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An Algorithm for the Solution of Second Order Linear Fuzzy System With Mechanical Applications

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Abstract. In this paper, we consider homogeneous and non-homogeneous second order linear fuzzy systems under granular differentiability. The concept of continuous n-dimensional fuzzy functions on the space of n-dimensional fuzzy numbers are introduced. Developed an algorithm for the solution of a non-homogeneous second order linear fuzzy system under granular differentiability. The proposed algorithm is applied to solve some well-known mechanical problems with fuzzy uncertainty.

1. Introduction

Mathematical models can be explained through fuzzy differential equations (FDE). The innovative work on system of fuzzy differential equations (SFDEs) extended from population models, bio informatics, quantum optics, and soft computing models. Second-order linear fuzzy systems (SLFS) are modeled by behaviors of many dynamical systems with uncertainty. SLFSs specifically appear in many spring-mass mechanical systems with uncertainty. Fard and Ghal-EH [3] proposed a numerical method to solve SFDEs under H-differentiability. Gasilov et al. [4] presented a solution method for SFDEs with fuzzy initial conditions. Mondal et al. [7] analyzed adaptive schemes to study the SFDEs. Barazandeh and Ghazanfari [1] obtained the solutions for SFDEs applying variation iteration technique. Keshavarz et al. [5] enhanced to obtain an analytical solution for SFDEs using gH-differentiability. Boukezzoula

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et al. [2] enhanced a method to solve the SFDEs with variables as fuzzy intervals. The limitations of previous methods for dealing with SFDEs are derivatives do not always exist, monotonicity of the uncertainty, doubling properties, unnatural behavior in modeling phenomenon, and multiplicity of solutions.

Piegat and Landowski [11] introduced horizontal membership function (HMF), and their applications. Piegat and Pluciński [12] was stated the difference between relative distance measure interval arithmetic (RDM-IA) yields a multidimensional answer while the results produced with SIA. Mazandarani et al. [6] elaborated the concept of HMF, granular differentiability (gr-differentiability) and granular integrability (gr-integrability). Najariyan and Zhao [9] offered a solution to the fuzzy dynamical system under gr-differentiability. Nagalakshmi et al. [8] generalized the concept of fuzzy numbers to n-dimensional fuzzy numbers and developed an algorithm to solve system of first-order FBVPs under the concept of gr-differentability.

In this manuscript, consider two types of SLFSs under gr-differentiability. The upcoming sections of this manuscript are along these lines. Section 2, presents basic definitions and propositions related to gr-differentiability of n-dimensional fuzzy valued function. Section 3, an algorithm is presented as a working method to solve SLFSs under gr-differentiability. In Section 4, we describe mechanical applications such as automobile two-axles, railway cars system, and spring-mass systems to highlight the proposed algorithm. Section 5, Conclusions and future works are analyzed.

2. Preliminaries

For a later discussion, this section provides some essential notations, definitions, and findings.
Suppose that the membership function, $q : R \to [0, 1]$ of a fuzzy subset of the real number set R, satisfies the following conditions:

(i) $q(t_0) = 1$ for at least one $t_0 \in R$.
(ii) $q(\lambda y + (1-\lambda)z) \geq min\{q(y), q(z)\}, \forall \lambda \in [0, 1],\ y, z \in R$.
(iii) q is upper semi continuous on R.
(iv) $cl\{t \in R; q(t) > 0\}$ is compact.

Then it is called a fuzzy number (FN). Here $q(t)$ is the membership degree of $t, \forall t \in R$. The λ-level sets of q are defined by $[q]^\lambda = \{t \in R : q(t) \geq \lambda\} = [q_l^\lambda, q_r^\lambda]$, for $0 < \lambda \leq 1$ and $[q]^0 = cl\{t \in R : q(t) > 0\}$. Let R_F denotes the space of FNs in R.
Refer to [6] for definitions, notations, and essential findings regarding HMFs, first-order granular derivative (gr-derivative), and granular integration (gr-integrations) of FNs in R.

Definition 2.1. *Suppose that $p, q \in R_F$, whose HMFs are $p_{gr}(\lambda, \alpha_p)$ and $q_{gr}(\lambda, \alpha_q)$ respectively. Then $r = p * q \in R_F$, such that $H(r) \triangleq p_{gr}(\lambda, \alpha_p) o\, q_{gr}(\lambda, \alpha_q)$, where "o" and "$*$" denotes any one of the operations addition, multiplication, subtraction and division in R and R_F, respectively and $0 \notin q_{gr}(\lambda, \alpha_q)$ if "$*$" denotes the division. That is*

(1) $H(p \oplus q) \triangleq p_{gr}(\lambda, \alpha_p) + q_{gr}(\lambda, \alpha_q)$,

(2) $H(p \otimes q) \triangleq p_{gr}(\lambda, \alpha_p) q_{gr}(\lambda, \alpha_q)$,

(3) $H(p \ominus q) \triangleq p_{gr}(\lambda, \alpha_p) - q_{gr}(\lambda, \alpha_q)$,

(4) $H(p \oslash q) \triangleq p_{gr}(\lambda, \alpha_p) \div q_{gr}(\lambda, \alpha_q)$,

(5) $H(k \odot q) \triangleq k \ q_{gr}(\lambda, \alpha_q)$,

where $k \in R$ and $p, q, r \in R_F$.

Definition 2.2. *[9] Let $f : [a, b] \to R_F$, be the FF. If there exists $\frac{d^2_{gr} f(t_0)}{dt^2} \in R_F$, such that*

$$\lim_{h \to 0} \frac{f'(t_0 + h) \ominus f'(t_0)}{h} = \frac{d^2_{gr} f(t_0)}{dt^2} = f''_{gr}(t_0),$$

then f is said to be second order gr-differentiable at a point $t_0 \in [a, b]$.

Theorem 2.1. *[9] Let $f : [a, b] \to R_F$. Then f is twice gr-differentiable if and only if its HMF is twice differentiable with respect to $t \in [a, b]$. Moreover,*

$$H\left(\frac{d^2_{gr} f(t)}{dt^2}\right) = \frac{\partial^2 f_{gr}(t, \lambda, \alpha_f)}{\partial t^2}.$$

Proposition 2.1. *Let $f : [a, b] \to R_F$ be a FF, with $[f(t)]^\lambda = \left[f_l^\lambda(t), f_r^\lambda(t)\right]$. The FF f is gr-differentiable twice on $[a, b]$ if and only if $(f_l^\lambda)'(t)$ and $(f_r^\lambda)'(t)$ are differentiable on $[a, b]$.*

Proof. Since $[f(t)]^\lambda = \left[f_l^\lambda(t), f_r^\lambda(t)\right]$, then $f_{gr}(t, \lambda, \alpha_g) = f_l^\lambda(t) + (f_r^\lambda(t) - f_l^\lambda(t))\alpha_f$, where λ, $\alpha_f \in [0, 1]$. From Definition 2.2 and Theorem 2.1, we have

$$\begin{aligned} &\text{Suppose that } f(t) \text{ is a gr-differentiable twice on } [a, b] \\ \iff& \frac{\partial^2 f_{gr}(t, \lambda, \alpha_f)}{\partial t^2} = (f_l^\lambda)''(t) + ((f_r^\lambda)''(t) - (f_l^\lambda)''(t))\alpha_f \\ \iff& (f_l^\lambda)'(t) \text{ and } (f_r^\lambda)'(t) \text{ are differentiable on } [a, b]. \end{aligned}$$

. □

Definition 2.3. *[8] Let $R_F^n = \underbrace{R_F \times R_F \times R_F \times \cdots \times R_F}_{n\,times}$, be the space of n-dimensional fuzzy vectors whose components are fuzzy numbers. Then the addition and scalar multiplication defined component wise as follows:*

If $u = (u_1, u_2, \cdots, u_n)$, $v = (v_1, v_2, \cdots, v_n) \in R_F^n$, then

(i) $u \oplus v = (u_1 \oplus v_1, u_2 \oplus v_2, \cdots, u_n \oplus v_n)$,

(ii) $k \odot u = (k \odot u_1, k \odot u_2, \cdots, k \odot u_n)$,

where $u_i, v_i \in R_F^n$, $i = 1, 2, \cdots, n$ and $k \in R$.

