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Generalization of Homotopy Analysis Method for q-Fractional Non-linear Differential Equations

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Abstract. This paper presents a generalization of the Homotopy analysis method (HAM) for finding the solutions of non-linear q-fractional differential equations (q-FDEs). This method shows that the series solution in the case of generalized HAM is more likely to converge than that on HAM. In order that it is applicable to solve immensely non-linear problems and also address a few issues, such as the impact of varying the auxiliary parameter, auxiliary function, and auxiliary linear operator on the order of convergence of the method. The generalized HAM method is more accurate than the HAM.

1. Introduction

Fractional calculus is concerned with generalizing integer order of integration and differentiation to any order. For several decades, fractional differential equations have sparked much interest because of their various applications in physics and engineering [11, 18, 20, 22]. In Particular, fractional derivatives are a powerful tool for describing memory and inherited properties in a wide variety of materials and processes and also proved half-order derivatives and integrals to be more beneficial than classical models in formulating different types of problems. The q-calculus [4, 8–10, 23] has a long history and can be traced back to Euler and Jacobi's. The q-calculus has many potential applications in various numerical methods, like orthogonal polynomials and number theory. Moreover, connections between mathematics and physics appear in quantum theory and general relativity theory.

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The fractional q-calculus or q-fractional calculus is an q-extension case of ordinary fractional calculus. New interests are emerging on this subject. Agarwal [1]and Al-Salam [26] invented the hypothesis and built various kinds of operators, like q–fractional integral derivatives. For basics, refer to [2]. Due to the extensive study of this subject, an incredible interest showed up from many authors in the study of (q-FDEs) and some applications [11, 16, 28]. During the past decade, mathematicians and physicists have made immense efforts to determine the numerical and analytical methods for solving FDEs and q-difference equations. As of late, semi-analytical methods have been widely applied to determine fractional and q-differential models for non-linear and linear cases. A few are the differential transformation method [32], successive approximation method [25], variational iteration method, homotopic perturbation method [6, 13], operational matrix method [24] and HAM for solving various FDEs and q-differential equations when calculating in h-calculus and in q-calculus. Coming to semi-analytical methods for q-FDEs, in 2013, authors Wu and Baleanu [30] invented the variational iteration method from the differential equation to q-FDEs. In 2019, Pin Lyu and Seakweng Von [21] found an efficient finite difference numerical method for q-FDEs. In 2021, B. Madhavi and G. Suresh Kumar [17] developed an operational matrix method to find the solution of q-FDEs using the Laguerre polynomial. In the same year, Ying Sheng and Zhang [31] provided some Results on the q-Calculus and q-FDEs.

Homotopy analysis method [3,19,27,33] is one of the most used semi-analytical techniques. Liao first discussed this method in 1992 in his Ph.D. monograph [12–15]. In this technique, the solution is entirely in series formation. It offers an easy way to ensure the solution's convergence is in series form, which is different from all other methods. It can be used to solve problems that are highly nonlinear problems. The perturbation results are applicable only for small physical parameters. However, homotopy has always been independent of any small or large parameters. This method also makes changing and controlling the solution series converges simple. Furthermore, the HAM provides excellent flexibility in terms of equation type and solution expressions of high-order equations, making it simple to obtain approximations at rather than high order. Because of these benefits, HAM attracts the attention of many researchers. It is used to solve various nonlinear problems, including nonlinear Riccati fractional differential equations, the vakhnenko equation, the glauert-jet problems, the KdV-burgers-kuramoto fractional equation, and the nonlinear heat transfer, and so on. M. A. El-Tawil and S.N. Huseen [29] present a method called the q-homotopy analysis method (q-HAM), a more popular HAM approach. The fundamental thought behind this technique is to incorporate a homotopy parameter, say n, which changes from 0 to $\frac{1}{n}$, and a nonzero auxiliary parameter h. As n approaches 1, the system undergoes a series of deformations, with each stage's solution similar to the previous stage.

The construction of the paper is as follows. In section 2, the fundamentals of q-fractional derivatives and integrals are provided. The HAM is presented. In section 3, for the solution of q-FDEs. In section 4, we produce a numerical result to illustrate the efficiency of the method. Finally, conclusions are presented in section 5.

