International Journal of Analysis and Applications
ISSN 2291-8639
Volume 6, Number 1 (2014), 18-27
http://www.etamaths.com

A WEAK CONTRACTION PRINCIPLE IN PARTIALLY ORDERED CONE METRIC SPACE WITH THREE CONTROL FUNCTIONS

BINAYAK S. CHOUDHURY[1], L. KUMAR[2], T. SOM[2,*] AND N. METIYA[3]

Abstract. In this paper we utilize three functions to define a weak contraction in a cone metric space with a partial order and establish that this contraction has necessarily a fixed point either under the continuity assumption or an order condition which we state here. The uniqueness of the fixed point is also derived under some additional assumptions. The result is supported with an example. The methodology used is a combination of order theoretic and analytic approaches.

1. Introduction

In this paper we consider a fixed point theorem in a space which is a generalization of metric space, namely, cone metric space. The space is introduced by allowing the metric to take up values in Banach spaces. Following the work of Huang et al in [19], fixed point theory has experienced a rapid growth in cone metric spaces. A review of this development is given in [22].

Weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber et al. in Hilbert spaces [1]. It was subsequently extended to metric spaces by Rhoades [30] and further generalized by several authors like Dutta and Choudhury [16], Popescu [28], Choudhury and Kundu [9] etc. The weak contraction principle has been recently extended to cone metric spaces [6]. Several weak contractive inequalities have been used in the fixed point theory in metric and cone metric spaces. References [7], [8], [10], [14], [32] are some examples of these works.

In an attempt to blend the order theoretic and analytic aspects of fixed point theory, several authors have created a number of fixed point results in partially ordered metric spaces. Some of these works are noted in [12], [18], [27], [29]. In cone metric spaces also, such efforts have been made in [2], [3], [11], [23] for examples. The purpose here is to establish weak contraction results in partially ordered cone metric spaces by using three control functions. Control functions first appeared in fixed point theory in the work of Khan et al. [25] and afterward this function and its generalizations have been used in a number of fixed point problems like [4], [5], [26], [31]. Our result is illustrated with an example.

2. Mathematical preliminaries

2010 *Mathematics Subject Classification.* 54H10, 54H25.
Key words and phrases. Partially ordered set; Cone metric space; Weak contraction; Control function; Fixed point.

Definition 2.1 [19] Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is nonempty, closed, and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R},\ \ a, b \geq 0,\ \ x, y \in P \Longrightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is naturally defined by $x \leq y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone P is said to be normal if there exists a real number $K \geq 1$ such that for all $x, y \in E$,
$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$.
The least positive number K satisfying the above statement is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that
$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y$,
for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose that E is a real Banach space with cone P in E with $\mathrm{int}P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition 2.2 [19] Let X be a nonempty set. Let the mapping $d : X \times X \longrightarrow E$ satisfies

(i) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x,\ y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3 [19] Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.

(i) If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, x is the limit of $\{x_n\}$. We denote this by $\lim_n x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.
(ii) If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbf{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.
(iii) If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

If P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$ [19].

Definition 2.4 Let $\psi : \mathrm{int}P \cup \{0\} \longrightarrow \mathrm{int}P \cup \{0\}$ be a function.

(i) We say ψ is strongly monotone increasing if for $x,\ y \in \text{int}P \cup \{0\}$
$x \leq y \iff \psi(x) \leq \psi(y)$.
(ii) ψ is said to be continuous at $x_0 \in \text{int}P \cup \{0\}$ if for any sequence $\{x_n\}$ in $\text{int}P \cup \{0\}$, $x_n \longrightarrow x_0 \implies \psi(x_n) \longrightarrow \psi(x_0)$.

The following is the definition of Altering distance function in cone metric space.

