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Best Proximity Point and Existence of the Positive Definite Solution for Matrix Equations

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Abstract. In this research, $\alpha - \psi - \theta$ contraction has been defined to find the best proximity point in partially ordered metric spaces. Proper support for the result has been given in the form of a suitable example. The third part is fully devoted to the positive definite solution of matrix equations.

1. Introduction and Preliminaries

The concept of the best proximity point was introduced by Basha [5] with the help of the Banach contraction principle. It may be impossible to find a fixed point for two non empty subsets $L, M \subseteq W$ and a mapping $S : L \to M$ (for example, when $L \cap M = \phi$). However, it is very interesting to find a point $x \in L$, where x and Sx are as close as possible; in other words, find an $x \in L$ which minimizes $\varrho(x, Sx)$. Such optimal approximate solutions are called "best proximity points for S." Letter on many Mathematicians [1–3, 6, 9, 10] established best proximity point results. In 2014, idea of θ contraction introduced by Jleli et al. [8] and defined generalization of Banach contraction. In this paper, we define $\alpha - \psi - \theta$ contraction and establish the best proximity point in partially ordered metric spaces. Moreover, as a consequence of the result, a fixed point result and the existence of a positive definite solution to matrix equations have been given.

In the whole paper, complete metric space and the best proximity point are abbreviated as CMS and BPP, respectively. The subsequent symbols used in our results are:

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Let (W, ϱ) be a metric space and C, D be non-empty subsets of W.

$$\varrho(C,\ D) = \inf\{\ \varrho(u_1, v_1) : u_1 \in C \ \ and\ v_1 \in D\},$$

$$C_0 = \{u_1 \in C :\ \varrho(u_1, v_1) = \varrho(C, D)\ \ for\ some\ v_1 \in D\},$$

$$D_0 = \{v_1 \in D :\ \varrho(u_1, v_1) = \varrho(C, D)\ \ for\ some\ u_1 \in C\}.$$

In 2012, Samet et al. [13] defined the following contraction:

$$\alpha(x_1, y_1)\varrho(Tx_1, Ty_1) \leq \psi(\varrho(x_1, y_1)),$$

where $\psi : [0, \infty) \to [0, \infty)$ satisfy the consequent conditions:

(1) ψ is non decreasing,

(2) $\sum_{m=1}^{\infty} \psi^m(t) < \infty \ \ forall\ t > 0$, where ψ^m is the m^{th} iterate of ψ and $\psi(t) < t$ for any $t > 0$,

T is α- admissible i.e. for all $x_1, y_1 \in W$,

$$\alpha(x_1, y_1) \geq 1 \Rightarrow \alpha(Tx_1, Ty_1) \geq 1,$$

where $\alpha : W \times W \to [0, \infty)$ is a mapping.

Jleli et al. [8] proposed θ contraction in 2014 as follows:

Definition 1.1. *[8] Let Θ be the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfy the conditions:*

θ_1. *θ is non decreasing,*

θ_2. *for every sequence $\{\alpha_n\} \subset (0, \infty)$,*

$$\lim_{n\to\infty} \theta(\alpha_n) = 1 \Leftrightarrow \lim_{n\to\infty} \alpha_n = 0^+,$$

θ_3. *there exists $s \in (0, 1)$ and $L \in (0, \infty)$ such that*

$$\lim_{\alpha\to 0} \frac{\theta(\alpha) - 1}{\alpha^s} = L$$

and prove the following results:

Theorem 1.1. *[8] Let (V, ϱ) be a CMS and $T : V \to V$ be a mapping, if there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that for all $u, v \in V$,*

$$\varrho(Tu, Tv) \neq 0 \ \ \Rightarrow \theta(\varrho(Tu, Tv)) \leq [\theta(\varrho(u, v))]^k. \tag{1.1}$$

Then T has a unique fixed point.

Also, in 2017, Ahmed et al. [4] used the subsequent weaker condition in place of condition (θ_3) : $(\theta_3^{'})$ θ is continuous on $(0, \infty)$.

In this order we denote Ψ the set all functions θ satisfy $\theta_1, \theta_2, \theta_3^{'}$.

