Int. J. Anal. Appl. (2023), 21:52

International Journal of Analysis and Applications

Well-Posedness and Exponential Stability of the Von Kármán Beam With Infinite Memory

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Abstract. In the present work, we consider a one-dimensional Von kármán beam with infinite memory, we establish the well-posedness of the system using semigroup theory and prove the exponential stability under some conditions on the kernel of the infinite memory term.

1. Introduction

Many studies have looked at nonlinear dynamical elasticity systems modeled by Von kármán equations, wich is one of the basic equations in mathematical models of physics (see [14, 20–22]). Their importance stems from the fact that many physical phenomena connected to the theory of oscillation are described by non-linear dynamic elastic models. This nonlinear elastic systems incorporating wave equations govern the propagation of waves, oscillations, and vibrations of membranes, plates, and shells. The entire Von kármán's model in contrast to other fundamental models like the Euler-Bernoulli, Raleigh, or Timoshenko is appropriate for considering both transverse and longitudinal displacements for vibrating slender bodies with large deflection (for more discussion see [5–7]).

In [18, 19] Lagnese et al investigated a one-dimensional Von kármán system listed below

$$\begin{cases} \rho A u_{tt} - \left[EAu_x + \frac{1}{2} w_x^2\right]_x, & \text{in } (0,L) \times \mathbb{R}_+, \\ \rho A w_{tt} - EI\left(w_{xx}\right)_{xx} - \left[EA\left(u_x + \frac{1}{2} w_x^2\right) w_x\right]_x, & \text{in } (0,L) \times \mathbb{R}_+. \end{cases} \tag{1.1}$$

Received: Apr. 9, 2023.

2020 *Mathematics Subject Classification.* 35B40, 93D23, 74F05, 93D15.

Key words and phrases. Von kármán beam; exponential stability; semi-group; Lyapunov functional; infinite memory.

https://doi.org/10.28924/2291-8639-21-2023-52

ISSN: 2291-8639

Here E is the Young's modulus, A is the cross-sectional area of the beam, L is the beam length, ρA represents the weight per unit length and EI is the beam stiffness or flexural rigidity. A substantial range of literature use this type of model, where they addressing the problems of existence, uniqueness and asymptotic behavior in time when some damping effects are considered , (See Refs. [4, 17, 25] and the references therein for more information).

In (1998) the system (1.1) was stabilized by Benabdallah and Teniou [14] by fusing the system using two heat equations: both the longitudinal and transverse components, respectively, they showed the unique solution decay exponentially by using Lyapunov functions. Many articles have looked into the stabilization of systems using boundary damping , see Favini et al [10], Puel and Tucsnak [24], and the references therein (see [3, 8, 15, 26]).

In [9] Djebabla and Tatar (2013) by linking the system, take into account the following full Von kármán beam in one dimension. (namely, the longitudinal component) with only one heat equation according to Green and Naghdi's theory [11–13]

$$\begin{cases} u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) \right]_x + \gamma \theta_{tx} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ w_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + d_2 w_{xxxx} + \delta w_t = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \theta_{tt} - \theta_{xx} + \mu_1 \theta_t + \gamma u_{tx} = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

where d_1, d_2, δ, μ_1 and γ are positive constants, with the boundary conditions

$$\begin{cases} u = 0, \quad w = 0, \quad \theta_x = 0, \quad x = 0, L, \quad t > 0, \\ w_x = 0, \quad x = 0, L, \quad t > 0, \end{cases} \tag{1.2}$$

and the initial data

$$\begin{cases} u(0, .) = u_0, \quad u_t(0, .) = u_1, \quad w(0, .) = w_0, \quad w_t(0, .) = w_1, \\ \theta(0, .) = \theta_0, \quad \theta_t(0, .) = \theta_1, \end{cases} \tag{1.3}$$

they succeeded an exponential decay result by using Lyapunov functions.

A natural weak damping term can be thought of as the integral in the infinite memory term. It appears as a memory component in the form of a convolution in the fundamental equation between constraint and deformation. In this context, Khochemane et all In [16] think about infinite memory in a porous-elastic system and a nonlinear damping term:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & x \in (0, 1), t > 0, \\ J\phi_{tt} - \delta \phi_{xx} + bu_x + a\phi + \int_0^\infty g(s)\phi_{xx}(t-s)ds + \alpha(t) f(\phi) = 0, & x \in (0, 1), t > 0, \end{cases}$$

where ϕ and u represent, respectively,the volume fraction and displacement of the solid elastic material, the parameter ρ is the mass density, and J equals the product of the equilibrated inertia by the mass density, and the function g is the relaxation function, the term $\alpha(t) f(\phi)$ is the nonlinear damping term.