Definition 2.4. *If $u = (u_1, u_2, \cdots, u_n) \in R_F^n$, as $u_i \in R_F$, $i = 1, 2, \cdots, n$. Then the HMF for $u \in R_F^n$ is defined by $u_{gr}(\lambda, \alpha_u) = (u_{1gr}(\lambda, \alpha_1), u_{2gr}(\lambda, \alpha_2), \cdots,$*
$u_{ngr}(\lambda, \alpha_n))$, where λ, $\alpha_1, \cdots, \alpha_n \in [0, 1]$.

Proposition 2.2. *Let u and v be two n-dimensional fuzzy vectors. Then u and v are said to be equal if and only if $H(u) = H(v)$, for all $\alpha_u = \alpha_v \in [0, 1]$.*

Proof. Since $u, v \in R_F^n$, then $u = (u_1, u_2, \cdots, u_n), v = (v_1, v_2, \cdots, v_n)$, for u_i, $v_i \in R_F$, $i = 1, 2, \cdots, n$.

$$\begin{aligned}\text{Consider, } u = v &\iff (u_1, u_2, \cdots, u_n) = (v_1, v_2, \cdots, v_n)\\ &\iff u_i = v_i,\ i = 1, 2, \cdots, n.\\ &\iff H(u_i) = H(v_i), \text{ for all } \alpha_{u_i} = \alpha_{v_i} \in [0, 1], i = 1, 2, \cdots, n.\\ &\iff (H(u_1), H(u_2), \cdots, H(u_n)) = (H(v_1), H(v_2), \cdots, H(v_n))\\ &\iff H(u) = H(v), \text{ for all } \alpha_u = \alpha_v \in [0, 1],\end{aligned}$$

where $\alpha_u \triangleq (\alpha_{u_1}, \alpha_{u_2}, \cdots, \alpha_{u_n})$ and $\alpha_v \triangleq (\alpha_{v_1}, \alpha_{v_2}, \cdots, \alpha_{v_n})$. □

Definition 2.5. *[8] Let $u, v \in R_F^n$. The function $\mathcal{D}_{gr}^n : R_F^n \times R_F^n \to R^+ \cup \{0\}$, defined by*

$$\mathcal{D}_{gr}^n(u, v) = \sup_{\lambda} \max_{\alpha_u, \alpha_v} \|u_{gr}(\lambda, \alpha_u) - v_{gr}(\lambda, \alpha_v)\|,$$

which is called a n-dimensional granular distance between two n-dimensional fuzzy vectors u and v, where $\|.\|$ represents Euclidean norm in R^n.

Proposition 2.3. *The function $\mathcal{D}_{gr}^n$ is a metric on the space of R_F^n.*

Proof. Suppose that R_F^n is a non-empty set and $\mathcal{D}_{gr}^n : R_F^n \times R_F^n \to R^+ \cup \{0\}$ is real-valued function.

(i) Consider,

$$\mathcal{D}_{gr}^n(u, v) = \sup_{\lambda} \max_{\alpha_u, \alpha_v} \|u_{gr}(\lambda, \alpha_u) - v_{gr}(\lambda, \alpha_v)\| > 0.$$

(ii) Consider,

$$\begin{aligned}\mathcal{D}_{gr}^n(u, v) = 0 &\iff \sup_{\lambda} \max_{\alpha_u, \alpha_v} \|u_{gr}(\lambda, \alpha_u) - v_{gr}(\lambda, \alpha_v)\| = 0\\ &\iff \|u_{gr} - v_{gr}\| = 0\\ &\iff u_{gr} - v_{gr} = 0\\ &\iff u_{gr} = v_{gr}\\ &\iff u = v.\end{aligned}$$

(iii) Consider,

$$\begin{aligned}\mathcal{D}_{gr}^n(u, v) &= \sup_{\lambda} \max_{\alpha_u, \alpha_v} \|u_{gr}(\lambda, \alpha_u) - v_{gr}(\lambda, \alpha_v)\|\\ &= \sup_{\lambda} \max_{\alpha_u, \alpha_v} \|v_{gr}(\lambda, \alpha_v) - u_{gr}(\lambda, \alpha_u)\|\\ &= \mathcal{D}_{gr}^n(v, u).\end{aligned}$$

(iv) Consider,

$$
\begin{aligned}
\mathcal{D}^n_{gr}(u,w) &= \sup_{\lambda} \max_{\alpha_u,\alpha_w} \|u_{gr}(\lambda,\alpha_u) - w_{gr}(\lambda,\alpha_w)\| \\
&= \sup_{\lambda} \max_{\alpha_u,\alpha_v,\alpha_w} \|u_{gr}(\lambda,\alpha_u) - v_{gr}(\lambda,\alpha_v) + v_{gr}(\lambda,\alpha_v) - w_{gr}(\lambda,\alpha_w)\| \\
&\leq \sup_{\lambda} \max_{\alpha_u,\alpha_v} \|u_{gr}(\lambda,\alpha_u)_{gr} - v_{gr}(\lambda,\alpha_v)\| + \sup_{\lambda} \max_{\alpha_v,\alpha_w} \|v_{gr}(\lambda,\alpha_v) - w_{gr}(\lambda,\alpha_w)\| \\
&= \mathcal{D}^n_{gr}(u,v) + \mathcal{D}^n_{gr}(v,w).
\end{aligned}
$$

From (i)-(iv), $\left(R^n_F, \mathcal{D}^n_{gr}\right)$ is a metric space. □

Theorem 2.2. *$\left(R^n_F, \mathcal{D}^n_{gr}\right)$ is a complete metric space (CMS).*

Proof. If any Cauchy sequence of n-dimensional fuzzy vectors in $\left(R^n_F, \mathcal{D}^n_{gr}\right)$ is convergent then the proof concluded.

Suppose that $u_m \in R^n_F$, $m \geq 1$ is a Cauchy sequence. Then for all $\epsilon_1 > 0$, there exists $N \geq 1$ such that $\mathcal{D}^n_{gr}(u_m, u_{m+p}) < \epsilon_1$, for all $m \geq 1$, $q \geq 1$.

$$
\begin{aligned}
&\mathcal{D}^n{}_{gr}(u_m, u_{m+p}) < \epsilon_1 \\
&\implies \sup_{\lambda} \max_{\alpha_{um},\alpha_{u_{m+p}}} \|u_{m_{gr}}(\lambda,\alpha_{u_m}) - u_{m+p_{gr}}(\lambda,\alpha_{u_{m+p}})\| < \epsilon_1 \\
&\implies \|u_{m_{gr}} - u_{m+p_{gr}}\| < \epsilon_1.
\end{aligned}
$$

Now $\{u_{m_{gr}}\}$ is a Cauchy sequence in the space of R^n. Clearly $\left\{u_{m_{gr}}\right\}$ is convergent in R^n and $u_{mi_{gr}}(\lambda,\alpha_{u_{mi}}) = u_{mi}{}^{\lambda}_{l} + (u^{\lambda}_{mi\,r} - u^{\lambda}_{mi\,l})\alpha_{u_{mi}}$, where $\lambda, \alpha_{u_{mi}} \in [0,1]$.
Since $u_{mi_{gr}}(\lambda,\alpha_{u_{mi}})$ is convergent, so that $u^{\lambda}_{mi\,l}$ and $u^{\lambda}_{mi\,r}$ are convergent.
Suppose that $\lim\limits_{n\to\infty} u^{\lambda}_{mi\,l} = u_i{}^{\lambda}_{l}$ and $\lim\limits_{n\to\infty} u^{\lambda}_{mi\,r} = u_i{}^{\lambda}_{r}$. Since $u^{\lambda}_{mi\,l} \leq u^{\lambda}_{mi\,r}$, so that $u^{\lambda}_{i\ l} \leq u^{\lambda}_{i\ r}$ for all $i = 1, 2, \cdots, n$. If $[u^{\lambda}_{i\ l}, u^{\lambda}_{i\ r}]$, $i = 1, 2, \cdots, n$ are λ-level sets of u_i, then proof will be complete. It is shown in the same manner in the proof of Theorem 4 [6], and therefore is left off. □

Lemma 2.1. *Suppose that $u, v, w, s \in R^n_F$ and $\mu \in R$, then the below results hold:*

(i) $\mathcal{D}^n_{gr}(u \oplus v, w \oplus s) \leq \mathcal{D}^n_{gr}(u,w) + \mathcal{D}^n_{gr}(v,s)$.
(ii) $\mathcal{D}^n_{gr}(\mu \odot u, \mu \odot v) = |\mu|\mathcal{D}^n_{gr}(u,v)$.
(iii) $\mathcal{D}^n_{gr}(u \oplus v, w \oplus v) \leq \mathcal{D}^n_{gr}(u,w)$.