Definition 2.5 A function $\psi : \text{int}P \cup \{0\} \longrightarrow \text{int}P \cup \{0\}$ is called an Altering distance function if the following properties are satisfied:

(i) ψ is strongly monotone increasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.6 [20] Let $(X,\ d)$ be a cone metric space, $T : X \longrightarrow X$ and $x_0 \in X$. Then the function T is continuous at x_0 if for any sequence $\{x_n\}$ in X, $x_n \longrightarrow x_0$ implies $Tx_n \longrightarrow Tx_0$.

Definition 2.7 [18] Let $(X, \preceq)$ be a partially ordered set and $T : X \longrightarrow X$ be a self map. We say that T is monotone non decreasing if $x,\ y \in X,\ x \preceq y \implies Tx \preceq Ty$.

Lemma 2.1. Let E be a real Banach space with cone P in E. Then

(i) if $a \leq b$ and $b \ll c$, then $a \ll c$ [21],
(ii) if $a \ll b$ and $b \ll c$, then $a \ll c$ [21],
(iii) if $0 \leq x \leq y$ and $a \geq 0$, where a is real number, then $0 \leq ax \leq ay$ [21],
(iv) if $0 \leq x_n \leq y_n$, for $n \in \mathbf{N}$ and $\lim_n x_n = x$, $\lim_n y_n = y$, then $0 \leq x \leq y$ [21],
(v) P is normal if and only if $x_n \leq y_n \leq z_n$ and $\lim_n x_n = \lim_n z_n = x$ imply $\lim_n y_n = x$ [13].

Lemma 2.2 [7] Let $(X,\ d)$ be a cone metric space with regular cone P such that $d(x,\ y) \in \text{int}P$, for $x,\ y \in X$ with $x \neq y$. Let $\phi : \text{int}P \cup \{0\} \longrightarrow \text{int}P \cup \{0\}$ be a function with the following properties:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t) \ll t$, for $t \in \text{int}P$ and
(iii) either $\phi(t) \leq d(x,\ y)$ or $d(x,\ y) \leq \phi(t)$, for $t \in \text{int}P \cup \{0\}$ and $x,\ y \in X$.

Let $\{x_n\}$ be a sequence in X for which $\{d(x_n,\ x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n,\ x_{n+1})\}$ is convergent to either $r = 0$ or $r \in$ int P.

Lemma 2.3 [10] Let $(X,\ d)$ be a cone metric space. Let $\phi : \text{int}P \cup \{0\} \longrightarrow \text{int}P \cup \{0\}$ be a function such that

(i) $\phi(t) = 0$ if and only if $t = 0$ and
(ii) $\phi(t) \ll t$, for $t \in \text{int}P$.

Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll \phi(c)$, for all $n, m > n_0$.

3. Main Results

Lemma 3.1. Let (X, d) be a cone metric space with regular cone P such that $d(x,\ y) \in \text{int}P$, for $x,\ y \in X$ with $x \neq y$. Let $\phi : \text{int}P \cup \{0\} \longrightarrow \text{int}P \cup \{0\}$ be a function with the following properties.

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t) \ll t$, for $t \in \text{int}P$ and

(iii) either $\phi(t) \leq d(x,\ y)$ or $d(x,\ y) \leq \phi(t)$, for $t \in \mathrm{int}P \cup \{0\}$ and $x,\ y \in X$.

Let $\{x_n\}$ be a sequence in X and $x \in X$ for which $\{d(x_n,\ x)\}$ is monotonic decreasing. Then $\{d(x_n,\ x)\}$ is convergent to either $r = 0$ or $r \in \text{int } P$.

Proof. Let $\{x_n\}$ be a sequence in X and $x \in X$ for which $\{d(x_n,\ x)\}$ is monotonic decreasing. Since cone P is regular and $0 \leq d(x_n,\ x)$, for all $n \in \mathbf{N}$, there exists $r \in P$ such that

$d(x_n,\ x) \longrightarrow r$ as $n \longrightarrow \infty$.

