Int. J. Anal. Appl. (2023), 21:78

International Journal of Analysis and Applications

Some Aspects of Rectifying Curves on Regular Surfaces Under Different Transformations

Sandeep Sharma*, **Kuljeet Singh**

School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, Jammu and Kashmir, India; kulljeet83@gmail.com

**Corresponding author:* sandeep.greater123@gmail.com

Abstract. An essential space curve in the study of differential geometry is the rectifying curve. In this paper, we studied the adequate requirement for a rectifying curve under the isometry of the surfaces. The normal components of the rectifying curves are also studied, and it is investigated that for rectifying curves, the Christoffel symbols and the normal components along the surface normal are invariant under the isometric transformation. Moreover, we also studied some properties for the first fundamental form of the surfaces.

1. Introduction

Differential geometry is the area of geometry that employs calculus to study the characteristics of curves and surfaces of all kinds. It primarily focuses on the features of a small subset of geometric configurations of curves and surfaces. Different types of curves are explored in differential geometry, but the regular curves are the most significant ones. The number of continuous derivatives is a characteristic that indicates how smooth a curve is. If a curve is differentiable and thus continuous everywhere, it is considered to be smooth. Similar to this, if a curve can be differentiated and has no zero derivative, it is said to be regular. In differential geometry, the study of regular maps is a key area of research. For more information on the regular curve, we can refer the reader to see [3].

There are many ways to categorise motions, but we'll concentrate on the ones that preserve particular geometrical characteristics. We categorise transformations generally into the following equivalence classes: conformal, isometric, homothetic, and non-conformal or general motion, depending on the

Received: May 8, 2023.

2020 *Mathematics Subject Classification.* 53A04, 53A15, 53A35, 53B30.

Key words and phrases. geodesic curvature; first fundamental form; orthonormal frame; isometry of surfaces.

https://doi.org/10.28924/2291-8639-21-2023-78
ISSN: 2291-8639

varying nature of the mean curvature (M) and the Gaussian curvature (G). In isometry, lengths and angles between curves on surfaces are both preserved. In terms of geometry, isometry preserves the Gaussian curvature's invariance while altering the mean curvature. The isometry between a helicoid and a catenoid, which suggests that they have the same G but different M, is one of the best known examples.

The most significant transformation is a conformal transformation, which preserves angles in terms of magnitude and direction but not always in terms of length. Conformal maps play a significant role in cartography. The stereographic projection, which maps a sphere onto a plane, is the most typical example of conformal transformation. In the year 1569, Gerardus Mercator initially used this conformal map characteristic to produce the legendary Mercator's globe map, the first conformal world map. We suggest that the reader watch an animated movie on conformal maps that was released by Bobenko and Gunn in 2018 along with Springer VideoMATH for additional details regarding the application of conformal maps [7]. Angles and distances between any pair of intersecting curves are not preserved in the context of general motion. The use of motion, transformation, and maps is for same throughout the paper.

The normal, rectifying, and osculating curves are the most often covered topics in differential geometry, therefore they are typically covered in every basic book on differential geometry of curves and surfaces. For more information on these topics, we can refer the reader to see [1, 3, 6]. In the Euclidean 3-Dimensional space $\mathbb{R}^3$, Chen et al. [4, 10] studied the motion of rectifying curves and investigated some of the basic properties of such curves. Shaikh et al. [1, 2, 6, 13] investigated the sufficient conditions for the invariance of the conformal image of osculating and normal curves on smooth immersed surfaces and found that there are various other properties of such curves that remain invariant under the isometry of surfaces. In the year 2003, Chen [4] came across the following query regarding rectifying curves: *What occurs when a space curve's position vector is always within the range of its rectifying plane*. It was found that, under the assumption that surfaces are isometric, the component of the position vector of a space curve along the surface normal remains constant.

This paper's primary objective is to expand on the work of Lone et al. [5, 12, 13], they studied the geometric invariants of normal curves under conformal transformation in the Euclidean space $\mathbb{E}^3$. In [5],the author explored the behaviour of the normal and tangential components of the normal curves under the same motion as well as the invariant characteristics of normal curves under conformal transformation. By motivating from the work of Shaikh and Ghosh in the recent papers [2, 6], where they studied regarding the geometric invariants properties of rectifying curves on smooth surface under isometry of the surfaces.