Proof. (i) From Definition 2.5, we have

$$
\begin{aligned}
&\mathcal{D}^n_{gr}(u \oplus v, w \oplus s) \\
&= \sup_{\lambda} \max_{\alpha_u,\alpha_v,\alpha_w,\alpha_s} \|(u_{gr}(\lambda,\alpha_u) + v_{gr}(\lambda,\alpha_v)) - (w_{gr}(\lambda,\alpha_w) + s_{gr}(\lambda,\alpha_s))\| \\
&= \sup_{\lambda} \max_{\alpha_u,\alpha_v,\alpha_w,\alpha_s} \|(u_{gr}(\lambda,\alpha_u) - w_{gr}(\lambda,\alpha_w)) + (v_{gr}(\lambda,\alpha_v) - s_{gr}(\lambda,\alpha_s))\|
\end{aligned}
$$

$$\le \sup_{\lambda} \max_{\alpha_u,\alpha_w} \|u_{gr}(\lambda,\alpha_u) - w_{gr}(\lambda,\alpha_w)| + \sup_{\lambda} \max_{\alpha_v,\alpha_s} |(v_{gr}(\lambda,\alpha_v) - s_{gr}(\lambda,\alpha_s))\|$$
$$= \mathcal{D}^n_{gr}(u,w) + \mathcal{D}^n_{gr}(v,s).$$

(ii) From Definition 2.5, we have

$$\begin{aligned}
\mathcal{D}^n_{gr}(\mu \odot u, \mu \odot v) &= \sup_{\lambda} \max_{\alpha_u,\alpha_v,\alpha_\mu} \|\mu u_{gr}(\lambda,\alpha_u) - \mu v_{gr}(\lambda,\alpha_v)\| \\
&= |\mu| \sup_{\lambda} \max_{\alpha_u,\alpha_v} \|u_{gr}(\lambda,\alpha_u) - v_{gr}(\lambda,\alpha_v)\| \\
&= |\mu|\, \mathcal{D}^n_{gr}(u,v).
\end{aligned}$$

(iii) From (i), we have

$$\begin{aligned}
&\mathcal{D}^n_{gr}(u \oplus v, w \oplus v) \\
&= \mathcal{D}^n_{gr}(u,w) + \mathcal{D}^n_{gr}(v,v) \\
&= \mathcal{D}^n_{gr}(u,w).
\end{aligned}$$

□

Proposition 2.4. *If $f : [a,b] \to R^n_F$ is a fuzzy function, then it is called an n-dimensional fuzzy valued function on $[a,b]$.*

Proof. Since $f : [a,b] \to R^n_F$ is a fuzzy function, then $f(t) \in R^n_F$, for all $t \in [a,b]$.
Therefore $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, for all $t \in [a,b]$ and $f_i(t) \in R_F,\ i = 1,2,\ldots,n$.
Thus $f(t)$ is a n-dimensional fuzzy vector for each $t \in [a,b]$ and hence $f : [a,b] \to R^n_F$, is a n-dimensional fuzzy valued function on $[a,b]$. □

Proposition 2.5. *If $f : [a,b] \to R^n_F$ is a n-dimensional fuzzy valued function, include $mn \in N$ distinct FNs, then the HMF of f is denoted by $H(f(t)) \triangleq f_{gr}(t,\lambda,\alpha_f)$, and interpreted as $f_{gr} : [a,b] \times \underbrace{[0,1] \times [0,1] \times [0,1] \times \cdots \times [0,1]}_{mn\, times} \to R^n$, in which $\alpha_f \triangleq (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m})$, where $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}$ are the mn RDM variables for u_{i_1}, u_{i_2}, …, u_{i_m} for $i = 1,2,\cdots,n$.*

Proof. Since $f : [a,b] \to R^n_F$ is a n-dimensional fuzzy valued function, so that $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, for all $t \in [a,b]$ and $f_i(t) \in R^n_F,\ i = 1,2,\ldots,n$.
Therefore

$$\begin{aligned}
H(f(t)) &= (H(f_1(t)), H(f_2(t)), \ldots, (f_n(t))) \\
f_{gr}(t,\lambda,\alpha_f) &= (f_{1_{gr}}(t,\lambda,\alpha_{1_1},\alpha_{1_2},\ldots,\alpha_{1_m}), f_{2_{gr}}(t,\lambda,\alpha_{2_1},\alpha_{2_2},\ldots,\alpha_{2_m}), \\
&\quad \ldots, f_{n_{gr}}(t,\lambda,\alpha_{n_1},\alpha_{n_2},\ldots,\alpha_{n_m}))
\end{aligned}$$

where $\alpha_f \equiv (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}) \in [0,1],\ i = 1,2,\cdots,n$. □

Definition 2.6. *Let $f : [a, b] \to R^n_F$ be a n-dimensional fuzzy valued function. The limit of $f(t)$ as $t \to p$ is $q \in R^n_F$, which is subject to following conditions:*

(i) *If $p \in (a, b)$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $|t - p| < \delta_1 \implies {D^n}_{gr}(f(t), q) < \epsilon_1$, and write it as $\lim_{t\to p} f(t) = q$.*

(ii) *If $p = b$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $0 < t - b < \delta_1 \implies {D^n}_{gr}(f(t), q) < \epsilon_1$, and write it as $\lim_{t\to b+} f(t) = q$.*

(iii) *If $p = c$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $0 < c - t < \delta_1 \implies {D^n}_{gr}(f(t), q) < \epsilon_1$, and write it as $\lim_{t\to c-} f(t) = q$.*

Definition 2.7. *Let $f : [a, b] \to R^n_F$ be a n-dimensional fuzzy valued function. The function $f(t)$ is said to be continuous at $t = p$ if $f(p) \in R^n_F$, which is subject to following conditions:*

(i) *If $p \in (a, b)$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $|t - p| < \delta_1 \implies {D^n}_{gr}(f(t), f(p)) < \epsilon_1$, and write it as $\lim_{t\to p} f(t) = f(p)$.*

(ii) *If $p = b$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $0 < t-b < \delta_1 \implies {D^n}_{gr}(f(t), f(b)) < \epsilon_1$, and write it as $\lim_{t\to b+} f(t) = f(b)$.*

(iii) *If $p = c$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $0 < c-t < \delta_1 \implies {D^n}_{gr}(f(t), f(c)) < \epsilon_1$, and write it as $\lim_{t\to c-} f(t) = f(c)$.*

Note 2.1. *[8] If $f, h : [a, b] \to R^n_F$ are n-dimensional fuzzy valued functions, then the granular distance is*

$$D_{gr}(f(t), h(t)) = \sup_{\lambda} \max_{\alpha_f,\alpha_h} \|f_{gr}(t, \lambda, \alpha_f) - h_{gr}(t, \lambda, \alpha_h)\|,$$

where $t \in [a, b] \subset R$ and $\lambda, \alpha_f, \alpha_h \in [0, 1]$.

Refer to [8] first-order gr-derivative, and gr-integration for n-dimensional fuzzy valued function. Now, we define second order gr-differentiability for n-dimensional fuzzy valued function.