If $d(x_n,\ x) = 0$, for some n then trivially $r = 0$. Hence we shall assume that $d(x_n,\ x) \neq 0$, for all $n \in \mathbf{N}$. Then according to the conditions of the lemma, $d(x_n,\ x) \in \text{int } P$, for all $n \in \mathbf{N}$.

Let $r \neq 0$.

Since P is a regular cone, it is also a normal cone. Let $B = \{t \in \text{int } P : \|t\| < \frac{\|r\|}{K}\}$, where K is the normal constant of the cone P. For every positive real number a with $a < \frac{\|r\|}{K}$ and $t \in \text{int } P$, $\frac{at}{\|t\|} \in B$. Therefore, B is non empty. Now we claim that for every $t \in B$,

$\phi(t) \leq d(x_n,\ x)$, for all $n \in \mathbf{N}$.

Otherwise, there exists $t_0 \in B$ for which we can find a positive integer m such that

$d(x_m,\ x) < \phi(t_0)$ (using the property (iii) of ϕ in the lemma).

Since $\{d(x_n,\ x)\}$ is monotonic decreasing, we have

$d(x_n,\ x) \leq d(x_m,\ x) < \phi(t_0)$, for all $n \geq m$,

which implies $d(x_n,\ x) < \phi(t_0)$, for all $n \geq m$.

Letting $n \longrightarrow \infty$ in the above inequality, by (iv) of lemma 2.1 and using a property of ϕ, we have

$r \leq \phi(t_0) \ll t_0$,

which implies $\|r\| \leq K\|t_0\|$, where K is the normal constant of cone P.

That is, $\|t_0\| \geq \frac{\|r\|}{K}$, which contradicts our assumption that $t_0 \in B$.

Hence for every $t \in B$,

$\phi(t) \leq d(x_n,\ x)$, for all $n \in \mathbf{N}$.

Letting $n \longrightarrow \infty$ in the above inequality, we have

$\phi(t) \leq r$.

Therefore, for every $t \in B$,

$r - \phi(t) \in P$; that is, $r = \phi(t) + q$, for some $q \in P$.

Now, $0 \leq q \ll \phi(t) + q$ (since $\phi(t) \in \text{int } P$, for every $t \in B$). Then by (i) of lemma 2.1,

$0 \ll \phi(t) + q$; that is, $0 \ll r$.

Therefore, $r \in \text{int } P$. Hence the proof is completed. □

Theorem 3.1. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a cone metric d in X for which the cone metric space $(X,\ d)$ is complete with regular cone P such that $d(x,\ y) \in \mathrm{int}P$, for $x,\ y \in X$ with $x \neq y$. Let $T : X \longrightarrow X$ be a continuous and non decreasing mapping such that for all comparable $x,\ y \in X$

$$\psi(d(Tx, Ty)) \leq \eta(d(x, y)) - \phi(d(x, y)), \tag{3.1}$$

where $\psi,\ \eta,\ \phi : \mathrm{int}P \cup \{0\} \longrightarrow \mathrm{int}P \cup \{0\}$ are such that ψ and η are continuous, ϕ is lower semi-continuous and also

(i) ψ is strongly monotonic increasing,
(ii) $\psi(t) = \eta(t) = \phi(t) = 0$ if and only if $t = 0$,
(iii) $\psi(t) - \eta(t) + \phi(t) > 0$ for all $t \in \mathrm{int}P$,
(iv) $\phi(t) \ll t$, for $t \in \mathrm{int}P$ and

(v) either $\phi(t) \leq d(x,\ y)$ or $d(x,\ y) \ll \phi(t)$, for $t \in \text{int}P \cup \{0\}$ and $x,\ y \in X$.