In 2017, Piri et al. [11] defined generalized Khan contraction.

Theorem 1.2. *[11] Let (W, ϱ) be a CMS and $A : W \to W$ be a mapping satisfies*

$$\varrho(Au,\ Av) \leq \begin{cases} k\frac{\varrho(u,Au)\varrho(u,Av)+\varrho(v,Av)\varrho(v,Au)}{max\{\varrho(u,Av),\varrho(Au,v)\}}, & if \quad max\{\varrho(u,Av),\varrho(Au,v)\} \neq 0, \\ 0, & if \quad max\{\varrho(u,Av),\varrho(Au,v)\} = 0, \end{cases}$$

where $k \in [0, 1)$ and $u, v \in W$, then A has a unique fixed point.

However, the mappings involved in all results were self mappings.

Definition 1.2. *[14] Let (C, D) be a pair of non-empty subsets of a metric space W with $C_0 \neq \phi$. Then, the pair (C, D) is said to have the weak $P - property$ if and only if*

$$\left.\begin{array}{c} \varrho(u_1,\ v_1) = \varrho(C,\ D) \\ \varrho(u_2,\ v_2) = \varrho(C,\ D) \end{array}\right\} \Rightarrow \varrho(u_1,\ u_2) \leq \varrho(\ v_1,\ v_2),$$

$where$ $u_1,\ u_2 \in C$ and $v_1, v_2 \in D$.

Definition 1.3. *[13] Let C, D be the subsets of metric space (W, ϱ). A non self mapping $A : C \to D$ is said to be $\alpha - proximal$ admissible if*

$$\left.\begin{array}{c} \alpha(v_1, v_2) \geq 1 \\ \varrho(u_1,\ Av_1) = \varrho(C,\ D) \\ \varrho(u_2,\ Av_2) = \varrho(C,\ D) \end{array}\right\} \Rightarrow\ \alpha(u_1,\ u_2) \geq 1,$$

$where$ $u_1,\ u_2,\ v_1, v_2 \in C$ and $\alpha : C \times C \to [0, \infty)$ be a function.

2. Main Results

Let C, D be two subsets of a partially ordered CMS $(V, \varrho, \preceq)$ and $\alpha : C \times C \to [0, \infty)$ be a function. A mapping $T : C \to D$ is said to be $\alpha - \psi - \theta$ contraction, if for $\theta \in \Psi$, there exists $\kappa \in (0, 1)$ and for every $x, y \in C$ with $\alpha(x, y) \geq 1,\ \varrho(Tx, Ty) > 0$, we have

$$\alpha(x, y)\theta[\varrho(Tx, Ty)] \leq [\psi(\theta(M(x, y)))]^{\kappa}, \tag{2.1}$$

where $M(x, y) = \max\{G(x, y), \varrho(x, y)\}$ and

$$G(x, y) = \begin{cases} \frac{\varrho(x,Tx)\varrho(x,Ty)+\varrho(y,Ty)\varrho(y,Tx)}{max\{\varrho(x,Ty),\varrho(Tx,y)\}}, & if \quad max\{\varrho(x,Ty),\varrho(Tx,y)\} \neq 0, \\ 0, & if \quad max\{\varrho(x,Ty),\varrho(Tx,y)\} = 0. \end{cases}$$

Theorem 2.1. *Let $(V, \varrho, \preceq)$ be a partially ordered CMS and C, D are closed subsets of V and let $T : C \to D$ be a $\alpha - \psi - \theta$ contraction satisfies*

(i) *T is $\alpha - proximal$ admissible,*
(ii) *$T(C_0) \subseteq D_0$ and the pair (C, D) satisfies week $P - property$,*
(iii) *T is continuous,*
(iv) *there exists $x_0, x_1 \in C_0, x_0 \preceq x_1$ with $\varrho(x_1, Tx_0) = \varrho(C, D)$ such that $\alpha(x_0, x_1) \geq 1$.*

Then there exists $x \in V$ such that $\varrho(x, Tx) = \varrho(C, D)$.

