 2\omega(f, \Delta_\sigma^{<s_\sigma>}(x)).$$

by choosing

$$\Delta_\sigma^{<s_\sigma>}(x)^{-1} = \sqrt{A_{2,\sigma}^{<s_\sigma>}(x)} = \left(\frac{-3\sigma x^2+3\sigma x+(s_\sigma-1)^2+3(s_\sigma-1)^2x^2-3(s_\sigma-1)^2x}{3(\sigma+s_\sigma-1)^2}\right)^{1/2}.$$

□

Corollary 3.2. *Since, $A_{2,\sigma}^{<s_\sigma>}(x) \sim \left(\frac{s_\sigma}{\sigma}\right)$, there holds the order $L_\sigma^{<s_\sigma>} f(x) - f(x) = O\left(\sqrt{\frac{s_\sigma}{\sigma}}\right)$ for rate of approximation. Moreover, $|L_\sigma^{<s_\sigma>} f(x) - f(x)| \sim \sqrt{\frac{s_\sigma}{\sigma}}\varphi(x)$.*

Corollary 3.3. *We have global rate of approximation as*

$$\|L_\sigma^{<s_\sigma>}(f) - f\| \sim \omega\left(f, \sqrt{\frac{s_\sigma}{\sigma}}\right).$$

Theorem 3.4. *If $f \in C[0,1]$, $0 \leq \lambda \leq 1, 0 < x < 1$ then there exists an absolute constant C such that*

$$|L_\sigma^{<s_\sigma>}(f, x) - f(x)| \leq CK_{\varphi^\lambda}\left(f, \sqrt{\frac{1}{q_\sigma}}\varphi^{1-\lambda}(x)\right).$$

where $q_\sigma = \frac{\sigma+s_\sigma-1}{s_\sigma}$

Proof. By the definition of $K_{\varphi^\lambda}(f, y)$ for fixed n, x, λ, we choose $g = g_{n,x,\lambda} \in W_{\varphi^\lambda}$, where $W_{\varphi^\lambda} = \{g; g \in AC(0, 1), \|\varphi g'\| \leq \infty, \|g'\| \leq \infty\}$. Then,

$$\|f - g\| + \frac{\varphi^{1-\lambda}(x)}{\sqrt{q_\sigma}}\|\varphi^\lambda g'\| \leq 2K_{\varphi^\lambda}\left(f, \sqrt{\frac{1}{q_\sigma}}\varphi^{1-\lambda}(x)\right).$$

Since, $L_\sigma^{<s_\sigma>} f_0, x) = 1$, we can write

$$|L_\sigma^{<s_\sigma>}(f, x) - f(x)| \leq 2\|f - g\| + |L_\sigma^{<s_\sigma>}(g, x) - g(x)| \tag{3.3}$$

By the expansion,

$$g(y) = g(x) + \int_x^y g'(u)\, du$$

and in view of lemma2.2, we have

$$\begin{aligned}|L_\sigma^{<s_\sigma>}(g, x) - g(x)| &= \left|L_\sigma^{<s_\sigma>}\left(\int_x^y g'(u)du; x\right)\right| \\ &\leq 2\sqrt{2}(\varphi^{-\lambda}(x))L_\sigma^{<s_\sigma>}(|y - x|(x)\|\varphi^\lambda g'\| \\ &\leq 2\sqrt{2}\|\varphi^\lambda g'\|[\varphi^{-\lambda}(x)L_\sigma^{<s_\sigma>}(|y - x|(x).\end{aligned}$$

Now estimating $L_\sigma^{<s_\sigma>}|y - x|(x)$ by Schwarz inequality it follows that

$$\begin{aligned}L_\sigma^{<s_\sigma>}(|y - x|(x) &\leq \sqrt{A_{2,\sigma}^{<s_\sigma>}(x)}\sqrt{L_\sigma^{<s_\sigma>}(f_0(x)} \\ &\leq \frac{\varphi(x)}{\sqrt{q_\sigma}}.\end{aligned}$$

We have

$$|L_\sigma^{<s_\sigma>}(g, x) - g(x)| \leq C\|\varphi^\lambda g'\|\varphi^{-\lambda}(x)\frac{\varphi(x)}{\sqrt{q_\sigma}}.$$