Further in [1], they investigated the invariant properties of osculating curves under the isometry of surfaces. But a natural question arises: What happens with the geometric properties of rectifying curves with respect to conformal transformation in Euclidean space $\mathbb{R}^3$.

In the present paper, we attempt to investigate the conformal image for rectifying curve on regular surfaces under the different transformation. Moreover, we also study some geometric properties for the first fundamental form of the surfaces, as well as the derivation for the normal components of a rectifying curve under the conformal transformation. This article demonstrated that the Christoffel symbols and normal component for the rectifying curves are also invariant under the isometry of the surfaces.

The format of this paper is as follows: Section 2 covers some fundamental definitions and information about dilation function, geodesic curvature, normal curvature, rectifying curves, and normal curves. Section 3 deals with the understanding of rectifying curves on regular surfaces and their conformal image under various transformations. We also examine the major results in this section.

2. Preliminaries

This section includes some essential information on rectifying curves, including their first fundamental form, geodesic and normal curvature, and some basic definitions. Let P and $\tilde{P}$ be two smooth and regular immersed surfaces in the Euclidean space $\mathbb{R}^3$, and $G : P \to \tilde{P}$ be a smooth map. A necessary and sufficient condition for G to be conformal is the first fundamental form quantities being proportional. In other words, the area element of P and $\tilde{P}$ are proportional to a differentiable function (factor), which is denoted by $\zeta(x, y)$ and is commonly known as dilation function. For more information on the dilation function, we can refer the reader to see [3, 8, 10]. A generalised class of certain motions is the conformal transformation, which is defined in the following way [8]:

- When the dilation factor $\zeta(x, y) = c$, where c is a constant with $c \neq \{0, 1\}$, then G is a homothetic transformation.
- When the function $\zeta(x, y) = 1$, then G becomes isometry.

Let $\delta : I \subset \mathbb{R} \to \mathbb{R}^3$ be unit speed smooth parametrized curve with at least fourth order continuous derivative, by an arc length parameter (r). Let the tangent, normal, and binormal of the curve δ is denoted by $\vec{t}$, $\vec{n}$ and $\vec{b}$ respectively. At each point on the curve $\delta(r)$, the vectors $\vec{t}$, $\vec{n}$, and $\vec{b}$ are mutually perpandicular to each other and the triplet $\{\vec{t}, \vec{n}, \vec{b}\}$ so forms an orthonormal frame.

Consider $\vec{t}'(r) \neq 0$, the unit normal vector $\vec{n}$ along the tangents at a point on the curve δ, then we can write $\vec{t}'(r) = \kappa(r)\vec{n}(r)$, where $\vec{t}'(r)$ is the derivative of $\vec{t}$ with respect to arc length parameter 'r' and $\kappa(r)$ is the curvature of $\delta(r)$. Also the binormal vector field is denoted by $\vec{b}$ and is defined by $\vec{b} = \vec{t} \times \vec{n}$, and we can write $\vec{b}'(r) = \tau(r)\vec{n}(r)$, where $\tau(r)$ is another curvature function known as torsion of the curve $\delta(r)$.

In [3, 11, 15], Serret-Frenet equations are given as follows:

$$\begin{aligned}\vec{t}'(r) &= \kappa(r)\vec{n}(r),\\ \vec{n}'(r) &= -\kappa(r)\vec{t}(r)+\tau(r)\vec{b}(r),\\ \vec{b}'(r) &= -\tau(r)\vec{n}(r),\end{aligned}$$

where the functions κ and τ are respectively called the curvature and torsion of the curve δ, satisfying the following conditions:

$$\vec{t}(r)=\delta'(r), \vec{n}(r)=\frac{\vec{t}'(r)}{\kappa(r)} \text{ and } \vec{b}(r)=\vec{t}(r)\times\vec{n}(r).$$

From the arbitrary point $\delta(r)$ on the curve δ, we see that the plane spanned by $\{\vec{t},\vec{n}\}$ is called the osculating plane, and the plane spanned by $\{\vec{t},\vec{b}\}$ is called the rectifying plane. In the same way the plane spanned by $\{\vec{n},\vec{b}\}$ is called the normal plane. Whenever we talk about the position vector of the curve, which defines the different kinds of curves [12, 16, 17] :

- It is possible to define a curve as being a normal curve if its position vector is in the normal plane.
- A curve is said to be a rectifying curve if its position vector is in the rectifying plane.
- If a curve's position vector is in the osculating plane, then the curve is said to be an osculating curve.