Proof. Let $x_o \in C_0$, since $T(C_0) \subseteq D_0$, there exists an element $x_1 \in C_0$ such that

$$\varrho(x_1, Tx_0) = \varrho(C, D) \ \ and \ x_0 \preceq x_1,$$

by the assumption (iv), $\alpha(x_0, x_1) \geq 1$. Again $x_1 \in C_0$ and $T(C_0) \subseteq D_0$, there exists $x_2 \in C_0$ such that

$$\varrho(x_2, Tx_1) = \varrho(C, D) \ and \ x_1 \preceq x_2.$$

By $\alpha - proximal$ admissibility of T, we have

$$\alpha(x_1, x_2) \geq 1,$$

continuing this process, we get

$$\varrho(x_{n+1}, Tx_n) = \varrho(C, D) \quad and \quad \alpha(x_n, x_{n+1}) \geq 1 \ \ for\ all\ n \in N, \tag{2.2}$$

where $x_0 \preceq x_1 \preceq x_2 \preceq x_3 \cdots \preceq x_n \preceq x_{n+1} \preceq \ldots$
Now, if there exists $n_0 \in N$ such that $x_{n_0} = x_{n_0+1}$, then we have

$$\begin{aligned}
\varrho(x_{n_0}, Tx_{n_0}) &= \varrho(x_{n_0+1}, Tx_{n_0}) \\
&= \varrho(C, D).
\end{aligned}$$

Then x_{n_0} is the best proximity point (BPP)of T.
Therefore, we assume that $x_n \neq x_{n+1}$, that is $\varrho(x_n, x_{n+1}) > 0$ for all $n \in N \cup \{0\}$.
By the week P-property of the pair (C, D) and from 2.1, 2.2, we have for all $n \in N$,

$$\begin{aligned}
1 &< \theta(\varrho(x_{n+1}, x_n)) = \theta(\varrho(Tx_n, Tx_{n-1})) \\
&\leq \alpha(x_n, x_{n-1})\theta(\varrho(Tx_n, Tx_{n-1}) \\
&\leq (\psi(\theta(M(x_n, x_{n-1}))))^{\kappa},
\end{aligned}$$

where

$$\begin{aligned}
M(x_n, x_{n-1}) &= \max\{G(x_n, x_{n-1}), \varrho(x_n, x_{n-1})\} \\
&= \max\{\frac{\varrho(x_{n-1}, Tx_{n-1})\varrho(x_{n-1}, Tx_n) + \varrho(x_n, Tx_{n-1})\varrho(x_n, Tx_n)}{max\{\varrho(x_{n-1}, Tx_n), \varrho(Tx_{n-1}, x_n)\}}, \varrho(x_n, x_{n-1})\} \\
&= \max\{\frac{\varrho(x_{n-1}, x_n)\varrho(x_{n-1}, x_{n+1})}{\varrho(x_{n-1}, x_{n+1})}, \varrho(x_n, x_{n-1})\} \\
&= \varrho(x_{n-1}, x_n),
\end{aligned}$$

so,

$$\begin{aligned}
1 &< \theta(\varrho(x_n, x_{n+1})) \leq (\psi(\theta(\varrho(x_n, x_{n-1}))))^{\kappa} \\
&\leq (\psi(\theta(\varrho(x_{n-1}, x_{n-2}))))^{\kappa^2} \\
&\leq (\psi(\theta(\varrho(x_{n-2}, x_{n-3}))))^{\kappa^3}
\end{aligned}$$

$$\begin{aligned} &\le \ \dots \\ &\quad\ \dots \\ &\le \ (\psi(\theta(\varrho(x_0, x_1))))^{\kappa^n}. \end{aligned}$$

Taking $n \to \infty$ we get

$$\theta(\varrho(x_n, x_{n+1})) \to 1,$$

therefore, by θ_2, we obtain

$$\lim_{n\to\infty} \varrho(x_n, x_{n+1}) = 0. \tag{2.3}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in C. Suppose, on the contrary that, if there exists $\epsilon > 0$, we can find the sequences $\{p_n\}$ and $\{q_n\}$ of natural numbers such that for $p_n > q_n > n$, we have

$$\varrho(x_{p_n}, x_{q_n}) \ge \epsilon. \tag{2.4}$$

Then,

$$\varrho(x_{p_{n-1}}, x_{q_n}) < \epsilon \ \ for\ all\ n \in N.$$

Thus, by triangular inequality and 2.4, we get

$$\begin{aligned} \epsilon &\le \varrho(x_{p_n}, x_{q_n}) \le \varrho(x_{p_n}, x_{p_{n-1}}) + \varrho(x_{p_{n-1}}, x_{q_n}) \\ &\le \varrho(x_{p_n}, x_{p_{n-1}}) + \epsilon. \end{aligned}$$