Using estimate (3.3), and the equivalence (1.2) it follows that

$$\begin{aligned}|L_\sigma^{<s_\sigma>}(f, x) - f(x)| &\leq C\left(\|f - g\| + \frac{\|\varphi^\lambda g'\|\varphi^{1-\lambda}(x)}{\sqrt{q_\sigma}}\right) \\ &\sim K_{\varphi^\lambda}\left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{q_\sigma}}\right) \sim \omega_{\varphi^\lambda}\left(f, \frac{\varphi^{1-\lambda}(x)}{\sqrt{q_\sigma}}\right).\end{aligned}$$

□

Remark 3.1. *For $\lambda = 0$, we have the ordinary local approximation*

$$|L_\sigma^{<s_\sigma>}(f, x) - f(x)| \sim \omega\left(f, \sqrt{\frac{1}{q_\sigma}}\varphi(x)\right)$$

For $\lambda = 1$, the global approximation is given by,

$$||L_\sigma^{<s_\sigma>}(f) - f|| \sim \omega_\varphi\left(f, \sqrt{\frac{1}{q_\sigma}}\right)$$

which is obtained in corollary 3.3.

Theorem 3.5. *If $f \in C[0,1], x \in [0,1]$, then the following inequality holds.*

$$|L_\sigma^{<s_\sigma>} f(x) - f(x)| \leq 2\sqrt{A_{2,\sigma}(x)}\omega\left(f', \sqrt{A_{2,\sigma}(x)}\right) + |f'(x)||A_{1,\sigma}(x)|,$$

where $A_{1,\sigma}(x)$ and $A_{2,\sigma}(x)$ are defined in corollary(2.1).

Proof. For $y, x \in [0,1]$

$$f(y) - f(x) = f'(x)(y-x) + \int_x^y (f'(s) - f'(x))\, ds.$$

By operating $L_\sigma^{<s_\sigma>} f(x)$ on both sides of the above equation, we obtain

$$L_\sigma^{<s_\sigma>} f(x) - f(x) = L_\sigma^{<s_\sigma>}((y-x)(x)f'(x) + L_\sigma^{<s_\sigma>}\left(\int_x^y (f'(s) - f'(x))ds; x\right).$$

Since, for any $\epsilon > 0$ and each $s \in [0,1]$,

$$|f(s) - f(x)| \leq \omega(f', \epsilon)\left(\frac{|s-x|}{\epsilon} + 1\right) \text{ where } f \in C[0,1],$$

therefore

$$\left|\int_x^y (f'(s) - f'(x))ds\right| \leq \omega(f', \epsilon)\left(\frac{(y-x)^2}{\epsilon} + |y-x|\right).$$

Hence,

$$\begin{aligned}|L_\sigma^{<s_\sigma>} f(x) - f(x)| &\leq |f'(x)||L_\sigma^{<s_\sigma>}(y - x(x)| + \omega(f', \epsilon)\Big(\frac{1}{\epsilon}L_\sigma^{<s_\sigma>}((y-x)^2(x) \\ &\quad + L_\sigma^{<s_\sigma>}(|y-x|(x)\Big). \\ &\leq |f'(x)||A_{1,\sigma}(x)| + \omega(f', \epsilon)\left(\frac{A_{2,\sigma}(x)}{\epsilon} + \sqrt{A_{2,\sigma}(x)}\right).\end{aligned}$$

The result follows by setting $\epsilon = \sqrt{A_{2,\sigma}(x)}$. □

Theorem 3.6. *Suppose that $f \in C^2[0,1], x \in [0,1]$. Then*

$$\left|L_\sigma^{<s_\sigma>} f(x) - f(x) - f'(x)A_{1,\sigma}(x) - \frac{1}{2}A_{2,\sigma}(x)\right| \sim \frac{\varphi^2(x)}{\sigma + s_\sigma - 1}\omega_\varphi\left(f'', \sqrt{\frac{1}{\sigma + s_\sigma - 1}}\right).$$