Definition 2.8. *Let $f : [a, b] \to R^n_F$, be the n-dimensional fuzzy valued function. If there exists $\frac{d^2_{gr} f(t_0)}{dt^2} \in R^n_F$, such that*

$$\lim_{h\to 0} \frac{f'(t_0 + h) \ominus f'(t_0)}{h} = \frac{d^2_{gr} f(t_0)}{dt^2} = f''_{gr}(t_0),$$

this limit is taken in the metric space (R^n_F, D^n_{gr}). Then f is said to be second order gr- differentiable at a point $t_0 \in [a, b]$.

Theorem 2.3. *Let $f : [a, b] \to R^n_F$ be a n-dimensional fuzzy valued function, then f is gr-differentiable if and only if its HMF is differentiable with respect to $t \in [a, b]$. Moreover,*

$$H\left(\frac{d^2_{gr} f(t)}{dt^2}\right) = \frac{\partial^2 f_{gr}(t, \lambda, \alpha_f)}{\partial t^2}.$$

Proof. Assuming that f is second order gr-differentiable then f is first order gr-differentiable and

$$H\left(\frac{d_{gr}f(t)}{dt}\right)=\frac{\partial f_{gr}(t,\lambda,\alpha_f)}{\partial t}.$$

for $t\in(a,b)$. Based on the Definition 2.6 and Definition 2.8, for all $\epsilon_1>0$, there exits $\delta_1>0$ such that $|h|<\delta_1 \implies D^n{}_{gr}(\frac{f'(t+h)\ominus f'(t)}{h},\frac{d^2_{gr}f(t)}{dt^2})<\epsilon_1$

$$\begin{aligned}
&\implies \sup_{\lambda}\max_{\alpha_f}\|\frac{f'_{gr}(t+h,\lambda,\alpha_f)-f'_{gr}(t,\lambda,\alpha_f)}{h}-\frac{d^2_{gr}f_{gr}(t,\lambda,\alpha_f)}{dt^2}\|<\epsilon_1\\
&\implies \|\frac{f'_{gr}(t+h,\lambda,\alpha_f)-f'_{gr}(t,\lambda,\alpha_f)}{h}-\frac{d^2_{gr}f_{gr}(t,\lambda,\alpha_f)}{dt^2}\|<\epsilon_1\\
&\implies \lim_{h\to 0}\frac{f'_{gr}(t+h,\lambda,\alpha_f)-f'_{gr}(t,\lambda,\alpha_f)}{h}=\frac{d^2_{gr}f_{gr}(t,\lambda,\alpha_f)}{dt^2}\\
&\implies \frac{\partial^2 f_{gr}(t,\lambda,\alpha_f)}{\partial t^2}=H\left(\frac{d^2_{gr}f(t)}{dt^2}\right).
\end{aligned}$$

□

Proposition 2.6. *Let $f:[a,b]\to R^n_F$ be a n-dimensional fuzzy valued function defined by $f(t)=(f_1(t),f_2(t),\cdots,f_n(t))$ for all $x\in[a,b]$ and $f_i(t)\in R_F$, with $[f_i(t)]^\lambda=\left[f^\lambda_{i_l}(t),f^\lambda_{i_r}(t)\right]$, $i=1,2,\cdots,n$. The n-dimensional fuzzy valued function f is gr-differentiable twice on $[a,b]$ if and only if $(f^\lambda_{i_l})'(t)$ and $(f^\lambda_{i_r})'(t)$ are differentiable on $[a,b]$, for all $i=1,2,\cdots,n$.*

Proof. Since $f_{gr}(t,\lambda,\alpha_f)=\left(f_{1_{gr}}(t,\lambda,\alpha_1),f_{2_{gr}}(t,\lambda,\alpha_2),\cdots,f_{n_{gr}}(t,\lambda,\alpha_n)\right)$, then

$$\begin{aligned}
f_{gr}(t,\lambda,\alpha_f)=&((f^\lambda_{1_l}(t)+(f^\lambda_{1_r}(t)-f^\lambda_{1_l}(t))\alpha_1),(f^\lambda_{2_l}(t)+(f^\lambda_{2_r}(t)-f^\lambda_{2_l}(t))\alpha_2),\cdots,\\
&(f^\lambda_{n_l}(t)+(f^\lambda_{n_r}(t)-f^\lambda_{n_l}(t))\alpha_n)),
\end{aligned}$$

where λ, $\alpha_f\triangleq(\alpha_1,\alpha_2,\cdots,\alpha_n)\in[0,1]$. From Definition 2.8 and Theorem 2.3, we have

Suppose that $f(t)$ is a gr-differentiable twice on $[a,b]$

$$\begin{aligned}
&\iff \frac{\partial^2 f_{gr}(t,\lambda,\alpha_f)}{\partial t^2}=(((f^\lambda_{1_l})''(t)+((f^\lambda_{1_r})''(t)-(f^\lambda_{1_l})'(t))\alpha_1),\\
&\quad((f^\lambda_{2_l})''(t)+((f^\lambda_{2_r})''(t)-(f^\lambda_{2_l})''(t))\alpha_2),\cdots,((f^\lambda_{n_l})''(t)+((f^\lambda_{n_r})''(t)-(f^\lambda_{n_l})''(t))\alpha_n))\\
&\iff (f^\lambda_{i_l})'(t)\text{ and }(f^\lambda_{i_r})'(t),\text{ are differentiable on }[a,b]\text{ for }i=1,2,\cdots,n.
\end{aligned}$$

□

Definition 2.9. *If a matrix $A=[a_{ij}]_{n\times m}$, for all $a_{ij}\in R_F$, $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$. Then that matrix A is called fuzzy matrix.*

Definition 2.10. *If $A=[a_{ij}]_{n\times m}$ is a fuzzy matrix, then the HMF of A is defined by $H(A)=[H(a_{ij})]_{n\times m}\triangleq[(a_{ij})_{gr}(\lambda,\alpha_{ij})]_{n\times m}$, where $\lambda,\alpha_{ij}\in[0,1]$, $i=1,2,\cdots,n$ and $j=1,2,\cdots,m$.*

3. An algorithm for the solution of system of second order linear fuzzy initial value problems under (SSLFDE) gr-differentiability

Consider a SSLFDEs,

$$Z''_{gr}(t) = A \otimes Z(t) \oplus F(t),\ with\, Z(t_0) = Z_0. \tag{3.1}$$

The matrix form of (3.1) is,

$$\begin{bmatrix} y''_{gr}(t) \\ z''_{gr}(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \oplus \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \tag{3.2}$$

$$\text{subject to, } \begin{bmatrix} y(t_0) \\ z(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \text{ and } \begin{bmatrix} y'(t_0) \\ z'(t_0) \end{bmatrix} = \begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix}. \tag{3.3}$$

The following algorithm describes the procedure to compute λ-cut solution of SSLFDEs (3.1) if it exists.

Step 1 : Applying HMF on both sides of (3.2) and (3.3), we get

$$\begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_y)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_z)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} a_{gr}(\lambda,\alpha_a) & b_{gr}(\lambda,\alpha_b) \\ c_{gr}(\lambda,\alpha_c) & d_{gr}(\lambda,\alpha_d) \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_y) \\ z_{gr}(t,\lambda,\alpha_z) \end{bmatrix} + \begin{bmatrix} f_{gr}(t,\lambda,\alpha_f) \\ g_{gr}(t,\lambda,\alpha_g) \end{bmatrix}, \tag{3.4}$$

$$\text{with, } \begin{bmatrix} y_{gr}(t_0) \\ z_{gr}(t_0) \end{bmatrix} = \begin{bmatrix} y_{0_{gr}}(\lambda,\alpha_{y_0}) \\ z_{0_{gr}}(\lambda,\alpha_{z_0}) \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(t_0) \\ z'_{gr}(t_0) \end{bmatrix} = \begin{bmatrix} y'_{0_{gr}}(\lambda,\alpha_{y'_0}) \\ z'_{0_{gr}}(\lambda,\alpha_{z'_0}) \end{bmatrix}, \tag{3.5}$$

where λ, α_z,α_f, α_g, α_a, α_b,α_c, α_d, α_{y_0}, α_{z_0},$\alpha_{y'_0}$, $\alpha_{z'_0} \in [0,1]$. Here, (3.4) and (3.5) taken as a ordinary second order system of differential equations.