Taking limit $n \to \infty$ and using 2.3, we get

$$\lim_{n\to\infty} \varrho(x_{p_n}, x_{q_n}) = \epsilon. \tag{2.5}$$

Again by triangular inequality, we have

$$\varrho(x_{p_n}, x_{q_n}) \le \varrho(x_{p_n}, x_{p_{n+1}}) + \varrho(x_{p_{n+1}}, x_{q_{n+1}}) + \varrho(x_{q_{n+1}}, x_{q_n}) \tag{2.6}$$

and

$$\varrho(x_{p_{n+1}}, x_{q_{n+1}}) \le \varrho(x_{p_{n+1}}, x_{p_n}) + \varrho(x_{p_n}, x_{q_n}) + \varrho(x_{q_n}, x_{q_n+1}). \tag{2.7}$$

Taking limit $n \to \infty$ and from 2.3, 2.5, we have

$$\lim_{n\to\infty} \varrho(x_{p_{n+1}}, x_{q_{n+1}}) = \epsilon, \tag{2.8}$$

so, equation 2.5 holds. Then by assumption $\alpha(x_{p_n}, x_{q_n}) \ge 1$, we get

$$\begin{aligned} 1 &\le \theta(\varrho(x_{p_{n+1}}, x_{q_{n+1}})) \le \theta(\varrho(Tx_{p_n}, Tx_{q_n})) \\ &\le \alpha(x_{p_n}, x_{q_n})\theta(\varrho(Tx_{p_n}, Tx_{q_n})) \\ &\le (\psi(\theta(M(x_{p_n}, x_{q_n}))))^{\kappa} \\ &< \theta(M(x_{p_n}, x_{q_n})), \end{aligned}$$

by taking limit as $n \to \infty$ in above inequality and using $[\theta_3']$ in equation 2.3, we get

$$\lim_{n\to\infty} \varrho(x_{p_n}, x_{q_n}) = 0 < \epsilon,$$

which is contraction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subseteq C$ and C is closed in a complete metric space, so we can find $x \in C$, such that $x_n \to x$.
Now, since T is continuous so, we have

$$Tx_n \to Tx.$$

This implies that

$$\varrho(x_{n+1}, Tx_n) \to \varrho(x, Tx),$$

since the sequence $\{\varrho(x_{n+1}, Tx_n)\}$ is a constant sequence with the value $\varrho(C, D)$. We deduce that

$$\varrho(C, D) = \varrho(x, Tx).$$

So, x is the best proximity point.

If we take $C = D = V$ and $\alpha(x, y) = 1$, we obtain the subsequent result:

Corollary 2.1. *Let $(V, \varrho, \preceq)$ be a complete metric space and $T : V \to V$ be a mapping satisfying*

$$\theta[\varrho(Tx, Ty)] \leq [\psi(\theta(M(x, y)))]^{\kappa},$$

where $M(x, y) = \max\{G(x, y), \varrho(x, y)\}$ and

$$G(x,y) = \begin{cases} \frac{\varrho(x,Tx)\varrho(x,Ty)+\varrho(y,Ty)\varrho(y,Tx)}{max\{\varrho(x,Ty),\varrho(Tx,y)\}}, & if \quad max\{\varrho(x,Ty),\varrho(Tx,y)\} \neq 0, \\ 0, & if \quad max\{\varrho(x,Ty),\varrho(Tx,y)\} = 0, \end{cases}$$

for all $x, y \in V$ with $\theta \in \Theta$ and $\kappa \in (0, 1)$, suppose that

(i) *T is continuous,*
(ii) *there exist $x_0 \in V$ such that $x_0 \preceq Tx_0$.*

Then T has a unique fixed point.