Firstly, try to investigate the properties of a rectifying curve on regular surfaces are invariant under conformal transformation.

If the position vector of a curve is located in the rectifying plane, then the curve is said to be a rectifying curve [8, 12, 15], i.e.,

$$\delta(r) = \mu_1(r)\vec{t}(r)+\mu_2(r)\vec{b}(r), \tag{2.1}$$

where μ_1 and μ_2 are two smooth functions. Let $\sigma : U \to P$ be the coordinate chart map on the regular surface P and the smooth parametrized unit speed curve $\delta(r) : I \to P$, where $I = (a\ ,\ b) \subset \mathbb{R}$ and $U \subset \mathbb{R}^2$.

As a result, the curve $\delta(r)$ is given by

$$\delta(r) = \sigma(x(r), y(r)). \tag{2.2}$$

By using chain rule to differentiate (2.2), with respect to r, we get

$$\delta'(r) = \sigma_x x' + \sigma_y y'. \tag{2.3}$$

Now, $\vec{t}(r) = \delta'(r)$. Then, from equation (2.3), we find that

$$\vec{t}(r) = \sigma_x x' + \sigma_y y'. \tag{2.4}$$

When we differentiate equation (2.4) again in terms of r, we obtained

$$\vec{t'}(r) = x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}.$$

If $\mathbb{N}$ is the normal to the surface P and $\kappa(r)$ is the curvature of the curve $\delta(r)$, then the normal vector $\vec{n}(r)$ can be written as

$$\vec{n}(r) = \frac{1}{\kappa(r)}(x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy}). \tag{2.5}$$

Now the binormal vector $\vec{b}(r)$ can be written as

$$\vec{b}(r) = \vec{t}(r) \times \vec{n}(r).$$

By substituting the value of $\vec{t}(r)$ and $\vec{n}(r)$ from equation (2.4) and (2.5) we obtained

$$\begin{aligned}\vec{b}(r) &= \frac{1}{\kappa(r)}[(\sigma_x x' + \sigma_y y') \times (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})],\\ &= \frac{1}{\kappa(r)}[(y''x' - y'x'')N + x'^3\sigma_x \times \sigma_{xx} + 2x'^2y'\sigma_x \times \sigma_{xy} + x'y'^2\sigma_x \times \sigma_{yy}\\ &\quad + x'^2y'\sigma_y \times \sigma_{xx} + 2x'y'^2\sigma_y \times \sigma_{xy} + y'^3\sigma_y \times \sigma_{yy}].\end{aligned} \tag{2.6}$$

2.1. **First Fundamental Form.** Let $\sigma = \sigma(x, y)$ represents the equation of a surface. Let us define $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$, $G = (\sigma_y \cdot \sigma_y)$. Then the expression $Edx^2 + 2Fdxdy + Gdy^2$ is called the first fundamental form, and E, F, and G are called the first fundamental form coefficient or the first fundamental form magnitude. Note that in the above expression, dx and dy cannot vanish together. We denote notion of $\sqrt{EG - F^2}$ by H. A necessary and sufficient condition for the surfaces P and $\tilde{P}$ to be isometric is that the first fundamental form magnitude are invariant, i.e., $\tilde{E} = E$, $\tilde{F} = F$, $\tilde{G} = G$. For more detail one can refer [5]. Some of the main results concerning the first fundamental form are given as follows:

Theorem 2.1. *Let P and $\tilde{P}$ be two regular surfaces in the Euclidean space $\mathbb{R}^3$ and E, F, G are the coefficients of the first fundamental form of the surfaces. Then*

(i) *The first fundamental form is the square of the metric.*
(ii) *The first fundamental form is positive definite form.*
(iii) $H = |\sigma_x \times \sigma_y|$.

Proof. Let $x = x(r)$, $y = y(r)$ be the curve on the surface $\sigma = \sigma(x, y)$. Let $Q(\sigma)$ and $R(\sigma + d\sigma)$ be two neighbouring points on the curve corresponding to the parameter (x, y) and $(x + dx, y + dy)$ respectively, such that arc $(QR) = ds$. Then

$$\begin{aligned} d\sigma &= \frac{\partial\sigma}{\partial x}dx + \frac{\partial\sigma}{\partial y}dy\\ &= \sigma_x dx + \sigma_y dy.\end{aligned}$$

Since Q and R are neighbouring points, therefore arc(QR) = chord (QR), i.e.,

$$\begin{aligned}
ds^2 &= |d\sigma|^2, \\
&= (d\sigma)\cdot(d\sigma), \\
&= (\sigma_x dx + \sigma_y dy)\cdot(\sigma_x dx + \sigma_y dy), \\
&= \sigma_x\cdot\sigma_x dx^2 + 2\sigma_x\cdot\sigma_y dxdy + \sigma_y\cdot\sigma_y dy^2, \\
&= Edx^2 + 2Fdxdy + Gdy^2.
\end{aligned}$$

This proves that the first fundamental form is the square of the metric.