2. Preliminaries

Definition 2.1. *[1] Let $\alpha > 0$, The R-Liouville type definition of q-fractional integral of a function $f(\chi)$ is defined as*

$$J_q^\alpha f(\psi) = \frac{1}{\Gamma_q(\alpha)} \int_0^\psi (\psi - qt)^{\alpha-1} f(t) \ d_q t, \tag{2.1}$$
$$J_q^0 f(\psi) = f(\psi).$$

Some properties of J_q^α [1] are as follows

$$J_q^\alpha J_q^\beta f(\psi) = J_q^{\alpha+\beta} f(\psi),$$
$$J_q^\alpha J_q^\beta f(\psi) = J_q^\beta J_q^\alpha f(\psi),$$
$$D_q^\alpha \psi^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+1-\alpha)} \psi^{\mu-\alpha},$$
$$I_q^\alpha \psi^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+1+\alpha)} \psi^{\mu+\alpha}.$$

Definition 2.2. *[1] Let $\alpha > 0$, The Caputo type definition of q-fractional integral of a function $f(\psi)$ is defined as*

$$D_q^\alpha f(\psi) = \frac{1}{\Gamma_q(m-\alpha)} \int_0^\psi (\psi - qt)^{m-\alpha-1} \frac{d_q^m}{d_q \chi^m} f(\psi) \ d_q t, \tag{2.2}$$

$(m-1) < \alpha < m$, $\psi > 0$. The basic properties of the Caputo q-fractional derivatives are as follows:

Let $f \in C_\mu^\eta$, $\mu \leq -1, \eta - 1 \leq \alpha \leq \eta, \eta \in N$, then

$$J_q^\alpha D_q^\alpha f(\psi) = f(\psi) - \sum_{\kappa=0}^{\eta-1} f^{(\kappa)}(0^+) \frac{\psi^\kappa}{\kappa_q!}. \tag{2.3}$$

Definition 2.3. *[1] The q-Leibniz rule for a q-derivative of a product of two functions*

$$D_q^m [f(\psi) g(\psi)] = \sum_{\kappa=0}^{m} \binom{\eta}{\kappa}_q D_q^{m-\kappa} f(q^\kappa \psi) D_q^m g(\psi), \tag{2.4}$$

and

$$D_q^{m-\kappa} f(q^\kappa \psi)|_{\psi=0} = a_{\kappa,m-\kappa}(q) D_q^{m-\kappa} f(\psi)|_{\psi=0}, \tag{2.5}$$

where

$$a_{\kappa,m-\kappa}(q) = \sum_{i=0}^{\kappa} \sum_{j=0}^{m-\kappa} i_q! \ \binom{\kappa}{i}_q \binom{m-\kappa}{j}_q q^{\frac{i(i-1)}{2}+ij}. \tag{2.6}$$

Theorem 2.1. *[19] Suppose the homotopy series is $\xi(\psi,\omega) = \sum\limits_{i=0}^{\infty} \nu_i(\psi)\omega^i$, then*

- (i) $D_q^m\Big[\xi(\psi,\omega)\Big] = \nu_m(\psi)\omega^m = \frac{1}{m_q!}\frac{d_q^m \xi}{d_q \omega^m}|_{\omega=0}$.
- (ii) $D_q^m\Big[\omega^\xi(\psi,\omega)\Big] = \nu_{m-1}(\psi)$.
- (iii) *If $\mathbb{L}$ is a linear operator independent of ω, then* $D_q^m\Big[\mathbb{L}\Big(\xi(\psi,\omega)\Big)\Big] = \mathbb{L}\Big[D_q^m(\xi(\psi,\omega))\Big]$.

(iv) *If $\nu_0(\psi)$ is the initial solution, then*
$D_q^m\Big[(1-\omega)\mathbb{L}\Big(\Upsilon(\psi,\omega)-\nu_0\Big)\Big] = \mathbb{L}\Big[\nu_m(\psi)-\Upsilon_m\nu_{m-1}(\psi)\Big]$.