Proof. By the Theorem 2.1, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subseteq V$ and V is a complete metric space, so we can find $x \in V$ such that $x_n \to x$. Now, we shall show that x is a fixed point of T.

$$\begin{aligned} \varrho(Tx, x) &= \lim_{n\to\infty} \varrho(Tx_n, x) \\ &= \lim_{n\to\infty} \varrho(x_{n+1}, x) \\ &= 0. \end{aligned}$$

Then x is a fixed point of T.
Uniqueness: let, if possible there are two fixed points x_1 and x_2 such that $x_1 \neq x_2$. Since x_1 and x_2

are fixed points, so $Tx_1 = x_1$ and $Tx_2 = x_2$.

$$\begin{aligned}
\theta[\varrho(Tx_1, Tx_2)] &\leq [\psi(\theta(M(x_1, x_2)))]^{\kappa} \\
\theta[\varrho(x_1, x_2)] &\leq [\psi(\theta(\max\{G(x_1, x_2), \varrho(x_1, x_2)\}))]^{\kappa} \\
&\leq [\psi(\theta(\max\{\frac{\varrho(x_1, Tx_1)\varrho(x_1, Tx_2) + \varrho(x_2, Tx_2)\varrho(x_2, Tx_1)}{max\{\varrho(x_1, Tx_2), \varrho(Tx_1, x_2)\}}, \varrho(x_1, x_2)\}))]^{\kappa} \\
&\leq [\psi(\theta(\max\{\frac{\varrho(x_1, x_1)\varrho(x_1, x_2) + \varrho(x_2, x_2)\varrho(x_2, x_1)}{max\{\varrho(x_1, x_2), \varrho(x_1, x_2)\}}, \varrho(x_1, x_2)\}))]^{\kappa} \\
&\leq [\psi(\theta(\varrho(x_1, x_2)))]^{\kappa} \\
&\leq [(\theta(\varrho(x_1, x_2)))]^{\kappa},
\end{aligned}$$

which is contradiction, so $x_1 = x_2$. Therefore, T has a unique fixed point.

Note. In this result, if $\psi(t) = t$ and $M(x, y) = \varrho(x, y)$, then we get theorem 1.1.

Example 2.1. Let $W = \{\ 0,\ 1,\ 2,\ 3\}$ with the usual order $\leq$, be a partially ordered set and let $\varrho : W \times W \to R$ be given as

$$\varrho(0,0) = \varrho(1,1) = \varrho((2,2) = \varrho(3,3) = 0, \varrho(0,1) = \varrho(1,0) = 2,$$

$$\varrho(0,2) = \varrho(2,0) = \frac{3}{2}, \varrho(0,3) = \varrho(3,0) = \frac{5}{2}, \varrho(2,3) = \varrho(1,3) = \frac{5}{2}, \varrho(1,2) = 3.$$

Consider $C = \{0,1\}, D = \{2,3\}$ and $T : C \to D$ defined by $T(0) = 2, T(1) = 3$. So, $\varrho(C, D) = \varrho(0,2) = \frac{3}{2}$. Also, $C_0 = \{0\}$ and $D_0 = \{2\}$. Clearly $T(C_0) \subseteq D_0$ and

$$\left.\begin{array}{l} \varrho(u_1,\ v_1) = \varrho(C,\ D) = \frac{3}{2} \\ \varrho(u_2,\ v_2) = \varrho(C,\ D) = \frac{3}{2} \end{array}\right\} \Rightarrow \varrho(u_1,\ u_2) \leq \varrho(\ v_1,\ v_2),$$

$where\ u_1,\ u_2 \in C\ \ and\ v_1, v_2 \in D$. Then, we have $u_1 = 0, v_1 = 2$ and $u_2 = 0, v_2 = 2$. In this case,

$$\varrho(0,0) = 0 = \varrho(2,2),$$

that is, the pair (C, D) has the weak P property.