In this paper, we study the following Von kármán system with infinite memory

$$\begin{cases} w_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + d_2 w_{xxxx} + \mu w_t (x, t) = 0, \\ u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) \right]_x - \int_0^\infty g(s) u_{xx}(t-s) ds = 0. \end{cases} \tag{1.4}$$

in $\Omega \times (0, \infty)$, where $\Omega = [0, L]$ and d_1, d_2, d and μ are positive constants and the function g is called the relaxation function. We complement system (1.4) with boundary conditions

$$\begin{cases} w(0, t) = w(1, t) = u_x(0, t) = u_x(1, t), \quad t > 0 \\ w_x = 0 \ \ at \ \ x = 0, \ L \ \ for \ \ any \ \ t > 0, \end{cases} \tag{1.5}$$

and the initial data

$$\begin{cases} u(0, .) = u_0, \quad u_t(0, .) = u_1, \\ w(0, .) = w_0, \quad w_t(0, .) = w_1, \\ w_t(x, t) = f_0(x, t) \ \ in \in (0, L), \end{cases} \tag{1.6}$$

2. Preliminaries

First, we introduce the following hypothesis that has been considered in many works such that [1,2]
(H1) $g : R^+ \to R^+$ is a C^1 function satisfying

$$g(0) > 0, \quad \delta - \int_0^\infty g(s) ds = l > 0, \quad \int_0^\infty g(s) ds = g_0.$$

(H2) There exists a non-increasing differentiable function $\alpha, \xi : R^+ \to R^+$ such that

$$g'(t) \leq -\xi(t) g(t), \ \forall t \geq 0.$$

Now, to prove that systems (1.4), (1.5), (1.6) are well posed using the semigroup theory we introduce the following new variable:

$$\begin{cases} \eta^t(x, s) = u(x, t) + u(x; t-s) \\ u_{xx}(x, t-s) = \eta^t_{xx}(x, s) - u_{xx}(x, t) \\ \eta^t_t + \eta^t_s = u_t \end{cases}$$

Therefore, the problem (1.4) is equivalent to

$$\begin{cases} w_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + d_2 w_{xxxx} + \mu w_t (x, t) = 0, \\ u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) \right]_x + g_0 u_{xx} - \int_0^\infty g(s) \eta^t_{xx} ds = 0, \\ \eta^t_t + \eta^t_s - u_t = 0. \end{cases} \tag{2.1}$$

With

$$g_0 = \int_0^\infty g(s) ds.$$

For any regular solution of (2.1), the energy $E(t)$, defined by

$$E(t) = \frac{1}{2} \int_0^L \left\{ w_t^2 + u_t^2 + d_2 w_{xx}^2 + d_1 \left(u_x + \frac{1}{2} (w_x)^2 \right)^2 \right\} dx + \frac{1}{2} (g \circ u_x)(t). \tag{2.2}$$

3. Well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (2.1) using the semigroup theory [23]. First, we introduce the vector function

Let $U = \left(w, w_t, u, u_t, \eta^t\right)^T$, then $U_t = \left(w_t, w_{tt}, u_t, u_{tt}, \eta_t^t\right)^T$. Introducing the vector function $v = w_t$, and $\psi = u_t$, system (2.1) can be written as

$$\begin{cases} U_t = \mathcal{A}U + \mathcal{F}(U), \\ U(0) = (w_0, w_1, u_0, u_1, \theta_0, q_0, f_0), \end{cases} \tag{3.1}$$

and the linear operator $\mathcal{A}$ is defined by:

$$\mathcal{A}\begin{pmatrix} w \\ v \\ u \\ \psi \\ \eta^t \end{pmatrix} = \begin{pmatrix} v \\ -d_2 w_{xxxx} - \mu v \\ \psi \\ l u_{xx} + \int_0^\infty g(s)\eta_{xx}^t ds \\ \psi - \eta_s^t \end{pmatrix}, \tag{3.2}$$

and

$$\mathcal{F}(U) = \begin{pmatrix} 0 \\ d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x \\ 0 \\ \frac{d_1}{2}(w_x)_x^2 \\ 0 \end{pmatrix}, \quad U_0 = \begin{pmatrix} w_0 \\ w_1 \\ u_0 \\ u_1 \\ \theta_0 \end{pmatrix}. \tag{3.3}$$

It is clear that $\mathcal{F}(U)$ is a continuous and uniformly Lipschitz operator.

And $\mathcal{H}$ is the energy space given by

$$\mathcal{H} := \left\{\left[H^4(0,L) \cap H_0^2(0,L)\right] \times H_0^1(0,L) \times \left[H^2(0,L) \cap H_0^2(0,L)\right] \times H_0^1(0,L) \times L_g\right\}.$$

$$L_g = \left\{\varphi : R_+ \to H_0^1(0,L), \int_0^L \int_0^\infty g(s)\varphi_x^2 ds dx \prec \infty\right\},$$

We equip the space L_g with the inner product through

$$\langle u, v\rangle_{L_g} = \int_0^L \int_0^\infty g(s) u_x(s) v_x(s) ds dx,$$

and we define on the Hilbert space $\mathcal{H}$ the inner product, for $U = \left(w, v, u, \psi, \eta^t\right)^T$, $\widetilde{U} = \left(\widetilde{w}, \widetilde{v}, \widetilde{u}, \widetilde{\psi}, \widetilde{\eta}^t\right)^T$

$$\begin{aligned}\left\langle U, \widetilde{U}\right\rangle &= \int_0^L v\widetilde{v} dx + \int_0^L \psi\widetilde{\psi} dx + d_2\int_0^L w_{xx}\widetilde{w}_{xx} dx \\ &+ l\int_0^L u_x\widetilde{u}_x dx + \left\langle \eta^t, \widetilde{\eta}^t\right\rangle.\end{aligned}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \left(w, v, u, \psi, \eta^t\right)^T \in \left[H^4(0,L) \cap H_0^2(0,L)\right] \times H_0^1(0,L) \\ \times \left[H^2(0,L) \cap H_0^2(0,L)\right] \times H_0^1(0,L) \times L_g \end{array} \right\}. \tag{3.4}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