(v) *If the zeroth order deformation equation is given by*
$(1-\omega)\mathbb{L}\Big(\xi(\psi,\omega)-\nu_0\Big) = hH(\psi,\omega)N[\xi(\psi,\omega)]$,
then the corresponding m'th order deformation equation for $m \geq 1$ is given by
$\mathbb{L}\Big[\nu_m(\psi,\omega)-\Upsilon_m\nu_{m-1}(\psi,\omega)\Big] = hH(\psi)D_q^{m-1}N\Big[\Upsilon(\psi,\omega)\Big]$, *where Υ_m is given by*
$$\Upsilon_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

(vi) $D_q^m\Big[\nu^2(\psi,\omega)\Big] = \sum_{\kappa=0}^m a_{\kappa,m-\kappa}(q)\nu_{m-\kappa}\nu_k$, *where $a_{\kappa,m-\kappa}(q)$ is specified by* (2.6) *and $m \geq 0$ is positive integer.*

3. Generalization of homotopy analysis method for q-FDEs

Consider the q-fractional nonlinear differential equation as

$$N\Big[\nu(\psi)\Big] = 0, \tag{3.1}$$

where N is a nonlinear q-fractional differential operator, $\nu(\chi)$ is treated as an unknown function and χ indicates independent variable. Now, we establish the zero-order deformation equation as follows:

$$(1-n\omega)\mathbb{L}\Big[\xi(\psi,\omega)-\nu_0(\psi)\Big] = \omega hH(\psi)N\Big[\xi(\psi,\omega)\Big]. \tag{3.2}$$

Here $\omega \in (0,1)$ and $n \geq 1$ $0 \leq \omega \leq \frac{1}{n}$ denotes the embedding parameter, $h \neq 0$ indicate as a control parameter, $H(\psi)$ considered as a non zero auxiliary function, $\xi(\psi,\omega)$ is unknown function and also ν_0 is consider as initial guess of $\xi(\psi)$.
When $\omega = 0$ and $\omega = \frac{1}{n}$, it is obvious that

$$\xi(\psi,0) = \nu_0(\psi), \;\; \xi(\psi,1) = \nu(\psi). \tag{3.3}$$

Which is similar to the given nonlinear q-fractional differential equation. So $\xi(\psi,\omega)$ is the solution of the given problem. When ω varies from initial zero to end one, then the solution $\xi(\psi,\omega)$ also varies from $\nu_0(\psi)$ *to* $\nu(\psi)$.
Now expanding $\xi(\psi,\omega)$ in q-Taylor series [4] with respected to the embedding parameter $\omega = \frac{1}{n}$, then

$$\xi(\psi,\omega) = \nu_0(\psi) + \sum_{m=1}^{\infty}\nu_m(\psi)\omega^m, \tag{3.4}$$

where

$$\nu_m(\psi) = \frac{1}{m_q!}\frac{\partial_q^m\xi(\psi,\omega)}{\partial_q\omega^m}|_{\omega=0}. \tag{3.5}$$

By assuming the initial guess $\nu_0(\psi)$, auxiliary linear operator $\mathbb{L}$, the embedded parameter ω, and the auxiliary function $H(\psi)$, accordingly taken then the series is convergent at $\omega = 1$, as from (3.3), we get

$$\xi(\psi, 1) = \nu(\psi) = \nu_0(\psi) + \sum_{m=1}^{\infty} \nu_m(\psi)(\frac{1}{n})^m, \tag{3.6}$$

where $\nu_m(\psi)$ can be determined by the higher-order deformation as follows. Define the vector form of $\nu_m(\psi)$ as

$$\overrightarrow{\nu_m(\psi)} = [\nu_0(\psi), \nu_1(\psi), \ldots, \nu_m(\psi)]. \tag{3.7}$$

from(3.2), then

$$\mathbb{L}\Big[\nu_m(\psi) - \Upsilon_m \nu_{m-1}(\psi)\Big] = hH(\psi)\Big[\Lambda_m(\nu_{m-1}(\psi)\Big], \tag{3.8}$$

where Υ_m is given by Theorem 2.1 and hence, the HAM series solution is

$$\nu(\psi) = \nu_0(\psi) + \nu_1(\psi) + \nu_2(\psi) + \nu_3(\psi) + \nu_4(\psi) + \ldots. \tag{3.9}$$

4. Numerical results

Example 4.1. *Let us take the q-fractional linear initial value problem*

$$D_q^{\alpha}\nu(\psi) + \nu(\psi) = 0, \;\; 0 < \alpha \leq 2, \tag{4.1}$$

and

$$\nu(0) = 1, \;\; \nu'(0) = 0. \tag{4.2}$$

The second initial condition is applicable for $\alpha > 1$. For $q \longrightarrow 1$, (4.1) becomes normal fractional problem.