Proof (ii): To prove the positive definiteness of the first fundamental form, it is sufficient to show that, for all real values of dx and dy, the expression for the first fundamental form is greater than 0. Now,

$$\begin{aligned}
Edx^2 + 2Fdxdy + Gdy^2 &= \frac{1}{E}[E^2dx^2 + 2EFdxdy + EGdy^2], \\
&= \frac{1}{E}[(Edx + Fdy)^2 + (EG - F^2)dy^2], \\
&= \frac{1}{E}[(Edx + Fdy)^2 + H^2dy^2], \\
&\geq 0
\end{aligned}$$

for all real values of dx and dy. If $\frac{1}{E}[(Edx + Fdy)^2 + H^2dy^2] = 0$

$\Rightarrow \quad Edx + Fdy = 0$ and $Hdy = 0$

$\Rightarrow \quad Edx + Fdy = 0$ and $dy = 0$, as $H \neq 0$

$\Rightarrow \quad Edx = 0$, and $dy = 0$

$\Rightarrow \quad dx = 0$, $\quad dy = 0$, which are not possible because dx and dy cannot vanish together.

Thus, $Edx^2 + 2Fdxdy + Gdy^2 > 0$, for all real values of dx and dy. It means that the first fundamental form for the surface is positive definite.

Proof (iii): We denote $\sqrt{EG - F^2}$ by H. Therefore, we can write

$$\begin{aligned}
H^2 &= EG - F^2, \\
&= (\sigma_x\cdot\sigma_x)(\sigma_y\cdot\sigma_y) - (\sigma_x\cdot\sigma_y)^2, \\
&= \sigma_x^2\sigma_y^2 - \sigma_x^2\sigma_y^2 cos^2(\theta), \\
&= \sigma_x^2\sigma_y^2(1 - cos^2(\theta)), \\
&= \sigma_x^2\sigma_y^2 sin^2(\theta), \\
&= (\sigma_x^2 \times \sigma_y^2)\cdot(\sigma_x^2 \times \sigma_y^2), \\
&= |\sigma_x \times \sigma_y|^2.
\end{aligned}$$

This proves (iii). □

Definition 2.1. *Let P and $\tilde{P}$ be two regular surfaces in $\mathbb{R}^3$. Then a diffeomorphism $G : P \to \tilde{P}$ is an isometry if G maps the curve of same length from P to $\tilde{P}$.*

Definition 2.2. *[4] Let P and $\tilde{P}$ be two regular surfaces in the Euclidean space $\mathbb{R}^3$ and $\delta(r)$ be a curve having arc length parametrization lies on the surface P. Then $\delta'(r)$ is perpandicular to the unit surface normal $\mathbb{N}$, and also $\delta'(r)$ and $\delta''(r)$ are perpandicular. Thus, δ'' can be represented as the linear combination of $\mathbb{N}$ and $\mathbb{N}\times\delta'$, i.e.,*

$$\delta'' = \kappa_n\mathbb{N} + \kappa_g\mathbb{N}\times\delta',$$

where the parameters κ_n and κ_g, which are commonly known as the normal and geodesic curvatures of the curve δ, and are given by

$$\begin{aligned}\kappa_n &= \delta''\cdot\mathbb{N},\\ \kappa_g &= \delta''\cdot(\mathbb{N}\times\delta').\end{aligned}$$

Now from Serret-Frenet equations we have $\vec{t}'(r) = \delta''(r) = \kappa(r)\vec{n}(r)$.
Now,

$$\begin{aligned}\kappa_n &= \delta''\cdot\mathbb{N},\\ &= \kappa(r)\vec{n}(r)\cdot\mathbb{N},\\ &= (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\cdot\mathbb{N},\\ &= (x''\sigma_x + y''\sigma_y + x'^2\sigma_{xx} + 2x'y'\sigma_{xy} + y'^2\sigma_{yy})\cdot(\sigma_x\times\sigma_y).\end{aligned}$$

By solving the above expression and using the properties of vectors, we find that

$$\kappa_n = x'^2X + 2x'y'Y + y'^2Z,$$

where X, Y, and Z are the magnitudes of second fundamental form [3, 11, 14].
This leads us to the conclusion that the curve $\delta(r)$ on the surface P, is asymptotic if and only if the normal curvature $\kappa_n = 0$ [9, 13].