The next result is our first main goal in this paper

Theorem 3.1. *Let $U \in \mathcal{H}$, for any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution $U \in C([0,\infty), \mathcal{H})$ for problem. Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), \mathcal{H})$.*

Proof. We will show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = \left(w, v, u, \psi, \eta^t\right)^T$

firstly we have:

$$\begin{aligned} \langle \mathcal{A}U, U \rangle &= \left\langle \begin{pmatrix} v \\ -d_2 w_{xxxx} - \mu v \\ \psi \\ l u_{xx} + \int_0^\infty g(s)\eta^t_{xx} ds \\ \psi - \eta^t_s \end{pmatrix}, \begin{pmatrix} w \\ v \\ u \\ \psi \\ \eta^t \end{pmatrix} \right\rangle \\ &= \int_0^L v\left(-d_2 w_{xxxx} - \mu v\right) dx - d_2 \int_0^L \varphi w_{xxxx} dx + d_1 \int_0^L \psi u_{xx} dx \\ &+ \int_0^L \psi \left(l u_{xx} + \int_0^\infty g(s)\eta^t_{xx} ds \right) dx \\ &+ d_2 \int_0^L w_{xx} v_{xx} dx + l \int_0^L \psi_x u_x dx \\ &+ \left\langle \eta^t, \psi - \eta^t_s \right\rangle . \end{aligned} \tag{3.5}$$

Using integration by parts, we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle = &-d_2 \int_0^L v w_{xxxx} dx - \mu \int_0^L v^2 dx \\ &+ l \int_0^L \psi u_{xx} dx + \int_0^L \psi \int_0^\infty g(s)\eta^t_{xx} ds dx \\ &+ d_2 \int_0^L v_{xx} w_{xx} dx + l \int_0^L \psi_x u_x dx \\ &+ \int_0^L \int_0^\infty g(s)\eta^t_x \left(\psi - \eta^t_{sx}\right) ds dx, \end{aligned}$$

thus,

$$\langle \mathcal{A}U, U \rangle = -\mu \int_0^L v^2 dx + \frac{1}{2}(g' \circ u_x)(t).$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective.

For this purpose, let

$$(f_1, f_2, f_3, f_4, f_5)^T \in H,$$

we seek

$$U = \left(w, v, u, \psi, \eta^t\right)^T \in D(\mathcal{A}),$$

solution of the following system of equations

$$\begin{cases} w - v = f_1, \\ v + d_2 w_{xxxx} + \mu v = f_2, \\ u - \psi = f_3, \\ \psi - l u_{xx} - \int_0^\infty g(s)\eta^t_{xx} ds = f_4, \\ \eta^t + \eta^t_s - \psi = f_5, \end{cases} \tag{3.6}$$

we obtain

$$\begin{cases} \eta^t = e^{-s}\int_0^L e^{\varsigma}\left(\psi + f_5(\varsigma)\right) d\varsigma, \\ v = w - f_1, \\ \psi = u - f_3, \end{cases}$$

$$\begin{cases} w - f_1 + d_2 w_{xxxx} + \mu w - \mu f_1 = f_2, \\ u - f_3 - l u_{xx} - \int_0^\infty g(s)\left(e^{-s}\int_0^s e^{\varsigma}\left(\psi + f_5(\varsigma)\right) d\varsigma\right)_{xx} ds = f_4, \end{cases}$$

then

$$\begin{aligned} -\int_0^\infty g(s)\left(e^{-s}\int_0^L e^{\varsigma}\left(\psi + f_5(\varsigma)\right) d\varsigma\right)_{xx} ds &= -\int_0^\infty g(s)\left(e^{-s}\int_0^s e^{\varsigma}\left(u - f_3 + f_5(\varsigma)\right) d\varsigma\right)_{xx} ds \\ &= -\int_0^\infty g(s)e^{-s}\int_0^s e^{\varsigma}\left(u_{xx} + (-f_3 + f_5)_{xx}\right) d\varsigma ds \\ &= -\int_0^\infty g(s)e^{-s}\int_0^s e^{\varsigma} u_{xx} d\varsigma ds \\ &\quad -\int_0^\infty g(s)e^{-s}\int_0^s e^{\varsigma}\left(-f_3 + f_5\right)_{xx} d\varsigma ds \\ &= -u_{xx}\int_0^\infty g(s)\left(1 - e^{-s}\right) ds \\ &\quad -\int_0^\infty g(s)e^{-s}\int_0^s e^{\varsigma}\left(-f_3 + f_5\right)_{xx} d\varsigma ds, \end{aligned}$$

we obtain

$$\begin{cases} (\mu + 1) w + d_2 w_{xxxx} = f_2 + (\mu + 1) f_1, \\ u - \left[l + \int_0^\infty g(s)\left(1 - e^{-s}\right) ds\right] u_{xx} = f_3 + f_4 + \int_0^\infty g(s)e^{-s}\int_0^s e^{\varsigma}\left(-f_3 + f_5\right)_{xx} d\varsigma ds, \end{cases}$$

where

$$\begin{cases} h_1 = f_2 + (\mu + 1) f_1, \\ h_2 = f_3 + f_4 + \int_0^\infty g(s)e^{-s}\int_0^s e^{\varsigma}\left(-f_3 + f_5\right)_{xx} d\varsigma ds. \end{cases}$$