Proof. By Taylor's expansion, we have the following equality satisfied for $f \in C[0,1]$

$$\big|f(y) - f(x) - (y-x)f'(x)\big| \leq \left|\int_x^y (y-\xi)f''(s)\, d\xi\right|.$$

Hence,

$$f(y) - f(x) - (y-x)f'(x) - \frac{f''(x)}{2}(y-x)^2 \leq \int_x^y (y-s)[f''(\xi) - f''(x)]\, d\xi.$$

Application of the operator $L_\sigma^{<s_\sigma>}$ on both the sides of the above equation yields

$$\begin{aligned}&\left|L_\sigma^{<s_\sigma>} f(x) - f(x) - f'(x)L_\sigma^{<s_\sigma>}(y - x(x) - \frac{f''(x)}{2}L_\sigma^{<s_\sigma>}((y-x)^2(x)\right| \\ &\leq L_\sigma^{<s_\sigma>}\left(\left|\int_x^y |y-\xi|[f''(\xi) - f''(x)]\, d\xi\right|; x\right).\end{aligned}$$

Using the estimation by [[4],p.337], we get

$$\left|\int_x^y |y-\xi|[f''(\xi)-f''(x)]\,d\xi\right| \le 2\|f''-g\|(y-x)^2 + 2\|\varphi g'\|\varphi^{-1}(x)|y-x|^3$$

for $x \in [0,1]$, $g \in \Omega_1$, where the class $\Omega_1 := \{\phi \in C[0,1] : \phi \in \mathrm{AC}_{loc}(0,1), \|\varphi\phi'\| < \infty\}$. Now, we have $L_\sigma^{<s_\sigma>}(y-x)^2(x) \sim \frac{\varphi^2(x)}{\sigma+s_\sigma-1}$ and $L_\sigma^{<s_\sigma>}(y-x)^4(x) \sim \frac{\varphi^4(x)}{(\sigma+s_\sigma-1)^2}$. Therefore, applying Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\left|L_\sigma^{<s_\sigma>}f(x) - f(x) - f'(x)A_{1,\sigma}(x) - \frac{1}{2}A_{2,\sigma}(x)\right| \\
&\le 2\|f''-g\|L_\sigma^{<s_\sigma>}((y-x)^2(x) + 2\|\varphi g'\|\varphi^{-1}(x)L_\sigma^{<s_\sigma>}(|y-x|^3(x) \\
&\le C\frac{\varphi^2(x)}{\sigma+s_\sigma-1}\|f''-g\| + 2\|\varphi g'\|\varphi^{-1}(x)\{L_\sigma^{<s_\sigma>}((y-x)^2(x)\}^{\frac{1}{2}} \times \{L_\sigma^{<s_\sigma>}((y-x)^4(x)\}^{\frac{1}{2}} \\
&\le C\frac{\varphi^2(x)}{\sigma+s_\sigma-1}\|f''-g\| + \frac{\varphi^2(x)}{\sigma+s_\sigma-1\sqrt{\sigma+s_\sigma-1}}\|\varphi g'\| \\
&\le C\left(\frac{\varphi^2(x)}{\sigma+s_\sigma-1}\|f''-g\| + \sqrt{\frac{1}{\sigma+s_\sigma-1}}\|\varphi g'\|\right) \\
&\le C\frac{\varphi^2(x)}{\sigma+s_\sigma-1}\omega_\varphi\left(f'', \sqrt{\frac{1}{\sigma+s_\sigma-1}}\right).
\end{aligned}$$

This completes the proof. □

Our last result is a Grüss type upper estimate for the operator $L_\sigma^{<s_\sigma>}f(x)$.