3. Conformal image of a rectifying curve

Consider two regular surfaces P and $\tilde{P}$ in the Euclidean space $\mathbb{R}^3$ and $\delta(r)$ is a rectifying curve that is located on the surface P. Then $\delta(r)$ can be written as:

$$\delta(r) = \mu_1(r)\vec{t}(r) + \mu_2(r)\vec{b}(r).$$

Now using equation (2.4) and (2.6) we get,

$$\begin{aligned}\delta(r) &= \mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}\{(x'y'' - x''y')\mathbb{N} + x'^3\sigma_x\times\sigma_{xx} + 2x'^2y'\sigma_x\times\sigma_{xy}\\ &\quad + x'y'^2\sigma_x\times\sigma_{yy} + x'^2y'\sigma_y\times\sigma_{xx} + 2x'y'^2\sigma_y\times\sigma_{xy} + y'^3\sigma_y\times\sigma_{yy}\}.\end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $G : P \to \tilde{P}$ be an isometry between two regular and smooth surfaces P and $\tilde{P}$ in the Euclidean space $\mathbb{R}^3$ and $\delta(r)$ be a rectifying curve on the surface P. Then the Christoffel symbols for $\delta(r)$ are invariant under G.*

Proof. Since $G : P \to \tilde{P}$ is an isometry and $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$, $G = (\sigma_y \cdot \sigma_y)$.
As E, F and G are the function of both x and y, then on differentiate with regard to x and y we find that

$$E_x = (\sigma_x \cdot \sigma_x)_x = 2\sigma_{xx} \cdot \sigma_x \qquad \Rightarrow \qquad \sigma_{xx} \cdot \sigma_x = \frac{E_x}{2} \tag{3.2}$$

Similarly, we can find

$$\left.\begin{aligned} \sigma_{xx} \cdot \sigma_y = F_x - \frac{E_y}{2},\ \ \sigma_{xy} \cdot \sigma_x = \frac{E_y}{2},\ \ \sigma_{xy} \cdot \sigma_y = \frac{G_x}{2},\\ \sigma_{yy} \cdot \sigma_x = F_y - \frac{G_x}{2},\ \ \sigma_{yy} \cdot \sigma_y = \frac{G_y}{2}. \end{aligned}\right\} \tag{3.3}$$

Let Γ^n_{lm}, where$\{l, m, n = 1, 2\}$, be the Christoffel symbols of second kind. Then we have,

$$\left.\begin{aligned} \Gamma^1_{11} &= \frac{1}{2H^2}\{GE_x + F[E_y - 2F_x]\},\\ \Gamma^2_{11} &= \frac{1}{2H^2}\{E[2F_x - E_y] - FE_y\},\\ \Gamma^2_{12} &= \frac{1}{2H^2}\{EG_x - FE_y\} = \Gamma^2_{21},\\ \Gamma^2_{22} &= \frac{1}{2H^2}\{EG_y + F[G_y - 2F_y]\},\\ \Gamma^1_{22} &= \frac{1}{2H^2}\{G[2F_y - G_x] - FG_y\},\\ \Gamma^1_{21} &= \frac{1}{2H^2}\{GE_y - FG_x\} = \Gamma^1_{12}, \end{aligned}\right\} \tag{3.4}$$

where $H = \sqrt{EG - F^2}$.
Now, for the conformal transformation these Christoffel symbols taking the form

$$\left.\begin{aligned} \tilde{\Gamma}^1_{11} = \Gamma^1_{11} + \Theta^1_{11},\ \ \tilde{\Gamma}^2_{11} = \Gamma^2_{11} + \Theta^2_{11},\ \ \tilde{\Gamma}^1_{12} = \Gamma^1_{12} + \Theta^1_{12},\\ \tilde{\Gamma}^2_{12} = \Gamma^2_{12} + \Theta^2_{12},\ \ \tilde{\Gamma}^1_{22} = \Gamma^1_{22} + \Theta^1_{22},\ \ \tilde{\Gamma}^2_{22} = \Gamma^2_{22} + \Theta^2_{22}, \end{aligned}\right\} \tag{3.5}$$