Now, we consider the following variational formulation

$$B\left((w, u); (w_1, u_1)\right) = L\left(w_1, u_1\right),$$

where $B:\left[H_0^2(0,1)\times H_0^1(0,1)\right]^2\to R$ is the bilinear form defined by

$$B\left((w,u);(w_1,u_1)\right)=(\mu+1)\int_0^L ww_1dx+d_2\int_0^L w_{xx}w_{1xx}dx+\int_0^L uu_1dx$$
$$+\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\int_0^L u_xu_{1x}dx,$$

and $L:H_0^2(0,1)\times H_0^1(0,1)\to R$ is the linear functional given by

$$L\left(w_1,u_1\right)=\int_0^L h_1w_1dx+\int_0^L h_2u_1dx,$$

now for $V=H_0^2(0,1)\times H_0^1(0,1)$ equipped with the norm

$$\|(w,u)\|_V^2=\|w\|_2^2+\|u\|_2^2+\|w_{xx}\|_2^2+\|u_x\|_2^2,$$

we have

$$\begin{aligned}B\left((w,u);(w,u)\right)&=\int_0^L w^2dx+\int_0^L w_{xx}^2dx+\int_0^L u^2dx+\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\int_0^L u_x^2dx\\&\geq\|w\|_2^2+\|w_{xx}\|_2^{2+}\|u\|_2^2+\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\|u_x\|_2^2\\&\geq M_0\left(\|w\|_2^2+\|u\|_2^2+\|w_{xx}\|_2^2+\|u_x\|_2^2\right)\\&=M_0\|(w,u)\|_V^2,\end{aligned}$$

thus B is coercive. On the other hand, by using Cauchy-Schwarz and Poincaré inequalities, we obtain

$$\begin{aligned}\left|B\left((w,u);(w_1,u_1)\right)\right|&\leq\|w\|_2\|w_1\|_2+d_2\|w_{xx}\|_2\|w_{1xx}\|_2+\|u\|_2\|u_1\|_2\\&\quad+\left[l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right]\|u_x\|_2\|u_{1x}\|_2\\&\leq\xi\left(\|w\|_2+\|u\|_2+\|w_{xx}\|_2+\|u_x\|_2\right)\\&\quad\times\left(\|w_1\|_2+\|u_1\|_2+\|w_{1xx}\|_2+\|u_{1x}\|_2\right)\\&\leq\xi\|(w,u)\|_V\|(w_1,u_1)\|_V.\end{aligned}$$

Similarly, we can show that

$$\left|L\left(w_1,u_1\right)\right|\leq\kappa\|(w_1,u_1)\|_V,$$

consequently, by the Lax-Milgram lemma the system has unique solution

$$(w,u)\in H_0^2(0,1)\times H_0^1(0,1),$$

satisfying

$$B\left((w,u);(w_1,u_1)\right)=L\left(w_1,u_1\right),\forall\left(w_1,u_1\right)\in V,$$

the substitution of w and u yields

$$(v,\psi)\in H_0^2(0,1)\times H_0^1(0,1).$$

Similarly, now by taking $u_1 = 0 \in H_0^2(0,1)$

$$\int_0^L u_1 u dx + \left[l + \int_0^\infty g(s)\left(1 - e^{-s}\right) ds \right] \int_0^L u_x u_{1x} dx = \int_0^L h_2 u_1 dx,$$

hence, we obtain

$$\left[l + \int_0^\infty g(s)\left(1 - e^{-s}\right) ds \right] \int_0^L u_x u_{1x} dx = \int_0^L \left(h_2 - u\right) u_1 dx,$$

by noting that $(h_2 - u) \in L^2(0,1)$, we obtain $u \in H^2(0,1) \cap H_0^1(0,1)$ and consequentely the form

$$\int_0^L \left[-\left[l + \int_0^\infty g(s)\left(1 - e^{-s}\right) ds \right] u_{xx} - h_2 + u \right] u_1 dx = 0, \forall u_1 \in H_0^1(0,1),$$

therefore, we obtain

$$-\left[l + \int_0^\infty g(s)\left(1 - e^{-s}\right) ds \right] u_{xx} + u = h_2,$$

similarly, if we take $w_1 = 0 \in H_0^2(0,1)$ using twice integration by parts we obtain

$$(\mu + 1)\int_0^L w w_1 dx + d_2 \int_0^L w_{xx} w_{1xx} dx = \int_0^L h_1 w_1 dx$$

$$d_2 \int_0^L w_{xx} w_{1xx} dx = \int_0^L \left(h_1 - (\mu + 1) w\right) w_1 dx$$

$$\int_0^L \left(d_2 w_{xxxx} + (\mu + 1) w - h_1\right) w_1 dx = 0, \forall w_1 \in H_0^2(0,1).$$

Therefore, we obtain

$$d_2 w_{xxxx} + (\mu + 1) w = h_1 \in L^2(0,1).$$

Consequentely, we get $w \in H^4(0,1) \cap H_0^2(0,1)$. Hence, there exists a unique $U \in D(A)$ such that $\lambda I - \mathcal{A}$ is satisfied. Therefore, the operator $\mathcal{A}$ is dissipative.