Now, apply the homotopy technique to the above problem, take into consideration the auxiliary linear operator $\mathbb{L} = D_q^{\alpha}$, and establish the zeroth-order deformation as

$$(1 - n\omega)\mathbb{L}\Big[\xi(\psi, \omega) - \nu_0(\psi)\Big] = \omega hH(\psi)\Big[D_q^{\alpha}\nu(\psi, \omega) + \nu(\psi, \omega)\Big]. \tag{4.3}$$

By using the initial conditions choose the initial guess,

$$\nu_0(\psi) = 1. \tag{4.4}$$

And let us also take $H(\psi) = 1$. Therefore, the mth-order deformation can be considered as

$$D_q^{\alpha}\Big[\nu_m(\psi) - \Upsilon_m \nu_{m-1}(\psi)\Big] = h\Big[D_q^{\alpha}(\nu_{m-1}(\psi) + \nu_{m-1}(\psi)\Big]. \tag{4.5}$$

Applying the J_q^{α} operator, which is the contrary operator of D_q^{α} on two sides of the mth-order deformation equation, we get

$$\nu_m(\psi) = \Upsilon_m \nu_{m-1}(\psi) + hJ_q^{\alpha}\Big[\Lambda_m(\nu_{m-1}(\psi))\Big]. \tag{4.6}$$

The initial terms of ν_m are as follows;

$$
\begin{aligned}
\nu_1(\psi) &= \frac{h\psi^{\alpha}}{\Gamma_q(\alpha+1)},\\
\nu_2(\psi) &= \frac{h(h+1)\psi^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{h^2\psi^{2\alpha}}{\Gamma_q(2\alpha+1)},\\
\nu_3(\psi) &= \frac{h(h+1)^2\psi^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{2h^2(h+1)\psi^{2\alpha}}{\Gamma_q(2\alpha+1)} + \frac{h^3\psi^{3\alpha}}{\Gamma_q(3\alpha+1)},\\
\nu_4(\psi) &= \frac{h(h+1)^3\psi^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{3h^2(h+1)^2\psi^{2\alpha}}{\Gamma_q(2\alpha+1)} + \frac{3h^3(h+1)\psi^{3\alpha}}{\Gamma_q(3\alpha+1)} + \frac{h^4\psi^{4\alpha}}{\Gamma_q(4\alpha+1)},\\
\nu_5(\psi) &= \frac{h(h+1)^4\psi^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{4h^2(h+1)^3\psi^{2\alpha}}{\Gamma_q(2\alpha+1)} + \frac{6h^3(h+1)^2\psi^{3\alpha}}{\Gamma_q(3\alpha+1)} + \frac{4h^4(h+1)\chi^{4\alpha}}{\Gamma_q(4\alpha+1)}\\
&\quad + \frac{h^5\psi^{5\alpha}}{\Gamma_q(5\alpha+1)},
\end{aligned}
$$

and so on. Hence the series solution of (4.1) *and* (4.2) *is*

$$
\begin{aligned}
\nu(\psi) &= \nu_0(\psi) + \frac{\nu_1(\psi)}{n} + \frac{\nu_2(\psi)}{n^2} + \frac{\nu_3(\psi)}{n^3}\cdots\\
&= 1 + \frac{h}{n}\left[1 + \left(\frac{n+h}{n}\right) + \left(\frac{n+h}{n}\right)^2 + \ldots\right]\frac{\psi^{\alpha}}{\Gamma_q(\alpha+1)}\\
&\quad + \left(\frac{h}{n}\right)^2\left[1 + 2\left(\frac{n+h}{n}\right) + 3\left(\frac{n+h}{n}\right)^2 + \cdots\right]\frac{\psi^{2\alpha}}{\Gamma_q(2\alpha+1)}\\
&\quad + \left(\frac{h}{n}\right)^3\left[1 + 3\left(\frac{n+h}{n}\right) + 6\left(\frac{n+h}{n}\right)^2 + \ldots\right]\frac{\psi^{3\alpha}}{\Gamma_q(3\alpha+1)} + \cdots .
\end{aligned}
\tag{4.7}
$$