Taking $\theta(u) = u + 1$ and $\psi(u) = \frac{999}{1000}u$ for all $u \geq 0$ and define $\alpha : W \times W \to [0, \infty)$ as follows,

$$\begin{cases} \alpha(u, v) = 1, & if (u, v) \in \{(0,0), (0,1), (1,1)\}, \\ \alpha(u, v) = 0, & if\ not. \end{cases}$$

Let $u_1, v_1, u_1\ and\ u_2$ in C such that

$$\begin{cases} \alpha(u_1, u_2) \geq 1, \\ \varrho(v_1,\ Tu_1) = \varrho(C,\ D) = \frac{3}{2}, \\ \varrho(v_2,\ Tu_2) = \varrho(C,\ D) = \frac{3}{2}. \end{cases}$$

Then we have $u_1 = v_1 = u_1 = u_2 = 0$. So,

$$\alpha(v_1, v_2) \geq 1,$$

that is, T is $\alpha-$ proximal admissible. By the symmetry of ϱ and α, it suffices to study the cases $(u=0, v=1)$ and $(u=v=0)$.
If $(u=0, v=1), u \leq v$,

$$\alpha(0,1)\theta(\varrho(T0,T1)) = \theta(\frac{11}{10}) = \frac{7}{2},$$

$$\begin{aligned} M(u,v) &= \max\{\frac{\varrho(u,Tv)\varrho(u,Tv)+\varrho(v,Tv)\varrho(v,Tu)}{max\{\varrho(u,Tv),\varrho(Tu,v)\}}, \varrho(u,v)\} \\ &= \max\{\frac{\varrho(0,T0)\varrho(0,T1)+\varrho(1,T1)\varrho(1,T0)}{max\{\varrho(0,T1),\varrho(T0,1)\}}, \varrho(0,1)\} \\ &= \max\{\frac{\varrho(0,2)\varrho(0,3)+\varrho(1,3)\varrho(1,2)}{max\{\varrho(0,3),\varrho(2,1)\}}, \varrho(0,1)\} \\ &= \max\{\frac{\frac{3}{2}\times\frac{5}{2}+\frac{5}{2}\times 3}{max\{\frac{5}{2},3)\}}, 2\} \\ &= \frac{15}{4}. \end{aligned}$$

So,

$$[\psi(\theta(M(0,1)]^{\kappa} = (\frac{999}{1000}\times\frac{19}{4})^{\kappa}.$$

Therefore, for $\kappa = .805$, we have

$$\frac{7}{2} = (\frac{999}{1000}\times\frac{19}{4})^{\kappa}.$$

If $(u=0, v=0)$, then

$$\alpha(0,0)\theta(\varrho(T0,T0)) = 1,$$

$$\begin{aligned} M(u,v) &= \max\{\frac{\varrho(u,Tv)\varrho(u,Tv)+\varrho(v,Tv)\varrho(v,Tu)}{max\{\varrho(u,Tv),\varrho(Tu,v)\}}, \varrho(u,v)\} \\ &= \max\{\frac{\varrho(0,T0)\varrho(0,T0)+\varrho(0,T0)\varrho(1,T0)}{max\{\varrho(0,T0),\varrho(T0,0)\}}, \varrho(0,1)\} \\ &= \max\{\frac{\frac{3}{2}\times\frac{3}{2}+\frac{3}{2}\times\frac{3}{2}}{max\{\frac{3}{2},\frac{3}{2})\}}, 0\} \\ &= 3. \end{aligned}$$

So,

$$[\psi(\theta(M(0,0)]^{\kappa} = (\frac{999}{1000}\times 4)^{\kappa}.$$

Therefore, for $\kappa = .005$, we have

$$\alpha(u,v)\theta[\varrho(Tu,Tv)] \leq [\psi(\theta(M(u,v)))]^{\kappa}$$

Hence, all the conditions of the theorem 2.1 are fulfilled. So T has a Best proximity point and it is $u=0$.