Theorem 3.7. *Let $\mathcal{M}, \mathcal{N} \in C[0,1]$ be such that $\mathcal{M}'', \mathcal{N}''$ exist in $[0,1]$. Then we have,*

$$\lim_{\sigma\to\infty}\frac{q_\sigma s_\sigma}{s_\sigma-1}\{L_\sigma^{<s_\sigma>}(\mathcal{M},\mathcal{N}(x) - \mathcal{M}(x)\mathcal{N}(x)\} = (1-2x)\mathcal{M}'(x)\mathcal{N}'(x).$$

Proof.

$$\begin{aligned}
&L_\sigma^{<s_\sigma>}(\mathcal{M};\mathcal{N}(x) \\
&= L_\sigma^{<s_\sigma>}(\mathcal{M}\mathcal{N})(x) - \mathcal{M}(x)\mathcal{N}(x) - (\mathcal{M}(x)\mathcal{N}(x))'L_\sigma^{<s_\sigma>}(f_1(t)-x)(x) \\
&- \frac{(\mathcal{M}(x)\mathcal{N}(x))''}{2!}L_\sigma^{<s_\sigma>}((f_1(t)-x)^2)(x) - \mathcal{N}(x)\{L_\sigma^{<s_\sigma>}(\mathcal{M}(x) - \mathcal{M}(x) \\
&- \mathcal{M}'(x)L_\sigma^{<s_\sigma>}(f_1(t)-x)(x) - \frac{\mathcal{M}''(x)}{2!}L_\sigma^{<s_\sigma>}((f_1(t)-x)^2)(x)\} - L_\sigma^{<s_\sigma>}(\mathcal{M})(x) \\
&\{L_\sigma^{<s_\sigma>}(\mathcal{N})(x) - \mathcal{N}(x) - \mathcal{N}'(x)L_\sigma^{<s_\sigma>}(f_1(t)-x)(x) - \frac{1}{2}L_\sigma^{<s_\sigma>}((f_1(t)-x)^2)(x)\}\mathcal{N}''(x) \\
&+ \frac{1}{2!}L_\sigma^{<s_\sigma>}((f_1(t)-x)^2)(x)\{\mathcal{M}(x)\mathcal{N}''(x) + 2\mathcal{M}'(x)\mathcal{N}'(x) - \mathcal{N}''(x)L_\sigma^{<s_\sigma>}(\mathcal{M})(x)\} \\
&+ L_\sigma^{<s_\sigma>}(f_1(t)-x)(x)\{\mathcal{M}(x)\mathcal{N}'(x) - \mathcal{N}'(x)L_\sigma^{<s_\sigma>}(\mathcal{M})(x)\}.
\end{aligned}$$

By uniform convergence of the operator $L_\sigma^{<s_\sigma>}\mathcal{M})(x)$ to the function $\mathcal{M}(x)$.

$$
\begin{aligned}
&(q_\sigma s_\sigma) L_\sigma^{<s_\sigma>}(\mathcal{M},\mathcal{N})(x)\\
&= (q_\sigma s_\sigma)\{L_\sigma^{<s_\sigma>}(\mathcal{M}\mathcal{N})(x) - L_\sigma^{<s_\sigma>}(\mathcal{M})(x) L_\sigma^{<s_\sigma>}(\mathcal{N})(x)\}\\
&= (q_\sigma s_\sigma)\mathcal{M}'(x)\mathcal{N}'(x) L_\sigma^{<s_\sigma>}((f_1(t)-x)^2)(x) + \frac{(\sigma + s_\sigma - 1)}{2!}\mathcal{N}''(x)\{\mathcal{M}(x) - L_\sigma^{<s_\sigma>}(\mathcal{M})(x)\}\\
&\times L_\sigma^{<s_\sigma>}((f_1(t)-x)^2)(x) + (\sigma + s_\sigma - 1)\mathcal{N}'(x)\{\mathcal{M}(x) - L_\sigma^{<s_\sigma>}(\mathcal{M})(x)\} L_\sigma^{<s_\sigma>}(f_1(t)-x)(x).
\end{aligned}
$$

The required result now follows by applying limit $\sigma \to \infty$ on both sides and from [3.2]. □

4. Numerical Verification and Applications

In our first example, we choose the function $g(x) := \frac{\sin(4x)}{\exp(2x)+1}, x \in [0,1]$ to be applied by the mappings $L_\sigma^{<s_\sigma>} g(x)$ for $\sigma = 25$ and $s_\sigma = 2, 5, 20$. First, we compare the function $g(x)$ with $L_\sigma^{<s_\sigma>} g(x)$ with respect to s_σ for a fixed degree, $\sigma = 25$ of the polynomials.