From where, we conclude that $\mathcal{A}$ is a maximal monotone operator. Now, we prove that the operator $\mathcal{F}$ is locally Lipschitz in $\mathcal{H}$. In fact, if $U = \left(w, v, u, \psi, \eta^t\right)^T$, $\widetilde{U} = \left(\widetilde{w}, \widetilde{v}, \widetilde{u}, \widetilde{\psi}, \widetilde{\eta}^t\right)^T$ belong to $\mathcal{H}$, we have

$$\left\| \mathcal{F}(U) - \mathcal{F}\left(\widetilde{U}\right) \right\|_H^2 = d_1 \left(|h|^2 + |g|^2 \right), \tag{3.7}$$

where $h = \left[\left(u_x + \frac{1}{2} w_x^2\right) w_x - \left(\widetilde{u}_x + \frac{1}{2}\widetilde{w}_x^2\right)\widetilde{w}_x\right]_x$and $g = \frac{1}{2}\left(w_x^2 - \widetilde{w}_x^2\right)_x$. Adding and subtracting the term $\left(u_x + \frac{1}{2} w_x^2\right)\widetilde{w}_x$ inside the norm $|h|$, we gets

$$\begin{aligned} |h| \leq \|w_x - \widetilde{w}_x\|_{L^\infty(0,L)} \left| u_x + \frac{1}{2} w_x^2 \right| + \|\widetilde{w}_x\|_{L^\infty(0,L)} |u_x - \widetilde{u}_x| \\ + \frac{1}{2} \|\widetilde{w}_x\|_{L^\infty(0,L)} |w_x + \widetilde{w}_x| \, \|w_x - \widetilde{w}_x\|_{L^\infty(0,L)}. \end{aligned} \tag{3.8}$$

Using the embedding of $H^1(0,L)$ into $L^\infty(0,L)$ one has from (3.8) that

$$|h| \leq k\left(\|U\|_H, \left\|\widetilde{U}\right\|_H \right) \left\| U - \widetilde{U} \right\|_H \tag{3.9}$$

Using once again the embedding of $H^{1}(0,L)$ into $L^{\infty}(0,L)$, one also sees that

$$|g| \leq k\left(\|U\|_{H}, \left\|\widetilde{U}\right\|_{H}\right)\left\|U-\widetilde{U}\right\|_{H}. \tag{3.10}$$

Combining (3.7), (3.9) and (3.10), it follows that $\mathcal{F}(U)$ is locally Lipschitz continuous in $\mathcal{H}$. Finally, by using the regularity theory to solve linear elliptic equations ensures the presence of unique solution $U \in D(\mathcal{A})$. Consequently, $\mathcal{A}$ is a dissipative operator. Hence, the result of Theorem 3.1 follows from the Lumer-Phillips theorem. □

4. Stability result

In this section, we use the multiplier technique, to study the stability result for the energy of solution of the system (2.1). First, we state and prove the following lemma.

Lemma 4.1. *Let (w,v,u,ψ,η^{t}) be the solution of (2.1). Then the energy functional $E(t)$, defined by (2.2) satisfies*

$$E'(t) \leq -\mu\int_0^L w_t^2dx + \frac{1}{2}(g'\circ u_x)(t). \tag{4.1}$$

Proof. Multiplying the first equation by w_t , the second equation by u_t, integrating over $(0,1)$, and summing them up we obtain

$$\frac{d}{2dt}\int_0^L w_t^2dx + d_1\left(u_x+\frac{1}{2}(w_x)^2\right)w_xw_{xt}dx + \frac{d_2d}{2dt}\int_0^L (w_{xx})^2dx + \mu\int_0^L w_t^2dx = 0$$
$$\frac{d}{2dt}\int_0^L u_t^2dx + d_1\left(u_x+\frac{1}{2}(w_x)^2\right)u_{xt}dx - \int_0^L u_t\int_0^\infty g(s)\eta_{xx}^t dsdx = 0$$

We estimate the last term as follows:

$$\begin{aligned}
-\int_0^L u_t\int_0^\infty g(s)\eta_{xx}^t dsdx &= -\int_0^L\left(\eta_t^t+\eta_s^t\right)\int_0^\infty g(s)\eta_{xx}^t dsdx\\
&= -\int_0^\infty g(s)\int_0^L \eta_{xx}^t\left(\eta_t^t+\eta_s^t\right)dxds\\
&= -\int_0^\infty g(s)\int_0^L \eta_{xx}^t\eta_t^t dxds - \int_0^\infty g(s)\int_0^L \eta_{xx}^t\eta_s^t dxds\\
&= \int_0^\infty g(s)\int_0^L \eta_x^t\eta_{tx}^t dxds + \int_0^\infty g(s)\int_0^L \eta_x^t\eta_{sx}^t dxds\\
&= \frac{d}{2dt}\int_0^L\int_0^\infty g(s)\left(\eta_x^t\right)^2dsdx + \frac{1}{2}\int_0^L\int_0^\infty g(s)\frac{d}{ds}\left(\eta_x^t\right)^2dsdx\\
&= \frac{d}{2dt}\int_0^L\int_0^\infty g(s)\left[u_x(t)-u_x(t-s)\right]^2dsdx - \frac{1}{2}\int_0^L\int_0^\infty g'(s)\left(\eta_x^t\right)^2dsdx\\
&= \frac{d}{2dt}\int_0^L\int_0^\infty g(s)\left[u_x(t)-u_x(t-s)\right]^2dsdx\\
&\quad -\frac{1}{2}\int_0^L\int_0^\infty g'(s)\left[u_x(t)-u_x(t-s)\right]^2dsdx\\
&= \frac{d}{2dt}(g\circ u_x)(t) - \frac{1}{2}(g'\circ u_x)(t).
\end{aligned}$$

□

Lemma 4.2. *Let $\left(w, v, u, \psi, \eta^t\right)$ be the solution of (2.1). Then the functional*

$$I_1(t)=\int_0^L\left(u_t u+\frac{1}{2} w_t w+\frac{\mu}{4} w^2\right) dx, \quad t \geq 0. \tag{4.2}$$

satisfies, the estimate

$$\begin{aligned} I_1^{\prime}(t)= & -d_1 \int_0^L\left(u_x+\frac{1}{2}\left(w_x\right)^2\right)^2 dx+\int_0^L u_t^2 dx \\ & -\frac{d_2}{2} \int_0^L w_{xx}^2 dx+g_0 \int_0^L u_x^2 dx \\ & +\frac{1}{2} \int_0^L w_t^2 dx+\int_0^L u \int_0^{\infty} g(s) \eta_{xx}^t ds dx \end{aligned} \tag{4.3}$$

Proof. Using Young's inequalitie we get

$$\begin{aligned} \int_0^L u \int_0^{\infty} g(s) \eta_{xx}^t ds dx & =-\int_0^L u_x \int_0^{\infty} g(s)\left(u_x(t)-u_x(t-s)\right) ds dx \\ & \leq \varepsilon_1 \int_0^L u_x^2 dx+\frac{1}{4 \varepsilon_1} \int_0^L\left(\int_0^{\infty} g(s)\left(u_x(t)-u_x(t-s)\right)\right)^2 ds dx \\ & \leq \varepsilon_1 \int_0^L u_x^2 dx+\frac{g_0}{4 \varepsilon_1}\left(g \circ u_x\right)(t), \end{aligned} \tag{4.4}$$

$$\begin{aligned} I_1^{\prime}(t) \leq & -d_1 \int_0^L\left(u_x+\frac{1}{2}\left(w_x\right)^2\right)^2 dx-\frac{d_2}{2} \int_0^L\left(w_{xx}\right)^2 dx \\ & +\left(g_0+\varepsilon_1\right) \int_0^L u_x^2 dx+\int_0^L u_t^2 dx \\ & +\frac{1}{2} \int_0^L w_t^2 dx .+\frac{g_0}{4 \varepsilon_1}\left(g \circ u_x\right)(t), \end{aligned}$$

□

Lemma 4.3. *Let $\left(w, v, u, \psi, \eta^t\right)$ be the solution of (2.1). Then the functional*

$$I_2(t):=-\int_0^L u_t \int_0^{\infty} g(s)\left(u_x(t)-u_x(t-s)\right) ds dx \qquad t \geq 0, \tag{4.5}$$

satisfies, the estimate

$$\begin{aligned} I_2^{\prime}(t) \leq & -\frac{g_0}{2} \int_0^L u_t^2 dx+\frac{d_1}{2} \int_0^L\left(u_x+\frac{1}{2}\left(w_x\right)^2\right)^2 dx \\ & +\left(g_1+d_1\right)\left(g \circ u_x(t)\right)+\frac{g_0}{2} \int_0^L u_x^2 dx \qquad t \geq 0, \\ & +\frac{g(0)}{2 g 0}\left(g^{\prime} \circ u_x\right)(t). \end{aligned} \tag{4.6}$$

Proof. firste we note that

$$\begin{aligned}\frac{d}{dt}\int_0^\infty g(s)(u(t)-u(t-s))ds &= \frac{d}{dt}\int_{-\infty}^t g(t-s)(u(t)-u(s))ds\\ &= \int_{-\infty}^t g^{'}(t-s)(u(t)-u(s))ds + \int_{-\infty}^t g(t-s)u(t)ds\\ &= g_0u_t(t) + \int_0^\infty g^{'}(s)(u(t)-u(t-s))ds,\end{aligned}$$