where

$$\left.\begin{aligned} \Theta^1_{11} &= \frac{EG\zeta_x - 2F^2\zeta_x + FE\zeta_y}{\zeta H^2}\\ \Theta^2_{11} &= \frac{EF\zeta_x - E^2\zeta_y}{\zeta H^2},\\ \Theta^1_{12} &= \frac{EG\zeta_y - FG\zeta_x}{\zeta H^2},\\ \Theta^2_{12} &= \frac{EG\zeta_x - FE\zeta_y}{\zeta H^2},\\ \Theta^1_{22} &= \frac{GF\zeta_y - G^2\zeta_x}{\zeta H^2},\\ \Theta^2_{22} &= \frac{EG\zeta_y - 2F^2\zeta_v + FG\zeta_u}{\zeta H^2}. \end{aligned}\right\} \tag{3.6}$$

Now for the isometry between the surface we have $\zeta = 1$, then on putting $\zeta = 1$ in equation (3.6), we find that all $\Theta^n_{lm} = 0$, for $l, m, n = \{1, 2\}$. On putting $\Theta^n_{l,m} = 0$ for $l, m, n = \{1, 2\}$, in equation (3.5) we get

$$\tilde{\Gamma}^n_{lm} = \Gamma^n_{lm}.$$

This proves that for an isometry, the Christoffel symbols are invariant. □

Theorem 3.2. *Let $G : P \to \tilde{P}$ be a conformal map between two regular and smooth surfaces P and $\tilde{P}$ in the Euclidean space $\mathbb{R}^3$ and $\delta(r)$ be a rectifying curve on the surface P. Then the normal component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$, satisfying the following conditions:*

$$\tilde{\delta}(r)\cdot\tilde{\mathbb{N}} - \zeta^4\delta(r)\cdot\mathbb{N} = \frac{\mu_2(r)}{\kappa(r)}\zeta^4 H^2(x'^3\Theta^2_{11} - y'^3\Theta^1_{22} + 2x'^2y'\Theta^2_{12} + x'y'^2\Theta^2_{22} + x'^2y'\Theta^1_{12} + 2x'y'^2\Theta^1_{12}).$$

Proof. Given that $\tilde{P}$ is the conformal image of P under the map G, and $\delta(r)$ is a rectifying curve on P. Let $\sigma(x, y)$ and $\tilde{\sigma}(x, y)$ be the surface patches of P and $\tilde{P}$ respectively, and $\tilde{\sigma}(x, y) = G \circ \sigma(x, y)$. We know that $E = (\sigma_x \cdot \sigma_x)$, $F = (\sigma_x \cdot \sigma_y)$, $G = (\sigma_y \cdot \sigma_y)$. Now for dilation function ζ we have

$$\tilde{E} = \zeta^2 E, \quad \tilde{F} = \zeta^2 F, \quad \tilde{G} = \zeta^2 G. \tag{3.7}$$

On differentiating the above terms with respect to both x and y, we get

$$\left.\begin{aligned} \tilde{E}_x &= 2\zeta\zeta_x E + \zeta^2 E_x, & \tilde{E}_y &= 2\zeta\zeta_y E + \zeta^2 E_y,\\ \tilde{F}_x &= 2\zeta\zeta_x F + \zeta^2 F_x, & \tilde{F}_y &= 2\zeta\zeta_y F + \zeta^2 F_y,\\ \tilde{G}_x &= 2\zeta\zeta_x G + \zeta^2 G_x, & \tilde{G}_y &= 2\zeta\zeta_y G + \zeta^2 G_y. \end{aligned}\right\} \tag{3.8}$$

Now for finding the normal component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$, we have