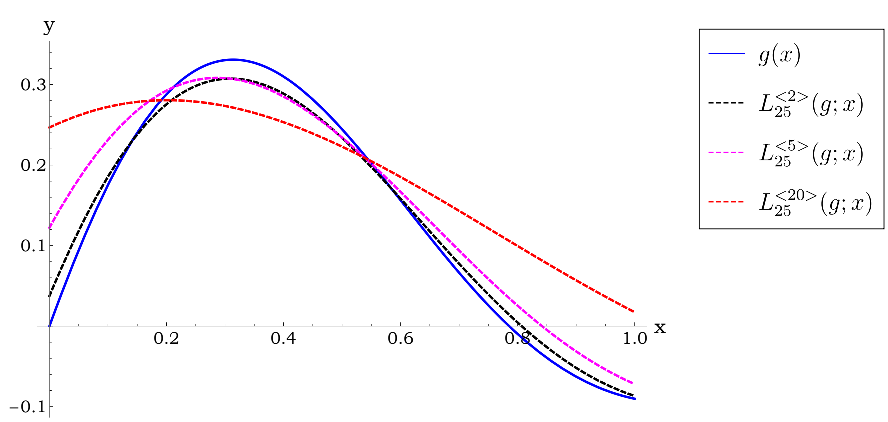

Figure 1. Function $g(x)$ versus $L_\sigma^{<s_\sigma>} g(x)$ for a fixed $\sigma = 25$.

The uniform degree of convergence, $\max_{0\le x\le 1} |L_\sigma^{<s_\sigma>} g(x) - g(x)|$ for $\sigma = 25; s_\sigma = 2, 5, 10$ and $s_\sigma = 20$ are found to be 0.0238, 0.0263759, 0.0375231 and 0.0622243 respectively.

By Figure 1, Table 1 and computations for $L_\sigma^{<s_\sigma>} g(x)$, it is straightforward that the better convergence is found for smaller values of s_σ with respect to σ. Therefore, we compare $g(x)$ with $L_\sigma^{<s_\sigma>} f(x)$ for $s_\sigma = 2$ and $\sigma = 10, 25, 50$.

Table 1. The values of $L_\sigma^{<s_\sigma>}g(x)$ with respect to the function $g(x) = \frac{\sin(4x)}{\exp(2x)+1}$ for $\sigma = 25$ and $s_\sigma = 2, 5$ and 10.

		$L_\sigma^{<s_\sigma>}g(x), \sigma = 25$			
x	$g(x)$	$s_\sigma = 2$	$s_\sigma = 5$	$s_\sigma = 10$	$s_\sigma = 20$
0	0	0.0374022	0.122212	0.19976	0.246637
0.1	0.175303	0.185672	0.231344	0.265402	0.272215
0.2	0.287884	0.275514	0.292409	0.297426	0.280508
0.3	0.330262	0.307098	0.307921	0.298851	0.273379
0.4	0.309893	0.288674	0.2851	0.27477	0.253239
0.5	0.244548	0.23377	0.234179	0.231598	0.222853
0.6	0.156353	0.158219	0.166673	0.176317	0.185163
0.7	0.0662661	0.0774502	0.0938214	0.1158	0.143113
0.8	-0.00980578	0.00438413	0.025337	0.056237	0.0994998
0.9	-0.062772	-0.0519022	-0.0314394	0.00271949	0.056845
1	-0.0902131	-0.0866977	-0.0719608	-0.0410263	0.0173014

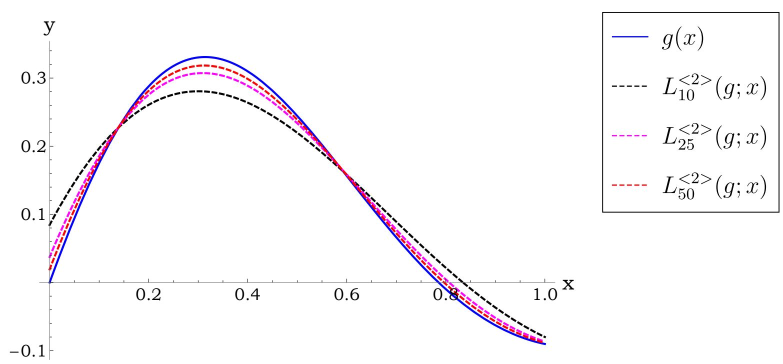

Figure 2. Function $g(x)$ versus $L_\sigma^{<s_\sigma>}g(x)$ for a fixed $s_\sigma = 2$.