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$. Since T is nondecreasing w.r.t. $\preceq$, we construct the sequence $\{x_n\}$ such that $x_n = Tx_{n-1} = T^n x_0$ and $x_0 \preceq Tx_0 \preceq T^2 x_0 \preceq ... \preceq T^n x_0 \preceq T^{n+1} x_0 \preceq ...$; that is

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq .... \tag{3.2}$$

Since $x = x_n$ and $y = x_{n+1}$ are comparable, from (3.1), we have

$$\psi(d(Tx_n, Tx_{n+1})) \leq \eta(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})). \tag{3.3}$$

Now, for all $n \geq 1$, we have

$$\begin{aligned}
&\psi(d(Tx_{n-1}, Tx_n)) - \psi(d(Tx_n, Tx_{n+1})) \\
&\qquad \geq \psi(d(Tx_{n-1}, Tx_n)) - \eta(d(x_n, x_{n+1})) + \phi(d(x_n, x_{n+1})) \\
&\qquad = \psi(d(x_n, x_{n+1})) - \eta(d(x_n, x_{n+1})) + \phi(d(x_n, x_{n+1})) \\
&\qquad \geq 0. \text{ (by (ii) and (iii))}
\end{aligned}$$

This implies that $\psi(d(Tx_n, Tx_{n+1})) \leq \psi(d(Tx_{n-1}, Tx_n))$. Then by (i), it follows that

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n), \text{ that is, } d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}).$$

Therefore, $\{d(x_n,\ x_{n+1})\}$ is a monotone decreasing sequence. Hence by lemma 2.2, there exists an $r \in \text{int}P \cup \{0\}$ such that

$$d(x_n,\ x_{n+1}) \longrightarrow r \text{ as } n \longrightarrow \infty. \tag{3.4}$$

Taking the limit as $n \to \infty$ in (3.3) and using (3.4), continuities of ψ, η and the lower semi continuity of ϕ, we have

$$\psi(r) \leq \eta(r) - \phi(r),$$

that is,

$$\psi(r) - \eta(r) + \phi(r) \leq 0,$$

which by (ii) and (iii) implies that $r = 0$.

Hence, we have

$$\lim_{n\to\infty} d(x_{n+1},\ x_n) \longrightarrow 0. \tag{3.5}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then by lemma 2.3, there exists a $c \in E$ with $0 \ll c$ such that $\forall\, n_0 \in \mathbb{N}$, $\exists\, n,\ m \in \mathbb{N}$ with $n > m > n_0$ such that $d(x_n,\ x_m) <\not\ll \phi(c)$. Hence by a property of ϕ in (v) of the theorem, $\phi(c) \leq d(x_n,\ x_m)$. Therefore, there exist sequences $\{n(k)\}$ and $\{m(k)\}$ in $\mathbb{N}$ such that for all positive integers k,

$$n(k) > m(k) > k \text{ and } d(x_{n(k)},\ x_{m(k)}) \geq \phi(c).$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(x_{n(k)},\ x_{m(k)}) \geq \phi(c)$$

and

$$d(x_{n(k)-1},\ x_{m(k)}) \ll \phi(c).$$

Now,

$$\phi(c) \leq d(x_{n(k)},\ x_{m(k)}) \leq d(x_{n(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{m(k)}),$$

that is,

$$\phi(c) \leq d(x_{n(k)},\ x_{m(k)}) \leq d(x_{n(k)},\ x_{n(k)-1}) + \phi(c).$$

Letting $k \longrightarrow \infty$ in the above inequality, using (3.5) and the property (v) of Lemma 2.1, we have

$$\lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)}) = \phi(c). \tag{3.6}$$

Again,

$$d(x_{n(k)},\ x_{m(k)}) \leq d(x_{n(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{m(k)+1}) + d(x_{m(k)+1},\ x_{m(k)})$$

and

$$d(x_{n(k)+1},\ x_{m(k)+1}) \leq d(x_{n(k)+1},\ x_{n(k)}) + d(x_{n(k)},\ x_{m(k)}) + d(x_{m(k)},\ x_{m(k)+1}).$$