$$\begin{aligned}
\delta(r)\cdot\mathbb{N} &= [\mu_1(r)(\sigma_x x' + \sigma_y y') + \mu_2(r)\frac{1}{\kappa(r)}\{(x'y'' - x''y')\mathbb{N} + x'^3\sigma_x\times\sigma_{xx} + 2x'^2y'\sigma_x\times\sigma_{xy}\\
&\quad + x'y'^2\sigma_x\times\sigma_{yy} + x'^2y'\sigma_y\times\sigma_{xx} + 2x'y'^2\sigma_y\times\sigma_{xy} + y'^3\sigma_y\times\sigma_{yy}\}]\cdot\mathbb{N},\\
&= \mu_1(r)(\sigma_x x' + \sigma_y y')\cdot\mathbb{N} + [\mu_2(r)\frac{1}{\kappa(r)}\{(x'y'' - x''y')\mathbb{N} + x'^3\sigma_x\times\sigma_{xx} + 2x'^2y'\sigma_x\times\sigma_{xy}\\
&\quad + x'y'^2\sigma_x\times\sigma_{yy} + x'^2y'\sigma_y\times\sigma_{xx} + 2x'y'^2\sigma_y\times\sigma_{xy} + y'^3\sigma_y\times\sigma_{yy}\}]\cdot\mathbb{N},\\
&= \mu_1(r)(\sigma_x x' + \sigma_y y')\cdot(\sigma_x\times\sigma_y) + \frac{\mu_2(r)}{\kappa(r)}\{(x'y'' - x''y')(EG - F^2) + x'^3(\sigma_x\times\sigma_{xx})\cdot(\sigma_x\times\sigma_y)\\
&\quad + 2x'^2y'(\sigma_x\times\sigma_{xy})(\sigma_x\times\sigma_y) + x'y'^2(\sigma_x\times\sigma_{yy})\cdot(\sigma_x\times\sigma_y) + x'^2y'(\sigma_y\times\sigma_{xx})\cdot(\sigma_x\times\sigma_y)\\
&\quad + y'^3(\sigma_y\times\sigma_{yy})\cdot(\sigma_x\times\sigma_y),\\
&= \frac{\mu_2(r)}{\kappa(r)}\{(x'y'' - x''y')(EG - F^2) + x'^3\{E(F_x - \frac{E_y}{2}) - \frac{FE_x}{2}\} + 2x'^2y'\{\frac{EG_x}{2} - \frac{FE_y}{2}\}\\
&\quad + x'y'^2\{\frac{EG_y}{2} - F(F_y - \frac{G_x}{2})\} + x'^2y'\{\frac{FG_x}{2} - \frac{GE_x}{2}\} + 2x'y'^2\{\frac{FG_x}{2} - \frac{GE_y}{2}\}\\
&\quad + y'^3\{\frac{FG_y}{2} - G(F_y - \frac{G_x}{2})\}\}.
\end{aligned} \tag{3.9}$$

Now on using equation (3.4) in equation (3.9), we obtained

$$\begin{aligned}\delta(r)\cdot\mathbb{N} &= \frac{\mu_2(r)}{\kappa(r)}(EG-F^2)\{(x'y''-x''y')+x'^3\Gamma^2_{11}+2x'^2y'\Gamma^2_{12}+x'y'^2\Gamma^2_{22}+x'^2y'\Gamma^1_{12}\\ &\quad +2x'y'^2\Gamma^1_{12}-y'^3\Gamma^1_{22}\}. \end{aligned} \tag{3.10}$$

Thus in view of (3.5), (3.7), and (3.10), the above equation can be written as

$$\tilde{\delta}(r)\cdot\tilde{\mathbb{N}}-\zeta^4\delta(r)\cdot\mathbb{N} = \frac{\mu_2(r)}{\kappa(r)}\zeta^4H^2(x'^3\Theta^2_{11}+2x'^2y'\Theta^2_{12}+x'y'^2\Theta^2_{22}+x'^2y'\Theta^1_{12}+2x'y'^2\Theta^1_{12}-y'^3\Theta^1_{22}).$$

□

Corollary 3.1. *Let $G : P \to \tilde{P}$ be an isometry between two regular and smooth surfaces P and $\tilde{P}$ in the Euclidean space $\mathbb{R}^3$ and $\delta(r)$ be a rectifying curve on the surface P. Then the normal component of the curve $\delta(r)$ along the surface normal $\mathbb{N}$ is invariant under the isometry G, i.e. $\tilde{\delta}(r)\cdot\tilde{\mathbb{N}} = \delta(r)\cdot\mathbb{N}$.*

Proof. Since for an isometry the dilation function $\zeta = 1$, if we put $\zeta = 1$ in the theorem (3.2), we obtained $\tilde{\delta}(r)\cdot\tilde{\mathbb{N}} = \delta(r)\cdot\mathbb{N}$. □

4. Conclusion

In this article, we investigate some geometric properties for the first fundamental form of the surfaces. We also introduce the invariant properties for a class of curves, namely rectifying curves, and their geometric invariance under isometric transformations. We came up with a derivation for the rectifying curves' normal components and discovered that both the Christoffel symbols and the normal components are isometrically invariant.