The fig 3 corresponds to the absolute error function $|L_\sigma^{<s_\sigma>}g(x) - g(x)|$ for $\sigma = 10, 25$ and $\sigma = 50$. The uniform degree of convergence in the case $s_\sigma = 2; \sigma = 10, 25, 50$ are 0.051022, 0.0238 and 0.0125919 respectively. By Table 2 clearly, an increment in the degree σ of polynomials $L_\sigma^{<s_\sigma>}g(x)$ leads to reduction in the absolute error. Figure 2 also verifies this observation.

Table 2. The values of $L_\sigma^{<s_\sigma>}g(x)$ with respect to the function $g(x) = \frac{\sin(4x)}{\exp(2x)+1}$ for $\sigma = 10, 25, 50$ and $s_\sigma = 2$

		$L_\sigma^{<s_\sigma>}g(x), s_\sigma = 2$			$\lvert L_\sigma^{<s_\sigma>}g(x) - g(x)\rvert$		
x	$f(x)$	$\sigma = 10$	$\sigma = 25$	$\sigma = 50$	$\sigma = 10$	$\sigma = 25$	$\sigma = 50$
0	0	0.0844836	0.0374022	0.0193417	0.0844836	0.0374022	0.0193417
0.1	0.175303	0.196936	0.185672	0.180847	0.0216328	0.0103687	0.00554453
0.2	0.287884	0.260778	0.275514	0.281405	0.0271063	0.01237	0.00647876
0.3	0.330262	0.280637	0.307098	0.318005	0.0496251	0.0231639	0.0122571
0.4	0.309893	0.263915	0.288674	0.298715	0.0459787	0.0212194	0.011178
0.5	0.244548	0.219777	0.23377	0.239011	0.0247712	0.0107776	0.00553703
0.6	0.156353	0.1582	0.158219	0.157545	0.0018466	0.00186588	0.00119166
0.7	0.0662661	0.0891005	0.0774502	0.072276	0.0228344	0.0111841	0.00600997
0.8	-0.00980578	0.0215798	0.00438413	-0.00240815	0.0313856	0.0141899	0.00739763
0.9	-0.062772	-0.0366978	-0.0519022	-0.0572768	0.0260742	0.0108698	0.00549518
1	-0.0902131	-0.0799688	-0.0866977	-0.0885667	0.0102442	0.00351539	0.00164636

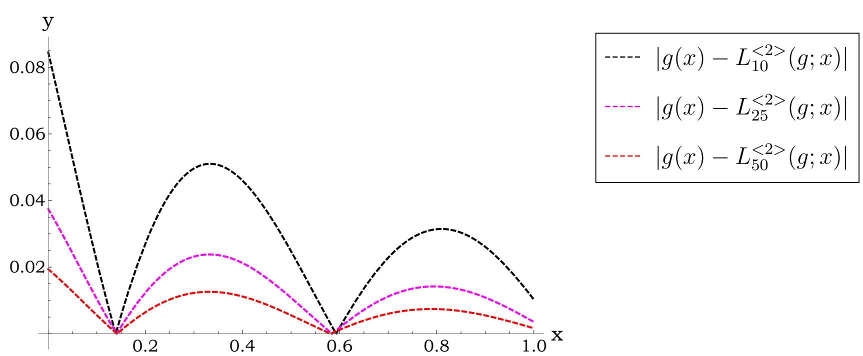

Figure 3. The absolute error $|g(x) - L_\sigma^{<s_\sigma>}g(x)|$ for $s_\sigma = 2, \sigma = 10, 25, 50$.