$$\begin{aligned}&I_2'(t) = -\int_0^L u_{tt}\int_0^\infty g(s)(u(t)-u(t-s))dsdx - \int_0^L u_t\frac{d}{dt}\int_0^\infty g(s)(u(t)-u(t-s))dsdx\\ &= -\int_0^L\left(d_1\left[\left(u_x+\frac{1}{2}(w_x)^2\right)\right]_x - g_0u_{xx} + \int_0^\infty g(s)\eta_{xx}^t ds\right)\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx\\ &- g_0\int_0^L u_t^2dx - \int_0^L u_t\int_0^\infty g^{'}(s)(u(t)-u(t-s))dsdx\\ &= -d_1\int_0^L\left[\left(u_x+\frac{1}{2}(w_x)^2\right)\right]_x\int_0^\infty g(s)(u(t)-u(t-s))dsdx\\ &+ g_0\int_0^L u_{xx}\int_0^\infty g(s)(u(t)-u(t-s))dsdx\\ &- \int_0^L\left(\int_0^\infty g(s)\eta_{xx}^t ds\right)\left(\int_0^\infty g(s)(u(t)-u(t-s))ds\right)dx\\ &- g_0\int_0^L u_t^2dx - \int_0^L u_t\int_0^\infty g^{'}(s)(u(t)-u(t-s))dsdx\\ &= d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)\int_0^\infty g(s)(u_x(t)-u_x(t-s))dsdx\\ &- g_0\int_0^L u_x\int_0^\infty g(s)(u_x(t)-u_x(t-s))dsdx\\ &+ \int_0^L\left(\int_0^\infty g(s)\eta_x^t ds\right)\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)dx\\ &- g_0\int_0^L u_t^2dx - \int_0^L u_t\int_0^\infty g^{'}(s)(u(t)-u(t-s))dsdx.\end{aligned}$$

By recalling Young's inequality, we get for any $\varepsilon_2 > 0$,

$$\begin{aligned}&d_1\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)\int_0^\infty g(s)(u_x(t)-u_x(t-s))dsdx\\ &\le d_1\varepsilon_2\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)^2dx\\ &+ \frac{d_1}{4\varepsilon_2}\int_0^L\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)^2dx\\ &\le d_1\varepsilon_2\int_0^L\left(u_x+\frac{1}{2}(w_x)^2\right)^2dx + \frac{g_0d_1}{4\varepsilon_2}\left(g\circ u_x(t)\right),\end{aligned}$$

$$-g_0\int_0^L u_x\int_0^\infty g(s)(u_x(t)-u_x(t-s))dsdx \le g_0\varepsilon_3\int_0^L u_x^2dx+\frac{g_0}{4\varepsilon_3}(g\circ u_x)(t),$$

$$\begin{aligned}-\int_0^L u_t\int_0^\infty g'(s)(u(t)-u(t-s))dsdx &= -\int_0^L \sqrt{g_0}u_t\int_0^\infty\left(\sqrt{\frac{1}{g_0}}g'(s)(u(t)-u(t-s))\right)dsdx\\ &\le \frac{g_0}{2}\int_0^L u_t^2dx+\frac{g(0)}{2g_0}(g'\circ u_x)(t),\end{aligned}$$

$$\begin{aligned}&\int_0^L\left(\int_0^\infty g(s)\eta_x^t ds\right)\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)dx\\ &=\int_0^L\left(\int_0^\infty g(s)(u_x(t)-u_x(t-s))ds\right)^2dx\\ &\le d_3(g\circ u_x)(t).\end{aligned}$$

Wich complete the proof. □

Now, we define the Lyapunov functional $L(t)$by

$$L(t)=NE(t)+N_1F_1(t)+N_2F_2(t),$$

where$N$, $N_1$and N_2 are positive constants.

Lemma 4.4. *Let (u,w) be the solution of Then, there exist two positive constants C_1 and C_2 such that the Lyapunov functional $L(t)$ satisfies*

$$C_1E(t)\le L(t)\le C_2E(t),\ \ \forall t\ge 0. \tag{4.7}$$

In other words, the functions E and L are equivalent.

Proof. Firstly we note that

$$\begin{aligned}\int_0^L u_x^2dx &\le 2\int_0^L\left(u_x+\frac{1}{2}\left(w_x^2\right)\right)^2dx+\frac{1}{2}\int_0^L w_x^2dx\\ &\le 2\int_0^L\left(u_x+\frac{1}{2}\left(w_x^2\right)\right)^2dx+\frac{1}{4}\int_0^L w_x^4dx,\\ &\le 2\int_0^L\left(u_x+\frac{1}{2}\left(w_x^2\right)\right)^2dx+\frac{L}{4}\int_0^L w_{xx}^2dx,\end{aligned}$$

we get

$$|L(t)-NE(t)|\le N_1\int_0^L\left|u_tu+\frac{1}{2}w_tw+\frac{\mu}{4}w^2\right|dx+N_2\int_0^L\left|u_t\int_0^\infty g(s)(u(t)-u(t-s))ds\right|dx.$$

By using Young's inequality, Cauchy–Schwarz inequality, and Poincaré's inequality, we obtain

$$
\begin{aligned}
|L(t) - NE(t)| \leq{}& [N_1\varepsilon_1 + N_2\varepsilon_3] \int_0^L u_t^2 dx \\
&+ \frac{N_1\varepsilon_2}{4} \int_0^L w_t^2 dx \\
&+ \frac{N_1\varepsilon_1}{2} \int_0^L \left(u_x + \frac{1}{2}\left(w_x^2\right)\right)^2 dx \\
&+ \left[\frac{N_1 L}{16\varepsilon_1} + \frac{N_1}{4\varepsilon_2} + \frac{N_1\mu}{4}\right] \int_0^L w_{xx}^2 dx \\
&+ \frac{d_3}{4\varepsilon 3}(g \circ u_x)(t).
\end{aligned}
$$