In future, one can study some other properties of the first and second fundamental forms of the surface. One can also make some new and intresting results about the conformal image of other classes of curves, namely osculating and normal curves under conformal transformation. Moreover, One can also check for these curves, the normal and tangential components, normal and geodesic curvature, and the Christoffel symbols of the first and second kinds are invariant under these transformations.

Ethical approval : There are no studies by any of the authors, either with human subjects or with animals, in the current article.

Data availability: Since no new data was created or examined in this study, so data sharing is not applicable to this work.

Author contribution: All authors have made equal contributions while preparing this article. The author read and approved the final manuscript.

Funding: The authors received no specific funding to support this study.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.A. Shaikh, M.S. Lone, P.R. Ghosh, Conformal image of an osculating curve on a smooth immersed surface, J. Geom. Phys. 151 (2020), 103625. https://doi.org/10.1016/j.geomphys.2020.103625.

[2] A.A. Shaikh, P.R. Ghosh, Rectifying curves on a smooth surface immersed in the Euclidean space, Indian J. Pure Appl. Math. 50 (2019), 883-890. https://doi.org/10.1007/s13226-019-0361-4.

[3] M.P. Do Carmo, Differential geometry of curves & surfaces: revised & updated, second ed., Dover Publications, Mineola, New York, (2016).

[4] B.Y. Chen, When does the position vector of a space curve always lie in its rectifying plane?, Amer. Math. Mon. 110 (2003), 147-152. https://doi.org/10.1080/00029890.2003.11919949.

[5] M.S. Lone, Geometric invariants of normal curves under conformal transformation in $\mathbb{E}^3$, Tamkang J. Math. 53 (2022), 75-87. https://doi.org/10.5556/j.tkjm.53.2022.3611.

[6] A.A. Shaikh, P.R. Ghosh, Rectifying and osculating curves on a smooth surface, Indian J. Pure Appl. Math. 51 (2020), 67-75. https://doi.org/10.1007/s13226-020-0385-9.

[7] A.I. Bobenko, C. Gunn, DVD-Video PAL, 15 minutes, in: Springer VideoMATH, Springer, 2018. https://www.springer.com/us/book/9783319734736.

[8] S. DESHMUKH, B.Y. Chen, S.H. Alshammari, On rectifying curves in Euclidean 3-space, Turk. J. Math. 42 (2018), 609-620. https://doi.org/10.3906/mat-1701-52.

[9] M. He, D.B. Goldgof, C. Kambhamettu, Variation of gaussian curvature under conformal mapping and its application, Computers Math. Appl. 26 (1993), 63-74. https://doi.org/10.1016/0898-1221(93)90086-b.

[10] B.Y. Chen, F. Dillen, Rectfying curve as centrode and extremal curve, Bull. Inst. Math. Acad. Sinica, 33 (2005), 77-90.

[11] K. Ilarslan, E. Nešović, Timelike and null normal curves in Minkowski space $\mathbb{E}^3_1$, Indian J. Pure Appl. Math. 35 (2004), 881-888.

[12] A.A. Shaikh, M.S. Lone, P.R. Ghosh, Rectifying curves under conformal transformation, J. Geom. Phys. 163 (2021), 104117. https://doi.org/10.1016/j.geomphys.2021.104117.

[13] A.A. Shaikh, M.S. Lone, P.R. Ghosh, Normal curves on a smooth immersed surface, Indian J. Pure Appl. Math. 51 (2020), 1343-1355. https://doi.org/10.1007/s13226-020-0469-6.

[14] F. Schwarz, Transformation to canonical form, in: Algorithmic Lie Theory for Solving Ordinary Differential Equations, 257-320, (2007).

[15] A. Yadav, B. Pal, Some characterizations of rectifying curves on a smooth surface in Euclidean 3-space, arXiv:2104.02907 [math.DG], (2021). https://doi.org/10.48550/arxiv.2104.02907.

[16] C. Camci, L. Kula, K. Ilarslan, Characterizations of the position vector of a surface curve in Euclidean 3-space, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. 19 (2011), 59-70.

[17] A.A. Shaikh, P.R. Ghosh, Curves on a smooth surface with position vectors lie in the tangent plane, Indian J. Pure Appl. Math. 51 (2020), 1097-1104. https://doi.org/10.1007/s13226-020-0452-2.