First, analyze the effect of the additional parameter h on series convergence by plotting the assumed h-curves for 5th-order approximation (4.7) *at $\psi = 0.5$, when $\alpha = 2$. We actually have the option of selecting the additional parameter h. To analyze the effect of h on the solution series, first study the convergence of some connected series, such as $D_q^{\alpha}\nu(\psi)$ for $q = 0.25, 0.5, 0.75, 1$ and $\alpha = 1.25, 1.5, 1.75, 2$. These curves are made up of a horizontal line segment, which represents the correct region of h that ensured the connected series' convergence. It is noticed that the correct region for h is $-3 < h < 1$ as shown in Fig. 4.1. As a result, the interval's midpoint. Hence, $h = -n$ is a proper choice for h where the numerical solution converges.*

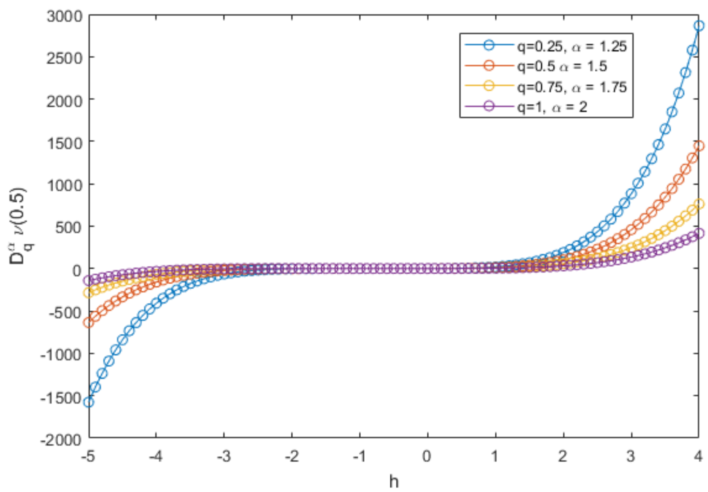

FIGURE 1. The h-curves of $D_q^\alpha \nu(\psi)$ at $\psi = 0.5$ for 5th-order approximations.

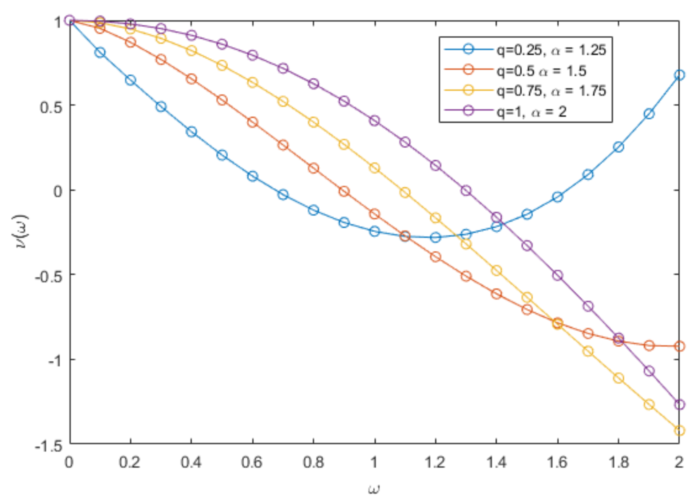

FIGURE 2. The 5th-order approximate solution $\nu(\psi)$ in [0, 2].

For $q = 1$, (4.1) and (4.2) becomes fractional initial value problem as in problem 1 [5] and the results are the same. If $q = 1$, $\alpha = 2$, $h = -n$, and $n = 1$ then the solution of (4.1) and (4.2) is $\nu(\psi) = \cos(\psi)$. The 5th-order approximations when $h = -n$, the solution ν is depicted in Fig. 4.1 for various values of q and α.

Example 4.2. *Let us take the q-fractional nonlinear initial value problem*

$$D_q^\alpha \nu = \nu^2 + 1, \ \ p-1 < \alpha < p, p \in N, \ \ 0 < \psi < 1, \tag{4.8}$$

with the initial condition

$$\nu^k(0) = 0, \ \ where \ \ k = 0, 1, \ldots, p-1. \tag{4.9}$$

The exact solution, when $\alpha = 1$, $q \longrightarrow 1$ is $\nu = \tan(\psi)$.