Letting $k \longrightarrow \infty$ in above inequalities, using (3.5) and (3.6), we have

$$\lim_{k\to\infty} d(x_{n(k)+1},\ x_{m(k)+1}) = \phi(c). \tag{3.7}$$

Since for $x = x_{n(k)}$ and $y = x_{m(k)}$ are comparable, from (3.1), we have

$$\begin{aligned}\psi(d(x_{n(k)+1},\ x_{m(k)+1})) &= \psi(d(Tx_{n(k)},\ Tx_{m(k)}))\\ &\leq \eta(d(x_{n(k)},\ x_{m(k)})) - \phi(d(x_{n(k)},\ x_{m(k)})).\end{aligned}$$

Letting $k \longrightarrow \infty$ in the above inequality, using (3.6), (3.7) and the continuities of ψ, η and the lower semi continuity of ϕ, we have

$$\psi(\phi(c)) \leq \eta(\phi(c)) - \phi(\phi(c)),$$

that is,

$$\psi(\phi(c)) - \eta(\phi(c)) + \phi(\phi(c)) \leq 0,$$

which by (ii) and (iii) implies that $\phi(c) = 0$. Now, by (ii), it follows that $c = 0$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $u \in X$ such that

$$x_n \longrightarrow u \text{ as } n \longrightarrow \infty. \tag{3.8}$$

Since T is continuous and $x_n \longrightarrow u$, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tu$$

and this proves that u is a fixed point of T. □

In our next theorem we relax the continuity assumption on T by imposing an order condition.

Theorem 3.2. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a cone metric d in X for which the cone metric space $(X,\ d)$ is complete with regular cone P such that $d(x,\ y) \in \text{int}P$, for $x,\ y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$ then $x_n \preceq x$, for all $n \in \mathbf{N}$. Let $T : X \longrightarrow X$ be a nondecreasing mapping such that for all comparable $x,\ y \in X$, (3.1) holds where the conditions upon ψ, η and ϕ are the same as in Theorem 3.1. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 3.1. Then we have $x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ...$, that is, $\{x_n\}$ is nondecreasing sequence. Also, this sequence converge to u. Then $x_n \preceq u$, for all $n \in \mathbf{N}$. Therefore, we can use the condition (3.1) for $x = u$, $y = x_n$ and so we have

$$\begin{aligned}\psi(d(Tu,\ x_{n+1})) &= \psi(d(Tu,\ Tx_n))\\ &\leq \eta(d(u,\ x_n)) - \phi(d(u,\ x_n)).\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, using the properties of ψ, η and ϕ, we have

$$\psi(d(Tu,\ u)) \leq 0.$$

It follows by a property of ψ that $d(Tu,\ u) = 0$; that is, $Tu = u$, that is, u is a fixed point of T. □

Theorem 3.3 In addition to the hypotheses of Theorem 3.1 and Theorem 3.2, in both of the theorems, suppose that for every $x,\ y \in X$ there exists a $z \in X$ such that $x \preceq z$ and $y \preceq z$. Then T has a unique fixed point.

Proof. It follows from the theorem 3.1 or theorem 3.2, the set of fixed points of T is non-empty. If possible, let $x,\ y \in X$ $(x \neq y)$ be two fixed points of T. We distinguish two cases:

Case 1.

Suppose that x and y are comparable. Without loss of generality we take $y \preceq x$.