Step 2 : Solving (3.4) and (3.5), we get

$$y_{gr}(t,\lambda,\alpha_y) \text{ and } z_{gr}(t,\lambda,\alpha_z). \tag{3.6}$$

Step 3 : Applying inverse HMF on both sides of (3.6), we get

$$[y(t)]^\lambda = [\inf_{\lambda\le\alpha\le1} \min_{\alpha_y} y_{gr}(t,\alpha,\alpha_y),\ \sup_{\lambda\le\alpha\le1} \max_{\alpha_y} y_{gr}(t,\alpha,\alpha_y)], \tag{3.7}$$

$$[z(t)]^\lambda = [\inf_{\lambda\le\alpha\le1} \min_{\alpha_z} z_{gr}(t,\alpha,\alpha_z),\ \sup_{\lambda\le\alpha\le1} \max_{\alpha_z} z_{gr}(t,\alpha,\alpha_z)], \tag{3.8}$$

which is the required λ-cut solution of SSLFDEs (3.1).

4. Mechanical applications

In this section, we describe mechanical applications [13] of which the uncertain information taken as fuzzy sets.

Example 4.1. *(Automobile with two axles) Now we have an automobile with two axles and distinct front and back suspension systems, we can examine a more realistic model. The suspension system of such a vehicle is seen in Figure 1 . We suppose that the car's body behaves similarly to a solid bar with the dimensions of mass M and length $l = l_1 + l_2$. Its centre of mass c, which is located at a distance l_1 from the front of the vehicle, has a moment of inertia I around it. The vehicle features suspension springs with Hooke's constants s_1 and s_2 for the front and back, respectively. Let $y(t)$ represent the car's vertical displacement from equilibrium while it is moving, and let $z(t)$ represent its angular displacement (in radians) from the horizontal. The equations may then be derived using Newton's laws of motion for linear and angular acceleration as follows:*

$$M \odot y''_{gr}(t) = -(s_1 + s_2) \odot y(t) \oplus (s_1 l_1 - s_2 l_2) \odot z(t),$$

$$I \odot z''_{gr}(t) = (s_1 l_1 - s_2 l_2) \odot y(t) \ominus (s_1 l_1^2 + s_2 l_2^2) \odot z(t),$$

with fuzzy initial values, $y(0) = y_0$, $z(0) = z_0$, $y'_{gr}(0) = y'_0$, $z'_{gr}(0) = z'_0$.

Suppose that $M = 75 lb.s^2/ft, l_1 = 7ft, l_2 = 3ft, s_1 = s_2 = 2000 lb/ft, I = 1000 ft.lb.s^2$ *and the* λ*-level sets of fuzzy initial values are* $[y_0]^\lambda = [z_0]^\lambda = [3+\lambda, 5-\lambda], [y'_0]^\lambda = [5+\lambda, 7-\lambda], [z'_0]^\lambda = [6+\lambda, 8-\lambda]$.

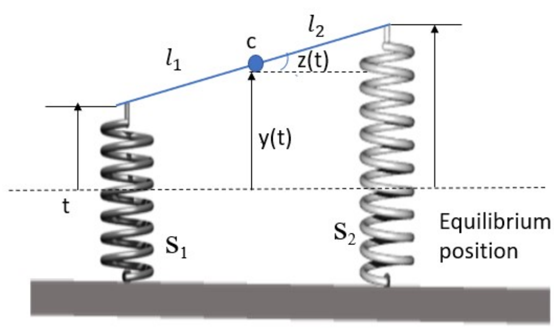


Figure 1. two axles car.

Then the matrix equation is,

$$\begin{bmatrix} y''_{gr}(t) \\ z''_{gr}(t) \end{bmatrix} = \begin{bmatrix} -53.33 & 106.67 \\ 8 & -116 \end{bmatrix} \odot \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \tag{4.1}$$

,

$$\text{subject to,} \quad \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix}. \tag{4.2}$$

Taking HMF on both sides of (4.1) *and* (4.2)*, we have*

$$\begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_y)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_z)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -53.33 & 106.67 \\ 8 & -116 \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_y) \\ z_{gr}(t,\lambda,\alpha_z) \end{bmatrix}, \tag{4.3}$$

$$\textit{subject to,} \quad \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} y_{0_{gr}}(\lambda,\alpha_1) \\ z_{0_{gr}}(\lambda,\alpha_1) \end{bmatrix} \textit{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} y'_{0_{gr}}(\lambda,\alpha_2) \\ z'_{0_{gr}}(\lambda,\alpha_3) \end{bmatrix}, \tag{4.4}$$

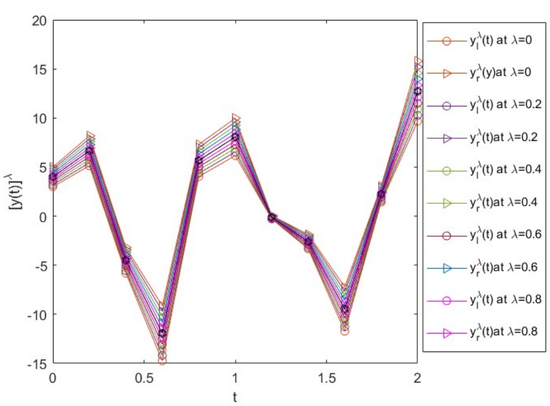
where $y_{0_{gr}}(\lambda,\alpha_1) = z_{0_{gr}}(\lambda,\alpha_1) = [3+\lambda+2(1-\lambda)\alpha_1]$, $y'_{0_{gr}}(\lambda,\alpha_2) = [5+\lambda+2(1-\lambda)\alpha_2]$, $z'_{0_{gr}}(\lambda,\alpha_3) = [6+\lambda+2(1-\lambda)\alpha_3]$*, where* λ, α_1, α_2, $\alpha_3 \in [0,1]$.
The solution for second order system of equations (4.3) *and* (4.4) *are*

$$y_{gr}(t,\lambda,\alpha_1,\alpha_2,\alpha_3) \textit{ and } z_{gr}(t,\lambda,\alpha_1,\alpha_2,\alpha_3). \tag{4.5}$$

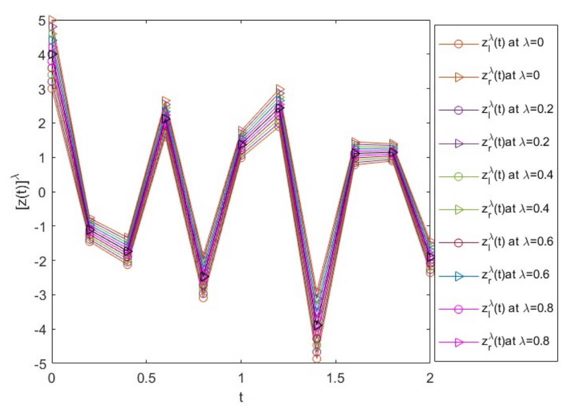
Applying inverse HMF on (4.5)*, we get*

$$[y(t)]^\lambda = [\inf_{\lambda\le\alpha\le1} \min_{\alpha_1,\alpha_2,\alpha_3} y_{gr}(t,\alpha,\alpha_1,\alpha_2,\alpha_3), \sup_{\lambda\le\alpha\le1} \max_{\alpha_1,\alpha_2,\alpha_3} y_{gr}(t,\alpha,\alpha_1,\alpha_2,\alpha_3)],$$
$$[z(t)]^\lambda = [\inf_{\lambda\le\alpha\le1} \min_{\alpha_1,\alpha_2,\alpha_3} z_{gr}(t,\alpha,\alpha_1,\alpha_2,\alpha_3), \sup_{\lambda\le\alpha\le1} \max_{\alpha_1,\alpha_2,\alpha_3} z_{gr}(t,\alpha,\alpha_1,\alpha_2,\alpha_3)].$$

The λ*-level sets solution is enumerated using MATLAB and is illustrated in Figure* 2



(a) λ-level sets of $y(t)$.