Now, apply the HAM technique to the given problem, consider $\mathbb{L} = D_q^{\alpha}$ is the auxiliary linear operator and establish the zeroth-order deformation as

$$(1-n\omega)\mathbb{L}\Big[\xi(\psi,\omega)-\nu_0(\psi)\Big] = \omega h H(\psi)\Big[N\Big(\xi(\psi,\omega)\Big)\Big]. \tag{4.10}$$

$$N\Big[\xi(\psi,\omega)\Big] = D_q^{\alpha}\nu(\psi,\omega)+\nu^2(\psi,\omega)-1. \tag{4.11}$$

choose the initial guess,

$$\nu_0 = \frac{\psi^{\alpha}}{\Gamma_q(\alpha+1)}. \tag{4.12}$$

And also choose $H(\psi)=1$. Hence the mth-order deformation can be given by

$$\mathbb{L}[\nu_m(\psi)-\Upsilon_m\nu_{m-1}(\psi)] = h[\Lambda_m(\nu_{m-1}(\psi))], \tag{4.13}$$

where

$$\Lambda_m(\nu_{m-1}(\psi)) = \frac{1}{(m-1)!}\frac{\partial_q^{m-1}N(\nu(\psi,\omega))}{\partial_q\omega^{m-1}}. \tag{4.14}$$

From Theorem 2.1, we have

$$\Lambda_m(\nu_{m-1}(\psi)) = D_q^{m}\nu_{m-1}-\sum_{j=0}^{m-1}\nu_j\nu_{m-1-j}-(1-\Upsilon_m). \tag{4.15}$$

Applying the J_q^{α} operator, which is the contrary operator of D_q^{α} on two side of the mth-order deformation equation, we get

$$\nu_m(\psi) = \Upsilon_m\nu_{m-1}(\psi)+hJ_q^{\alpha}[\Lambda_m(\nu_{m-1}(\psi))]. \tag{4.16}$$

The first 5 terms of the series solution of ν_m are given below;

$$\begin{aligned}
\nu_0(\psi) &= \Theta_0\psi^{\alpha},\\
\nu_1(\psi) &= -h\Theta_1\psi^{3\alpha},\\
\nu_2(\psi) &= -h(h+1)\Theta_1\psi^{3\alpha}+h^2\Theta_2\psi^{5\alpha},\\
\nu_3(\psi) &= -h(h+1)^2\Theta_1\psi^{3\alpha}+2h^2(1+h)\Theta_2\psi^{5\alpha}-h^3\Theta_3\psi^{7\alpha},\\
\nu_4(\psi) &= -h(h+1)^3\Theta_1\psi^{3\alpha}+3h^2(1+h)^2\Theta_2\psi^{5\alpha}-3h^3(1+h)\Theta_3\psi^{7\alpha}+h^4\Theta_4\psi^{9\alpha},
\end{aligned}$$

and so on, where

$$\begin{aligned}
&\Theta_0 = \frac{1}{\Gamma_q(\alpha+1)},\ \ \Theta_1 = \frac{\Gamma_q(2\alpha+1)}{\Gamma_q(3\alpha+1)}{\Theta_0}^2,\ \ \Theta_2 = \frac{\Gamma_q(4\alpha+1)}{\Gamma_q(5\alpha+1)}(2\Theta_0\,\Theta_1),\\
&\Theta_3 = \frac{\Gamma_q(6\alpha+1)}{\Gamma_q(7\alpha+1)}(2\Theta_0\,\Theta_2+{\Theta_1}^2),\ \ \Theta_4 = \frac{\Gamma_q(8\alpha+1)}{\Gamma_q(9\alpha+1)}(2\Theta_0\,\Theta_3+2\Theta_1\,\Theta_2).
\end{aligned}$$

When $q = 1$, the same result will be appear in [5] the series solution of the q-FDE is given by

$$\begin{aligned} \nu(\psi) &= \nu_0(\psi) + \nu_1(\psi) + \cdots \\ &= \Theta_0\psi^\alpha - \frac{h}{n}\Big[1 + (n+h) + (n+h)^2 + \cdots\Big]\Theta_1\chi^{3\alpha} \\ &\quad + \left(\frac{h}{n}\right)^2\Big[1 + 2(n+h) + 3(n+h)^2 + \cdots\Big]\Theta_2\psi^{5\alpha} \\ &\quad - \left(\frac{h}{n}\right)^3\Big[1 + 3(n+h) + 3(n+h)^2 + \cdots\Big]\Theta_3\omega^{7\alpha} + \cdots . \end{aligned} \tag{4.17}$$