Then $T^n y = y \preceq x = T^n x$, for $n = 0,\ 1,\ 2, ...$
By the condition (3.1), we have for all $n \geq 1$,

$$\psi(d(x,\ y)) = \psi(d(T^n x,\ T^n y)) \leq \eta(d(T^{n-1}x,\ T^{n-1}y)) - \phi(d(T^{n-1}x,\ T^{n-1}y)),$$

that is,

$$\psi(d(x,\ y)) \leq \eta(d(x,\ y)) - \phi(d(x,\ y)),$$

that is,

$$\psi(d(x,\ y)) - \eta(d(x,\ y)) + \phi(d(x,\ y)) \leq 0,$$

which by (ii) and (iii) implies that $d(x,\ y) = 0$; that is, $x = y$.
Case 2.
If x and y are not comparable, then there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$. Monotonicity of T implies that $T^n x = x \preceq T^n z$ and $T^n y = y \preceq T^n z$, for $n = 0,\ 1,\ 2, ...$
By the condition (3.1), we have for all $n \geq 1$,

$$\psi(d(T^n z,\ x)) = \psi(d(T^n z,\ T^n x)) \leq \eta(d(T^{n-1}z,\ T^{n-1}x)) - \phi(d(T^{n-1}z,\ T^{n-1}x)),$$

that is,

$$\psi(d(T^n z,\ x)) \leq \eta(d(T^{n-1}z,\ x)) - \phi(d(T^{n-1}z,\ x)). \tag{3.9}$$

Now, for all $n \geq 1$, we have

$$\begin{aligned}
&\psi(d(T^{n-1}z,\ x)) - \psi(d(T^n z,\ x)) \\
&= \psi(d(T^{n-1}z,\ T^{n-1}x)) - \psi(d(T^n z,\ T^n x)) \\
&= \psi(d(T^{n-1}z,\ T^{n-1}x)) - \psi(d(T(T^{n-1}z),\ T(T^{n-1}x))) \\
&\geq \psi(d(T^{n-1}z,\ T^{n-1}x)) - \eta(d(T^{n-1}z,\ T^{n-1}x)) + \phi(d(T^{n-1}z,\ T^{n-1}x)) \\
&= \psi(d(T^{n-1}z,\ x)) - \eta(d(T^{n-1}z,\ x)) + \phi(d(T^{n-1}z,\ x)) \\
&\geq 0. \text{ (by (ii) and (iii))}
\end{aligned}$$

This implies that $\psi(d(T^n z,\ x)) \leq \psi(d(T^{n-1}z,\ x))$. Then by (i), it follows that

$$d(T^n z,\ x) \leq d(T^{n-1}z,\ x),$$

Therefore, $\{d(T^n z,\ x)\}$ is a monotone decreasing sequence. Hence by lemma 3.1, there exists an $r \in \text{int}P \cup \{0\}$ such that

$$\lim_{n\to\infty} d(T^n z,\ x) = r. \tag{3.10}$$

Taking the limit as $n \to \infty$ in (3.9) and using (3.10), continuities of ψ, η and the lower semi continuity of ϕ, we have

$$\psi(r) \leq \eta(r) - \phi(r),$$

that is,

$$\psi(r) - \eta(r) + \phi(r) \leq 0,$$

which by (ii) and (iii) implies that $r = 0$.
Hence

$$\lim_{n\to\infty} d(T^n z,\ x) = 0.$$

Analogously, it can proved that

$$\lim_{n\to\infty} d(T^n z,\ y) = 0.$$

Finally, the uniqueness of the limit gives us $x = y$.

From above two cases we have that fixed point of T is unique. □

Example 3.1 Let $X = [0,\ 1]$ with usual order $\preceq$ be a partially ordered set. Let $E = \mathbb{R}^2$, with usual norm, be a real Banach space. We define $P = \{(x, y) \in E : x, y \geq 0\}$. The partial ordering $\leq$ with respect to the cone P be the partial ordering in E. Then P is a regular cone.
Let $d : X \times X \longrightarrow E$ be given as