3. Application to Matrix Equations

In this part, we will use the subsequent symbols:
$C(m)$ represents the collection of $m \times m$ complex matrices, $H(m) \subset C(m)$ represents the collection of the $m \times m$ hermitian matrices, $\wp(m) \subset H(m)$ represents the collection of $m \times m$ positive definite matrices, $H_1(m) \subset H(m)$ is the set of positive semi definite matrices of $m \times m$. In addition, $U_1, V_1 \in C(m)$. So, if $U_1 \in \wp(m)$ this means that $U_1 \succ 0$ and $U_1 \succeq 0$, means $U_1 \in H(m)$. Moreover, $U_1 \succeq V_1 (U_1 \preceq V_1)$ is replaced by $U_1 - V_1 \succeq 0 (U_1 - V_1 \preceq 0)$. The spectral norm of the matrix B is denoted by the notation $||.||$, i.e.,

$$||B|| = \sqrt{\lambda^+(B^*B)},$$

where $\lambda^+(B^*B)$ is the largest eigenvalue of B^*B and B^* is the traconjugate of B. We write

$$||B||_Y = \sum_{j=1}^{m} S_j(B),$$

where $S_j(B)$ is the singular value of $B \in C(m)$. For a given $G \in \wp(m)$, we denoted the modified norm by

$$||B||_{Y,G} = ||G^{\frac{1}{2}} B G^{\frac{1}{2}}||_Y.$$

The set H(m) equipped with the metric induced by $||.||$ is CMS. Furthermore, H(m) is Poset with partial order $\preceq$, where $U_1 \preceq V_1 \Leftrightarrow V_1 \preceq U_1$. In this Part, we use

$$\varrho(U_1, V_1) = ||V_1 - U_1||_{Y,G} = tr(G^{\frac{1}{2}}(V_1 - U_1)G^{\frac{1}{2}}).$$

We assume that the subsequent nonlinear matrix equation is

$$U = G \pm \sum_{j=1}^{n} B_j^* \tau(U) B_j. \tag{3.1}$$

Where $G \in \wp(m)$, $B_j, j = 1, 2, \ldots n$, are arbitrary $m \times m$ matrices and $\tau : H(m) \to H(m)$ is continuous mapping, which maps $\wp(m)$ into $\wp(m)$. Consider τ is order preserving, that is , if

$$C, D \in H(m) \Rightarrow \tau(C) \preceq \tau(D), \quad where\ C \preceq D.$$

Lemma 3.1. *[12] Let $C \succeq 0$ and $D \succeq 0$ be $m \times m$ matrices. Then $0 \le tr(CD) \le ||C||.tr(D)$.*

Theorem 3.1. *Let $T : H(m) \to H(m)$ be continuous (order preserving) mapping, which maps $\wp(m)$ into $\wp(m)$ and $G \in \wp(m)$. Consider that*

(i) *for all $U \preceq V$ and $M > 1$,*

$$\varrho(\tau(U), \tau(V)) \le \frac{\varrho(T(U), T(V))(\theta(tr(M(U,V))))^{\frac{1}{2}}}{M^{\frac{1}{2}}\theta(tr(T(U) - T(V)))},$$

where $M(U,V) = \max\{G(U,V), \varrho(U,V)\}$ *and*

$$G(U,V) = \begin{cases} \frac{\varrho(U,TU)\varrho(U,TV)+\varrho(V,TV)\varrho(V,TU)}{max\{\varrho(U,TV),\varrho(TU,V)\}}, & if \quad max\{\varrho(U,TV),\varrho(TU,V)\} \neq 0, \\ 0, & if \quad max\{\varrho(U,TV),\varrho(TU,V)\} = 0, \end{cases}$$

(ii) $0 < \sum_{j=1}^{n} B_j^* \tau(G) B_j \leq G$,

hold. Then 3.1 *has a positive definite solution* $\bar{U} \in \wp(m)$.

Proof. Define $T : H(m) \to H(m)$ by

$$T(U) = G \pm \sum_{j=1}^{n} B_j^* \tau(U) B_j, \tag{3.2}$$

and $\psi(v) = v/M$, then solution of 3.1 is a fixed point of T. Let $U, V \in H(m)$ with $U \preceq V$, then $T(U) \preceq T(V)$.