(b) λ-level sets of $z(t)$.

Figure 2. The black curve gives the solution at $\lambda = 1$ for the system (4.1) and (4.2).

Example 4.2. *(One springs-two railway cars system) Figure* 3 *represents one spring supporting two railway cars of masses* M_1 *and* M_2 *respectively system to one other. If all two of the two cars rightward displacements from their respective equilibrium positions are positive, then the spring is* *extended by$y(t)$. The motion equations for the two cars are generated as follows:*

$$M_1 \odot y''_{gr}(t) = -s \odot y(t) \oplus s \odot z(t),$$

$$M_2 \odot z''_{gr}(t) = s \odot y(t) \ominus s \odot z(t),$$

with fuzzy initial values, $y(0) = y_0$, $z(0) = z_0$, $y'_{gr}(0) = y'_0$, $z'_{gr}(0) = z'_0$.

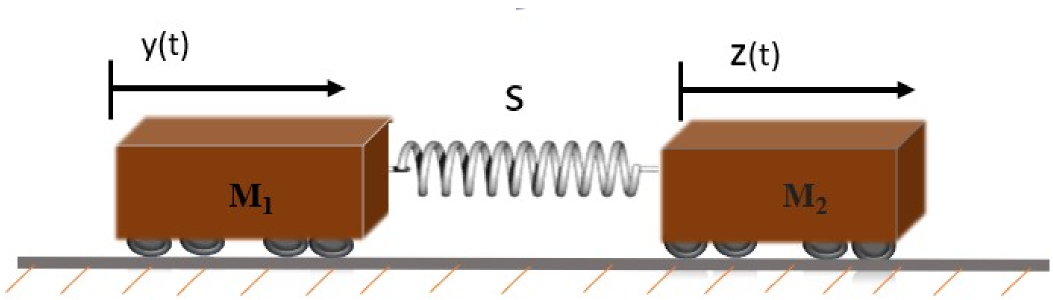


Figure 3. One springs-two cars systems.

Suppose that $M_1 = 1lb.s^2/ft$, $M_2 = 1lb.s^2/ft$, *and the* λ*-level sets of spring constant and fuzzy initial values are* $[s]^\lambda = [1+\lambda, 3-\lambda]$, $[y_0]^\lambda = [z_0]^\lambda = [\lambda, 2-\lambda]$, $[y'_0]^\lambda = [1+\lambda, 3-\lambda]$, $[z'_0]^\lambda = [\lambda, 2-\lambda]$. *Then the matrix equation is,*

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \odot \begin{bmatrix} y''_{gr}(t) \\ z''_{gr}(t) \end{bmatrix} = \begin{bmatrix} -s & s \\ s & -s \end{bmatrix} \otimes \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \tag{4.6}$$

$$\text{subject to, } \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix}. \tag{4.7}$$

Taking HMF on both sides of (4.6) *and* (4.7)*, we have*

$$\begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_y)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_z)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -s_{gr}(\lambda,\alpha_1) & s_{gr}(\lambda,\alpha_1) \\ s_{gr}(\lambda,\alpha_1) & -s_{gr}(\lambda,\alpha_1) \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_y) \\ z_{gr}(t,\lambda,\alpha_z) \end{bmatrix}, \tag{4.8}$$

$$\text{subject to } \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} y_{0_{gr}}(\lambda,\alpha_2) \\ z_{0_{gr}}(\lambda,\alpha_2) \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} y'_{0_{gr}}(\lambda,\alpha_1) \\ z'_{0_{gr}}(\lambda,\alpha_2) \end{bmatrix}, \tag{4.9}$$

here $s_{gr}(\lambda,\alpha_1) = y'_{0_{gr}}(\lambda,\alpha_1) = [1+\lambda+2(1-\lambda)\alpha_1]$, $y_{0_{gr}}(\lambda,\alpha_2) = z_{0_{gr}}(\lambda,\alpha_2) = z'_{0_{gr}}(\lambda,\alpha_2) = [\lambda+2(1-\lambda)\alpha_2]$, *where* λ, $\alpha_1\,\alpha_2 \in [0,1]$.

$$\Longrightarrow \begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_1,\alpha_2)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_1,\alpha_2)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -(1+\lambda+2(1-\lambda)\alpha_1) & 1+\lambda+2(1-\lambda)\alpha_1 \\ 1+\lambda+2(1-\lambda)\alpha_1 & -(1+\lambda+2(1-\lambda)\alpha_1) \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_1,\alpha_2) \\ z_{gr}(t,\lambda,\alpha_1,\alpha_2) \end{bmatrix}, \tag{4.10}$$

$$\textit{subject to, } \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} \lambda+2(1-\lambda)\alpha_1 \\ \lambda+2(1-\lambda)\alpha_1 \end{bmatrix} \textit{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} 1+\lambda+2(1-\lambda)\alpha_2 \\ \lambda+2(1-\lambda)\alpha_1 \end{bmatrix}. \tag{4.11}$$

The solution for second order system of equations (4.10) *and* (4.11) *is*

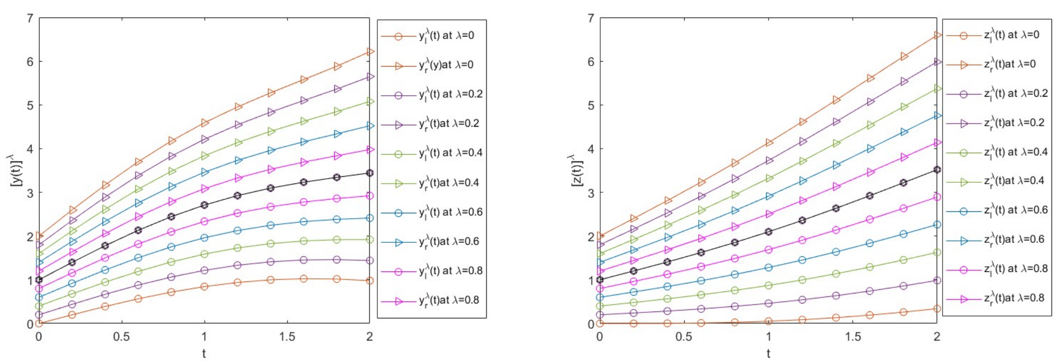
$$y_{gr}(t,\lambda,\alpha_1,\alpha_2) \text{ and } z_{gr}(t,\lambda,\alpha_1,\alpha_2). \tag{4.12}$$

Applying inverse HMF on (4.12)*, we get*

$$[y(t)]^\lambda = [\inf_{\lambda\le\alpha\le 1} \min_{\alpha_1,\alpha_2} y_{gr}(t,\alpha,\alpha_1,\alpha_2), \sup_{\lambda\le\alpha\le 1} \max_{\alpha_1,\alpha_2} y_{gr}(t,\alpha,\alpha_1,\alpha_2)],$$

$$[z(t)]^\lambda = [\inf_{\lambda\le\alpha\le 1} \min_{\alpha_1,\alpha_2} z_{gr}(t,\alpha,\alpha_1,\alpha_2), \sup_{\lambda\le\alpha\le 1} \max_{\alpha_1,\alpha_2} z_{gr}(t,\alpha,\alpha_1,\alpha_2)].$$

The λ*-level sets solution is enumerated using MATLAB and is illustrated in Figure* 4



(a) λ-level sets of $y(t)$. (b) λ-level sets of $z(t)$.

Figure 4. The black curve gives the solution at $\lambda = 1$ for the system (4.6) and (4.7).

Example 4.3. *(Two springs-two mass systems with external fuzzy force) Figure* 5 *represents two springs supporting two masses to one other. If all the two masses rightward displacements from their respective equilibrium positions are positive, then*

(i) *The first spring is extended by* $y(t)$.