$d(x,\ y) = (\mid x - y \mid,\ \mid x - y \mid)$, for $x,\ y \in X$.
Then $(X,\ d)$ is a complete cone metric space with the required properties of theorems 3.1 and 3.2.
Let $\psi,\ \eta,\ \phi : \text{int } P \cup \{0\} \longrightarrow \text{int } P \cup \{0\}$ be defined respectively as follows:
for $t = (x,\ y) \in \text{int } P \cup \{0\}$,

$$\psi(t) = \begin{cases} 0, \text{ if } \ x = 0 \text{ and } \ y = 0, \\ (x,\ y), \text{ if } \ 0 < x \leq 1 \text{ and } \ 0 < y \leq 1, \\ (x^2,\ y), \text{ if } \ x > 1 \text{ and } \ 0 < y \leq 1, \\ (x,\ y^2), \text{ if } \ 0 < x \leq 1 \text{ and } \ y > 1, \\ (x^2,\ y^2), \text{ if } \ x > 1 \text{ and } \ y > 1, \end{cases}$$

and

$\eta(t) = (v,\ v)$ and $\phi(t) = (\frac{v^2}{2},\ \frac{v^2}{2})$, where $v = \min\{x,\ y\}$.

Then ψ, η and ϕ have the properties mentioned in theorems 3.1 and 3.2.

$Tx = x - \frac{x^2}{2}$, for $x \in X$.

Then T has the required properties mentioned in theorems 3.1 and 3.2.
Without loss of generality we take $x,\ y \in X$ with $x > y$.
Now,

$$\begin{aligned} &\psi(d(Tx,\ Ty)) = \psi(d(x - \frac{x^2}{2},\ y - \frac{y^2}{2})) \\ &= \psi(((x - y) - \frac{(x-y)(x+y)}{2},\ (x - y) - \frac{(x-y)(x+y)}{2})) \\ &\qquad [\text{ since } 0 \leq (x-y) - \frac{(x-y)(x+y)}{2} \leq 1\ ] \\ &= ((x - y) - \frac{(x-y)(x+y)}{2},\ (x - y) - \frac{(x-y)(x+y)}{2}) \\ &\qquad [\text{ since } (x - y) \leq (x + y)\ ] \\ &\leq ((x - y) - \frac{(x-y)^2}{2},\ (x - y) - \frac{(x-y)^2}{2}) \\ &= (x - y,\ x - y) - (\frac{(x-y)^2}{2},\ \frac{(x-y)^2}{2}) \\ &= \eta((x - y,\ x - y)) - \phi((\frac{(x-y)^2}{2},\ \frac{(x-y)^2}{2})) \\ &= \eta(d(x,\ y)) - \phi(d(x,\ y)). \end{aligned}$$

Hence the conditions of theorems 3.1 and 3.2 are satisfied and it is seen that 0 is a fixed point of T.

Remark 3.1. It has been found that several fixed point problems in the cone metric spaces are reducible to problems of metric spaces ([15], [17], [22], [24]). This is not possible in general. Particularly, weak contraction is not transferable to a corresponding weak contraction in the generated metric space and, therefore, is not derived from the results of weak contractions in metric spaces. In fact there is even no assurance that a cone metric space inequality will generate an inequality condition in metric spaces, although, it does in several important cases as has been pointed out in ([15], [17], [22], [24]). Moreover, there is a problem when a partial ordering is defined on a cone metric space. In a partially ordered cone metric space with specific relations of the cone metric with the ordering, there is no natural way of transferring these relations to metric spaces. Our problem is outside the scope of those described in ([15], [17], [22], [24]).

References

[1] Ya. I. Alber and S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, Oper. Theory Adv. Appl. 98 (1997), 7-22.

[2] I. Altun, B. Damjanović, D. Djorić, Fixed point and common fixed point theorems on ordered cone metric spaces, Appl. Math. Lett. 23 (2010), 310-316.

[3] I. Altun, V. Rakočević, Ordered cone metric spaces and fixed point results, Comput. Math. Appl. 60 (2010), 1145-1151.

[4] B. S. Choudhury, A common unique fixed point result in metric spaces involving generalised altering distances, Math. Commun. 10 (2005), 105-110.