We get the exact solution if we take $h = -n$. First, analyze the effect of the additional parameter h on series convergence by plotting the assumed h-curves for 5th-order approximation (4.17) *at $\psi = 0.5$, when $\alpha = 1$. We actually have the option of selecting the additional parameter h. To analyze the effect of h on the solution series, we first study the convergence of some connected series, such as $D_q^\alpha\nu(\psi)$ for $q = 0.25, 0.5, 0.75, 1$ and $\alpha = 1.25, 1.5, 1.75, 2$. These curves are made up of a plane segment that represents the correct region of h that ensured the connected series' convergence. It is noticed that the correct region for h is $-2 < h < 0$ as shown in Fig. 4.2. As a result, the interval's midpoint. Hence, $h = -n$ is a proper choice for h where the numerical solution converges.*

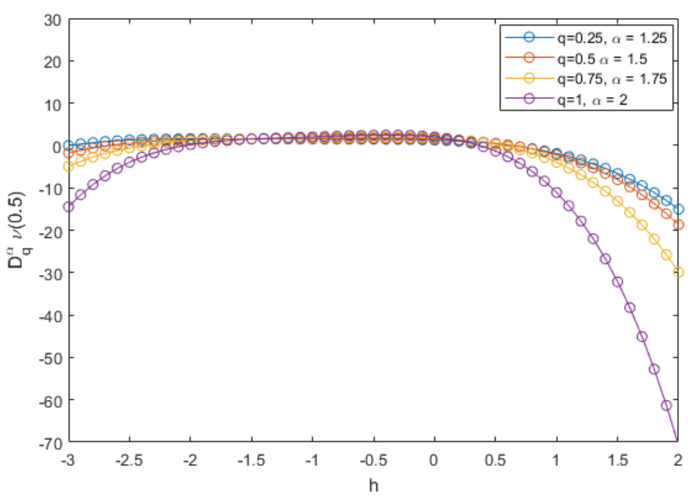

FIGURE 3. The h-curves of $D_q^\alpha\nu(\psi)$ at $\psi = 0.5$ for 5th-order approximations.

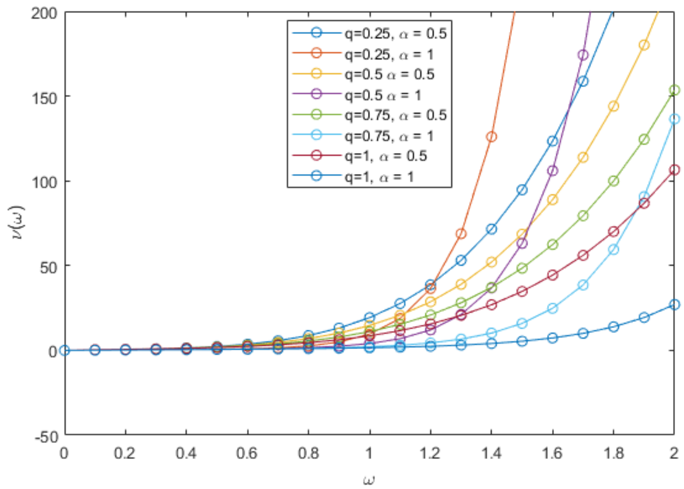

FIGURE 4. The 5th-order approximate solution $\nu(\psi)$ in [0, 2].

For $q = 1$, (4.8) and (4.9) becomes fractional initial value problem as in problem 1 [5] and the results are the same. If $q = 1$, $\alpha = 1$, $h = -n$, and $n = 1$ then the solution of (4.8) and (4.9) is $\nu(\psi) = \tan(\psi)$. For 5th-order approximations and $h = -n$, the approximate solution of ν is depicted in Fig. 4.2 for various values of q and α.

5. Conclusions

The HAM was successfully applied to obtain both the exact and analytical solutions to non-linear q-FDEs. The q-Taylor series expanded non-linear terms involving radical powers. The convergence of the series solution was obtained from the h-curves of $D_q^\alpha \nu(\chi)$ for a fixed value of χ and various values of α and q. The dependability of HAM, as well as the reduction in computations, allow it can be used for a wider range of applications.

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