(ii) *The second spring is extended by* $z(t) \ominus y(t)$.

The motion equations for the two masses are generated as follows:

$$M_1 \odot y''_{gr}(t) = -s_1 \odot y(t) \oplus s_2 \odot (z(t) \ominus y(t)),$$
$$M_2 \odot z''_{gr}(t) = -s_2 \odot (z(t) \ominus y(t)) + f(t),$$
with fuzzy initial values, $y(0) = y_0$, $z(0) = z_0$, $y'_{gr}(0) = y'_0$, $z'_{gr}(0) = z'_0$.

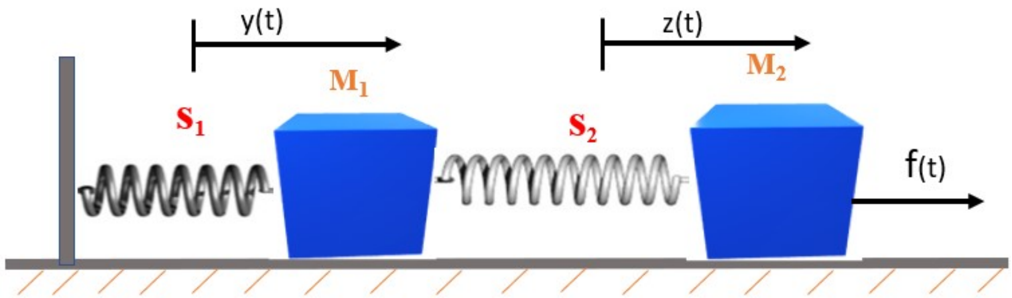


Figure 5. Two springs-two masses systems.

The matrix form of system of equations is

$$\begin{bmatrix} y''_{gr}(t) \\ z''_{gr}(t) \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \otimes \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \oplus \begin{bmatrix} 0 \\ p\cos(10t) \end{bmatrix}, \tag{4.13}$$

$$\text{subject to, } \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix}, \tag{4.14}$$

where λ*-cut set of coefficients and initial values are* $s_1 = -3$, $s_2 = 1$, $s_3 = 1$, $s_4 = -1$,$[y_0]^\lambda = [z_0]^\lambda = [y'_0]^\lambda = [\lambda, 2-\lambda]$, $[z'_0]^\lambda = [p]^\lambda = [1+\lambda, 3-\lambda]$. *Taking HMF on both sides of* (4.13) *and* (4.14)*, we have*

$$\begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_y)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_z)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_y) \\ z_{gr}(t,\lambda,\alpha_z) \end{bmatrix} + \begin{bmatrix} 0 \\ p_{gr}(\lambda,\alpha_1)\cos(10t) \end{bmatrix}, \tag{4.15}$$

$$\text{subject to, } \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} y_{0_{gr}} \\ z_{0_{gr}} \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(t_0) \\ z'_{gr}(t_0) \end{bmatrix} = \begin{bmatrix} y'_{0_{gr}} \\ z'_{0_{gr}} \end{bmatrix}, \tag{4.16}$$

where $p_{gr}(\lambda,\alpha_1) = [1+\lambda+2(1-\lambda)\alpha_1]$, $y_{0_{gr}}(\lambda,\alpha_2) = z_{0_{gr}}(\lambda,\alpha_2) = y'_{0_{gr}}(\lambda,\alpha_2) = [\lambda+2(1-\lambda)\alpha_2]$, $z'_{0_{gr}}(\lambda,\alpha_2) = [1+\lambda+2(1-\lambda)\alpha_1]$, *where* λ, $\alpha_1\,\alpha_2 \in [0,1]$.

$$\implies \begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_1,\alpha_2)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_1,\alpha_2)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_1,\alpha_2) \\ z_{gr}(t,\lambda,\alpha_1,\alpha_2) \end{bmatrix} + \begin{bmatrix} 0 \\ [1+\lambda+2(1-\lambda)\alpha_1]\cos(10t) \end{bmatrix}, \tag{4.17}$$

$$\textit{with, } \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} \lambda+2(1-\lambda)\alpha_2 \\ \lambda+2(1-\lambda)\alpha_2 \end{bmatrix} \textit{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} \lambda+2(1-\lambda)\alpha_2 \\ 1+\lambda+2(1-\lambda)\alpha_1 \end{bmatrix}. \tag{4.18}$$

The solution for system of equations (4.17) and (4.18) is

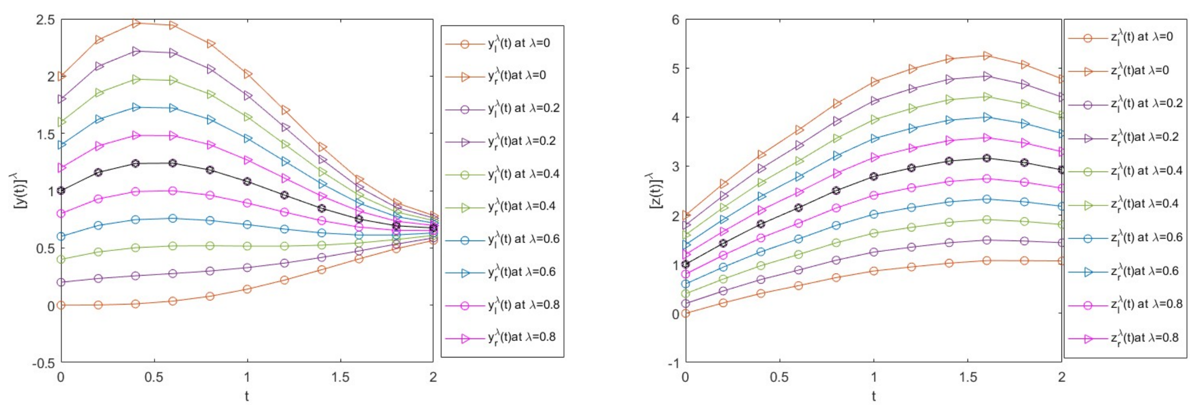
$$y_{gr}(t,\lambda,\alpha_1,\alpha_2) \textit{ and } z_{gr}(t,\lambda,\alpha_1,\alpha_2). \tag{4.19}$$

Applying inverse HMF on (4.19), we get

$$[y(t)]^\lambda = [\inf_{\lambda\le\alpha\le1}\min_{\alpha_1,\alpha_2} y_{gr}(t,\alpha,\alpha_1,\alpha_2), \sup_{\lambda\le\alpha\le1}\max_{\alpha_1,\alpha_2} y_{gr}(t,\alpha,\alpha_1,\alpha_2)],$$

$$[z(t)]^\lambda = [\inf_{\lambda\le\alpha\le1}\min_{\alpha_1,\alpha_2} z_{gr}(t,\alpha,\alpha_1,\alpha_2), \sup_{\lambda\le\alpha\le1}\max_{\alpha_1,\alpha_2} z_{gr}(t,\alpha,\alpha_1,\alpha_2)].$$

The λ-level sets solution is enumerated using MATLAB and is illustrated in Figure 6



(a) λ-level sets of $y(t)$. (b) λ-level sets of $z(t)$.

Figure 6. The black curve gives the solution at $\lambda = 1$ for the system (4.13) and (4.14).

Example 4.4. *(Three springs-two mass systems with external fuzzy force) Three springs supporting two masses on both sides and one another is depicts in Figure 7. Assume that there is no friction as the masses move and that each spring abides by Hooke's law. Let $f(t)$ be the fuzzy force applying on mass M_1 at time $t \geq 0$. If all the two masses rightward displacements (from their individual equilibrium positions) are positive, then*

(i) *The first spring is extended by* $y(t)$.

(ii) *The second spring is extended by* $z(t) \ominus y(t)$.

(iii) *The third spring is compressed by* $z(t)$.