So

$$
\begin{aligned}
|L(t) - NE(t)| &\leq c \int_0^L \left\{ w_t^2 + u_t^2 + d_2 w_{xx}^2 + d_1 \left(u_x + \frac{1}{2}(w_x)^2\right)^2 \right\} dx + c(g \circ u_x)(t) \\
&\leq cE(t).
\end{aligned}
$$

then

$$
(N - c)E(t) \leq L(t) \leq (N + c)E(t).
$$

Consequently, By choosing N large enough, we obtain the estimate (4.7). □

Now, we are ready to state and prove the main result of this section.

Theorem 4.1. *Let $\left(w, v, u, \psi, \eta^t\right)$ be the solution of (2.1). Then the energy functional (2.2) satisfies,*

$$
E(t) \leq k_0 e^{-k_1 t}, \ \forall t \geq 0, \tag{4.8}
$$

where k_0 and k_1 are two positive constants.

Proof.

$$
\begin{aligned}
L'(t) \leq{}& -\left[\frac{N_2 g_0}{2} - N_1\right] \int_0^L u_t^2 dx \\
&- \left[N\mu - \frac{N_1}{2}\right] \int_0^L w_t^2 dx \\
&- \left[d_1 N_1 - \frac{N_2 d_1}{2} - 2N_1(g_0 + \varepsilon_1) - N_2 g_0\right] \int_0^L \left(u_x + \frac{1}{2}\left(w_x^2\right)\right)^2 dx \\
&- \left[\frac{N_1 d_2}{2} - \frac{N_1 L}{4}(g_0 + \varepsilon_1) - \frac{N_2 g_0 L}{8}\right] \int_0^L w_{xx}^2 dx \\
&- \left[\frac{N_1 g_0}{4\varepsilon_1} + (N_2 g_0 + d_3)\right] (g \circ u_x)(t) \\
&- \left[\frac{N}{2} + \frac{N_2 g(0)}{2g_0}\right] (g' \circ u_x)(t).
\end{aligned}
$$

By setting

$$\varepsilon_1 = \frac{1}{N_1},$$

we obtain

$$\begin{aligned} L'(t) \leq & -\left[\frac{N_2 g_0}{2} - N_1\right] \int_0^L u_t^2 dx \\ & - \left[N\mu - \frac{N_1}{2}\right] \int_0^L w_t^2 dx \\ & - \left[d_1 N_1 - \frac{N_2 d_1}{2} - 2N_1 g_0 - 1 - N_2 g_0\right] \int_0^L \left(u_x + \frac{1}{2}\left(w_x^2\right)\right)^2 dx \\ & - \left[\frac{N_1 d_2}{2} - \frac{N_1 L}{4} g_0 - \frac{L}{4} - \frac{N_2 g_0 L}{8}\right] \int_0^L w_{xx}^2 dx \\ & - \left[\frac{N_1^2 g_0}{4} + (N_2 g_0 + d_3)\right] (g \circ u_x)(t) \\ & - \left[\frac{N}{2} + \frac{N_2 g(0)}{2g_0}\right] (g' \circ u_x)(t), \end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_1 large enough such that

$$\alpha_1 = N\mu - \frac{N_1}{2} > 0,$$

then we choose N_2 large enough such that

$$\alpha_2 = \frac{N_2 g_0}{2} - N_1 > 0,$$

$$\alpha_3 = d_1 N_1 - \frac{N_2 d_1}{2} - 2N_1 g_0 - 1 - N_2 g_0 > 0,$$

$$\alpha_4 = \frac{N_1 d_2}{2} - \frac{N_1 L}{4} g_0 - \frac{L}{4} - \frac{N_2 g_0 L}{8} > 0,$$

$$\alpha_5 = \frac{N_1^2 g_0}{4} + (N_2 g_0 + d_3) > 0,$$

Finally, once N_1 and N_2, is fixed, we choose N large enough so that

$$\alpha_6 = \frac{N}{2} + \frac{N_2 g(0)}{2g_0} > 0,$$

we obtain

$$L^{'}(t) \leq -\frac{1}{2}\int_0^1 \left\{\alpha_1 w_t^2 + \alpha_2 u_t^2 + \alpha_3 \left(u_x + \frac{1}{2}\left(w_x^2\right)\right)^2 + \alpha_4 w_{xx}^2\right\} - \frac{\alpha_5}{2}(g \circ u_x)(t),$$

By (2.2), we obtain

$$L^{'}(t) \leq -\sigma_0 E(t) \quad , \forall t \geq 0, \tag{4.9}$$

for some $\sigma_0 > 0$. A combination of (4.7) and (4.9) gives

$$L^{'}(t) \leq -k_1 L(t) \quad , \forall t \geq 0, \tag{4.10}$$

A simple integration of (4.10) over $(0, t)$ yields

$$L(t) \leq L(0)\, e^{-k_1 t}, \ \forall t \geq 0. \tag{4.11}$$

Finally, by combining (4.7) and (4.11) we obtain (4.8), which completes the proof. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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