In our next example we consider the function $h(x) = x^2\exp(-x), x \in [0,1]$. For a fixed degree $\sigma = 20$ we compare $h(x)$ with the polynomials $L_\sigma^{<s_\sigma>}h)(x)$ for $s_\sigma = 2, 4, 8$ and 10. Again, we get the similar observation, i.e. the magnitude of the quantity s_σ/σ controls the convergence of the sequence $L_\sigma^{<s_\sigma>}h)(x)$. Similarly, we compare $h(x)$ keeping s_σ very small $s_\sigma = 2$ and $\sigma = 10, 30$ and 50.

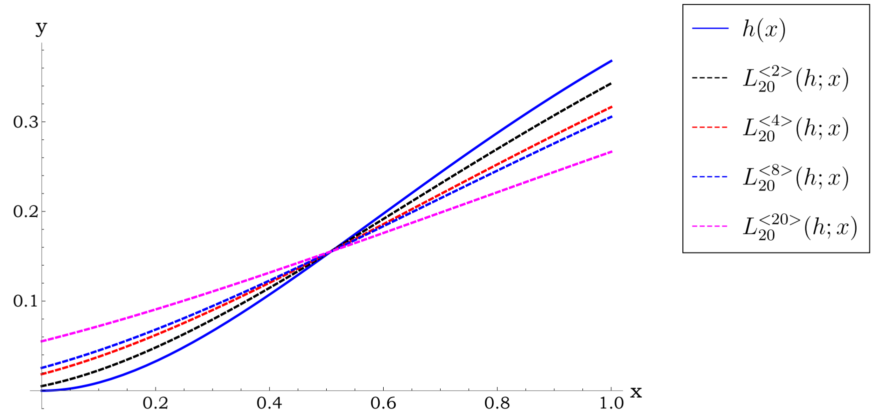

Figure 4. Function $h(x)$ versus $L_{\sigma}^{<s_{\sigma}>}h)(x)$ for a fixed $\sigma = 20$.

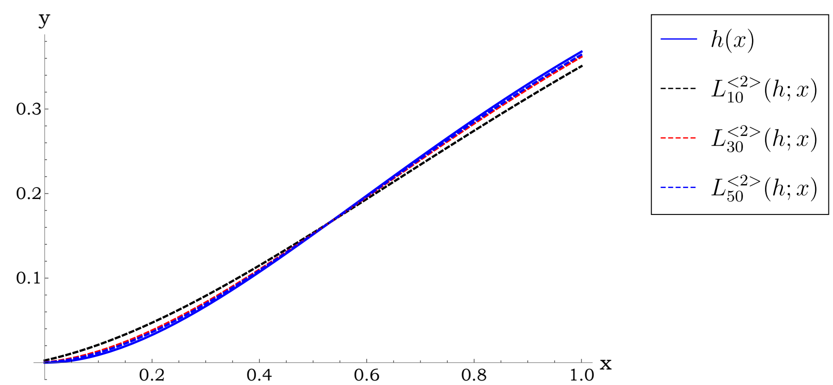

Figure 5. Function $h(x)$ versus $L_{\sigma}^{<s_{\sigma}>}h)(x)$ for a fixed $s_{\sigma} = 2$.

Finally, we consider a non-smooth function $g(x)$ as given below and it is shown that the method $L_{\sigma}^{<s_{\sigma}>}f(x)$ is suitable to approximate non-smooth functions with a bounded derivative.

$$g(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases} \tag{4.1}$$

We plot $g(x)$ versus $L_{\sigma}^{<s_{\sigma}>}g(x)$ for $s_{\sigma} = 10, 30, 60, 100; \sigma = 100$ in fig 6 and $s_{\sigma} = 2; \sigma = 10, 40, 70, 100$ in fig 7.