The motion equations for the two masses are generated as follows:

$$M_1 \odot y''_{gr}(t) = -s_1 \odot y(t) \oplus s_2 \odot (z(t) \ominus y(t)) + f(t),$$

$$M_2 \odot z''_{gr}(t) = -s_2 \odot (z(t) \ominus y(t)) - s_3 \odot z(t),$$

with fuzzy initial values, $y(0) = y_0$, $z(0) = z_0$, $y'_{gr}(0) = y'_0$, $z'_{gr}(0) = z'_0$.

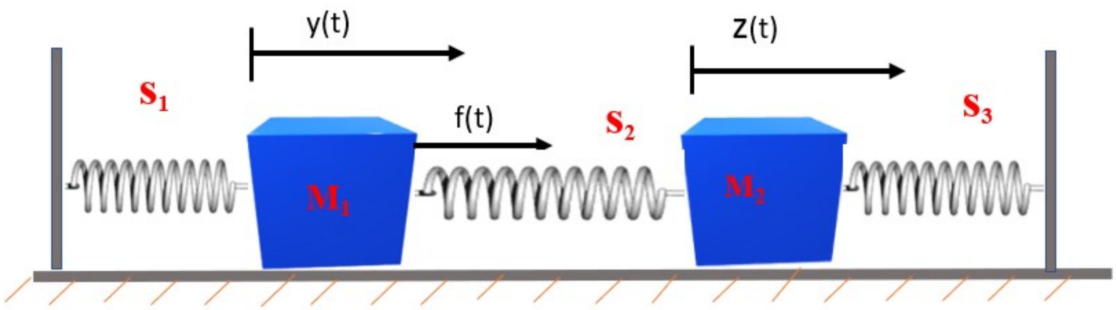


Figure 7. Three springs-two masses systems.

The matrix form of system of equations is

$$\begin{bmatrix} y''_{gr}(t) \\ z''_{gr}(t) \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \otimes \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \oplus \begin{bmatrix} p\cos(10t) \\ 0 \end{bmatrix}, \tag{4.20}$$

$$\textit{subject to, } \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \textit{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} y'_0 \\ z'_0 \end{bmatrix}, \tag{4.21}$$

where λ*-cut set of coefficients and initial values are* $s_1 = -3$, $s_2 = 1$, $s_3 = 1$, $s_4 = -3$, $y_0 = 1$, $z_0 = 1$, $[y'_0]^\lambda = [\lambda, 2-\lambda]$, $[z'_0]^\lambda = [p]^\lambda = [1+\lambda, 3-\lambda]$. *Taking HMF on both sides of* (4.20) *and* (4.21), *we have*

$$\begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_y)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_z)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_y) \\ z_{gr}(t,\lambda,\alpha_z) \end{bmatrix} + \begin{bmatrix} p_{gr}(\lambda,\alpha_2)\cos(10t) \\ 0 \end{bmatrix}, \tag{4.22}$$

$$\textit{subject to, } \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} y_{0_{gr}} \\ z_{0_{gr}} \end{bmatrix} \textit{ and } \begin{bmatrix} y'_{gr}(t_0) \\ z'_{gr}(t_0) \end{bmatrix} = \begin{bmatrix} y'_{0_{gr}} \\ z'_{0_{gr}} \end{bmatrix}, \tag{4.23}$$

where $p_{gr}(\lambda,\alpha_2) = [1+\lambda+2(1-\lambda)\alpha_2]$, $y'_{0_{gr}}(\lambda,\alpha_1) = [\lambda+2(1-\lambda)\alpha_1]$, $z'_{0_{gr}}(\lambda,\alpha_2) = [1+\lambda+2(1-\lambda)\alpha_1]$, *where* λ, $\alpha_1\,\alpha_2 \in [0,1]$.

$$\implies \begin{bmatrix} \frac{\partial^2 y_{gr}(t,\lambda,\alpha_1,\alpha_2)}{\partial t^2} \\ \frac{\partial^2 z_{gr}(t,\lambda,\alpha_1,\alpha_2)}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_{gr}(t,\lambda,\alpha_1,\alpha_2) \\ z_{gr}(t,\lambda,\alpha_1,\alpha_2) \end{bmatrix} + \begin{bmatrix} [1+\lambda+2(1-\lambda)\alpha_2]\cos(10t) \\ 0 \end{bmatrix}, \tag{4.24}$$

$$\text{with } \begin{bmatrix} y_{gr}(0) \\ z_{gr}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} y'_{gr}(0) \\ z'_{gr}(0) \end{bmatrix} = \begin{bmatrix} \lambda+2(1-\lambda)\alpha_1 \\ 1+\lambda+2(1-\lambda)\alpha_2 \end{bmatrix}. \tag{4.25}$$

The solution for system of equations (4.24) *and* (4.25) *is*

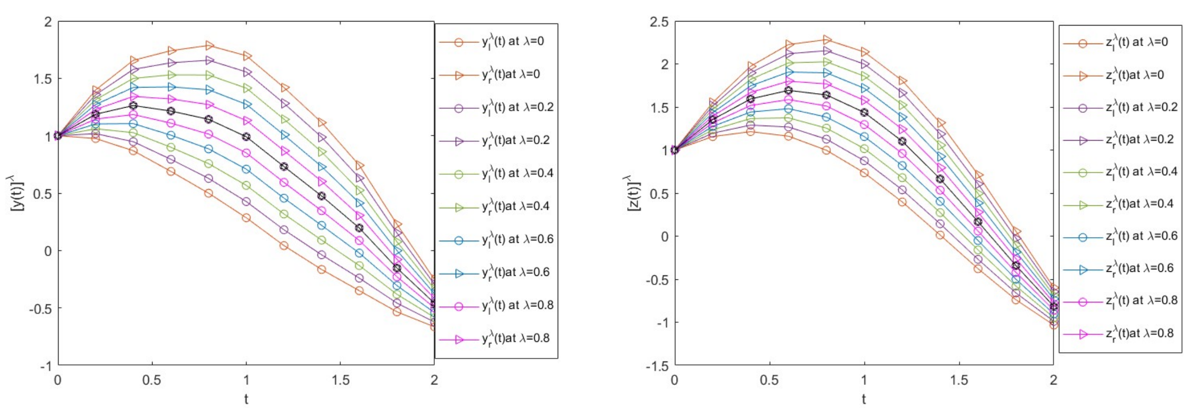
$$y_{gr}(t,\lambda,\alpha_1,\alpha_2) \text{ and } z_{gr}(t,\lambda,\alpha_1,\alpha_2). \tag{4.26}$$

Applying inverse HMF on (4.26)*, we get*

$$[y(t)]^\lambda = [\inf_{\lambda\le\alpha\le 1} \min_{\alpha_1,\alpha_2} y_{gr}(t,\alpha,\alpha_1,\alpha_2), \sup_{\lambda\le\alpha\le 1} \max_{\alpha_1,\alpha_2} y_{gr}(t,\alpha,\alpha_1,\alpha_2)],$$

$$[z(t)]^\lambda = [\inf_{\lambda\le\alpha\le 1} \min_{\alpha_1,\alpha_2} z_{gr}(t,\alpha,\alpha_1,\alpha_2), \sup_{\lambda\le\alpha\le 1} \max_{\alpha_1,\alpha_2} z_{gr}(t,\alpha,\alpha_1,\alpha_2)].$$

The λ-level sets solution is enumerated using MATLAB and is illustrated in Figure 8



(a) λ-level sets of $y(t)$. (b) λ-level sets of $z(t)$.

Figure 8. The black curve gives the solution at $\lambda = 1$ for the system (4.20) and (4.21).

5. Conclusions

This paper mainly deals with determining solutions of SSLFDEs and applications to some mechanical problems. The granular differentiability is extended to n-dimensional fuzzy valued functions. The SSLFDEs with fuzzy initial conditions are investigated under gr-differentiability. An algorithm is developed to determine the solutions of SSLFDEs with fuzzy initial conditions. Some mechanical problems as automobiles with two axles, railway cars systems, and mass-spring systems with fuzzy initial conditions are demonstrated for the effective implementation of the algorithm. In the future, this work will be extended for higher-order SFDEs with fuzzy initial and boundary conditions.

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