$$\begin{aligned}
\varrho(T(U), T(V)) &= ||T(V) - T(U)||_{Y,G} \\
&= tr(G^{\frac{1}{2}}(T(V) - T(U))G^{\frac{1}{2}}) \\
&= tr(\sum_{j=1}^{n} B_j^* G^{\frac{1}{2}}(\tau(V) - \tau(U))G^{\frac{1}{2}} B_j) \\
&= \sum_{j=1}^{n} tr(B_j^* G^{\frac{1}{2}}(\tau(V) - \tau(U))G^{\frac{1}{2}} B_j) \\
&= \sum_{j=1}^{n} tr(B_j^* G B_j(\tau(V) - \tau(U))) \\
&= \sum_{j=1}^{n} tr(B_j^* G B_j G^{\frac{1}{2}} G^{\frac{-1}{2}}(\tau(V) - \tau(U))G^{\frac{1}{2}} G^{\frac{-1}{2}}) \\
&= \sum_{j=1}^{n} tr(G^{\frac{-1}{2}} B_j^* G B_j G^{\frac{-1}{2}} G^{\frac{1}{2}}(\tau(V) - \tau(U))G^{\frac{1}{2}}) \\
&= tr(\sum_{j=1}^{n} G^{\frac{-1}{2}} B_j^* G B_j G^{\frac{-1}{2}})(G^{\frac{1}{2}}(\tau(V) - \tau(U))G^{\frac{1}{2}}),
\end{aligned}$$

by lemma 3.1, we get

$$\begin{aligned}
\varrho(T(U), T(V)) &= ||\sum_{j=1}^{n} G^{\frac{-1}{2}} B_j^* G B_j G^{\frac{-1}{2}}||.tr(G^{\frac{1}{2}}(\tau(V) - \tau(U))G^{\frac{1}{2}}) \\
&= ||\sum_{j=1}^{n} G^{\frac{-1}{2}} B_j^* G B_j G^{\frac{-1}{2}}||.||\tau(V) - \tau(U)||_{Y,G} \\
\varrho(T(U), T(V)) &= ||\sum_{j=1}^{n} G^{\frac{-1}{2}} B_j^* G B_j G^{\frac{-1}{2}}||.\varrho(\tau(V), \tau(U)).
\end{aligned}$$

So, by condition (i) and (ii), we get

$$\begin{aligned}
\varrho(T(U),T(V)) &\leq \frac{\varrho(T(U),T(V))(\theta(tr(M(U,V))))^{\frac{1}{2}}}{M^{\frac{1}{2}}\theta(tr(T(U)-T(V)))} \\
\theta(tr(T(U)-T(V))) &\leq \frac{(\theta(tr(M(U,V))))^{\frac{1}{2}}}{M^{\frac{1}{2}}} \\
\theta(tr(T(U)-T(V))) &\leq (\frac{(\theta(tr(M(U,V))))}{M^{\frac{1}{2}}})^{\frac{1}{2}} \\
\theta(tr(T(U)-T(V))) &\leq (\psi(\theta(M(U,V))))^{\frac{1}{2}}.
\end{aligned}$$

Hence, by corollary 2.1, T has a fixed point. Therefore, matrix equation 3.1 has a unique solution $\bar{U} \in \wp(m)$.

Numerical Experiment:

Example 3.1. Consider the matrix equation

$$U = G + \sum_{j=1}^{2} B_j^* \tau(U) B_j, \tag{3.3}$$

where G, B_1 and B_2 are given by

$$G = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.0241 & 0.047 & 0.047 \\ 0.047 & 0.0241 & 0.0241 \\ 0.047 & 0.0241 & 0.0241 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.58 & 0.0671 & 0.58 \\ 0.0671 & 0.58 & 0.0671 \\ 0.58 & 0.0671 & 0.58 \end{pmatrix}.$$

Define $\theta(u) = u + 1$ and $T(u) = \frac{u}{9}$. Then conditions (i) and (ii) of Theorem 3.1 are satisfied for $M = 2$.

By using the iteration

$$U_{n+1} = G + \sum_{j=1}^{2} B_j^* U_n B_j$$

with

$$U_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

After 15 iterations, we get the unique solution

$$\bar{U} = \begin{pmatrix} 1005.154 & 237.819 & 1001.821 \\ 237.819 & 64.151 & 237.514 \\ 1001.821 & 237.514 & 1004.516 \end{pmatrix}$$

of the matrix equation 3.3.

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