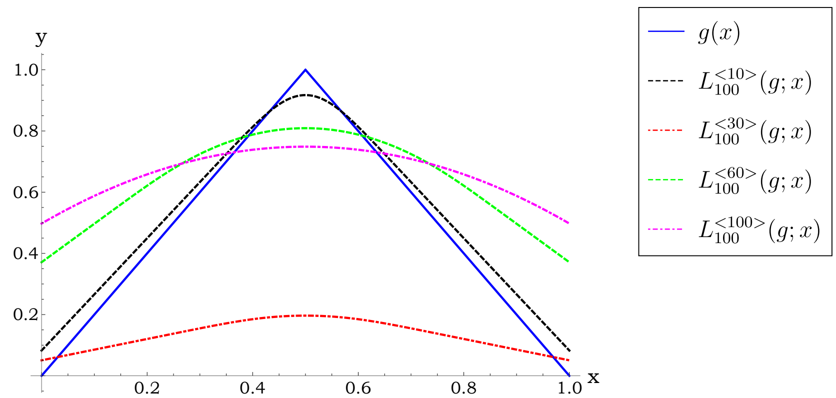

Figure 6. Function $g(x)$ in (4.1)
versus $L_{\sigma}^{<s_\sigma>}g(x)$ for a fixed $\sigma = 100$.

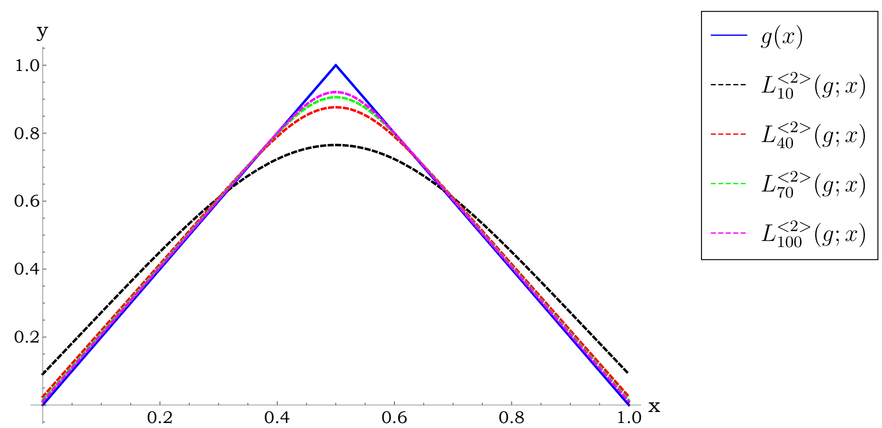

Figure 7. Function $g(x)$ in (4.1)
versus $L_{\sigma}^{<s_\sigma>}g(x)$ for a fixed $s_\sigma = 2$.

In the end, we purpose a numerical quadrature formula using the method $L_{\sigma}^{<s_\sigma>}f(x)$. Let $f \in C[0,1]$. Then,

$$\int_0^1 f(x)\,dx \approx \frac{(f(0)+f(1)) + 2\sum_{r=1}^{\sigma-1} f\left(\frac{r+s_\sigma-1}{\sigma+s_\sigma-1}\right)}{2(\sigma+1)(s_\sigma-1)}. \tag{4.2}$$

In order to verify the quadrature rule (4.2), we choose the functions $f(x) = e^{-x^2}\log(x+1)$ and $8\sqrt{x(1-x)}, 0 \le x \le 1,\ s_\sigma = 2$ and $\sigma = 10, 100, 200, 400, 500$ and 1000.

Remark 4.1. *Since $\int_0^1 8\sqrt{x}\sqrt{(1-x)}\,dx = \pi$, the second row in the Table3 provides approximations for π correct up to three digits of decimals. Further, using $\sigma = 13000$, we obtain $\pi = 3.14159$ correct up to five digits.*

Table 3. The values of $I = \int_0^1 \varphi(x)\,dx$ with respect to the quadrature rule (4.2) for various values of σ.

$\varphi(x)$	Exact value of I	$\sigma = 10$	$\sigma = 100$	$\sigma = 200$	$\sigma = 400$	$\sigma = 500$	$\sigma = 1000$
$e^{-x^2}\log(x+1)$	0.249883	0.24946	0.249775	0.249856	0.249876	0.249879	0.249882
$8\sqrt{x(1-x)}$	3.14159	3.11072	3.13048	3.13763	3.14018	3.14058	3.14124

5. Conclusion

It has been observed by corollary3.1 and corollary3.3, that the degree of uniform approximation of the operators $L_\sigma^{<s_\sigma>} f(x)$ is controlled by the quantity s_σ/σ. The smaller the ratio s_σ/σ implies smaller absolute error. Moreover, the arbitrariness of the positive sequence (s_σ) provides a flexible method of approximation.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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