[5] B. S. Choudhury, K. Das, A coincidence point result in Menger spaces using a control function, Chaos Solitons Fractals 42 (2009), 3058-3063.

[6] B. S. Choudhury, N. Metiya, Fixed points of weak contractions in cone metric spaces, Nonlinear Anal. 72 (2010), 1589-1593.

[7] B. S. Choudhury, N. Metiya, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Comput. Math. Appl. 60 (2010), 1686-1695.

[8] B. S. Choudhury, P. Konar, B. E. Rhoades, N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlinear Anal. 74 (2011), 2116-2126.

[9] B. S. Choudhury, A. Kundu, $(\psi,\ \alpha,\ \beta)$ - Weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25 (2012), 6 - 10.

[10] B. S. Choudhury, N. Metiya, Coincidence point and fixed point theorems in ordered cone metric spaces, J. Adv. Math. Stud. 5 (2012), 20-31.

[11] B. S. Choudhury, N. Metiya, Fixed point and common fixed point results in ordered cone metric spaces, An. St. Univ. Ovidius Constanta 20 (2012), 55-72.

[12] L. Ćirić, M. Abbas, R. Saadati, N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, Appl. Math. Comput. 217 (2011), 5784-5789.

[13] K. Deimling, Nonlinear Functional Analysis, Springer-Verlage, 1985.

[14] D. Dorić, Common fixed point for generalized $(\psi,\ \varphi)$-weak contractions, Appl. Math. Lett. 22 (2009), 1896-1900.

[15] W. S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. 72 (2010), 2259-2261.

[16] P. N. Dutta, B. S. Choudhury, A generalisation of contraction principle in metric spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 406368.

[17] A. A. Harandi, M. Fakhar, Fixed point theory in cone metric spaces obtained via the scalarization method, Comput. Math. Appl. 59 (2010), 3529-3534.

[18] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009), 3403-3410.

[19] L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[20] D. Ilić, V. Rakoćević, Common fixed point for maps on cone metric space, J. Math. Anal. Appl. 341 (2008), 876-882.

[21] S. Janković, Z. Kadelburg, S. Radenović, B. E. Rhoades, Assad-Kirk-Type fixed point theorems for a pair of nonself mappings on cone metric spaces, Fixed Point Theory Appl. 2009 (2009), Article ID 761086.

[22] S. Janković, Z. Kadelburg, S. Radenović, On cone metric spaces: A survey, Nonlinear Anal. 74 (2011), 2591-2601.

[23] Z. Kadelburg, M. Pavlović, S. Radenović, Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces, Comput. Math. Appl. 59 (2010), 3148-3159.

[24] Z. Kadelburg, S. Radenovic, V. Rakocevic, A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett. 24 (2011), 370-374.

[25] M. S. Khan, M. Swaleh, S. Sessa, Fixed points theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984), 1-9.

[26] S. V. R. Naidu, Some fixed point theorems in metric spaces by altering distances, Czechoslovak Math. J. 53 (2003), 205-212.
[27] J. J. Nieto, R. Rodroguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sinica 23 (2007), 2205-2212.
[28] O. Popescu, Fixed points for $(\psi,\ \phi)$ - weak contractions, Appl. Math. Lett. 24 (2011), 1-4.
[29] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.
[30] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (200l), 2683-2693.
[31] K. P. R. Sastry, G. V. R. Babu, Some fixed point theorems by altering distances between the points, Ind. J. Pure. Appl. Math. 30 (1999), 641-647.
[32] Q. Zhang, Y. Song, Fixed point theory for generalized $\phi-$ weak contractions, Appl. Math. Lett. 22 (2009), 75-78.

[1]Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah - 711103, West Bengal, India

[2]Department of Applied Mathematics, Indian Institute of Technology, (Banaras Hindu University), Varanasi - 221005, India

[3]Department of Mathematics, Bengal Institute of Technology, Kolkata - 700150, West Bengal, India

[*]Corresponding author