

The Security Assignment Problem and Its Solution

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Abstract. In this paper we derived a new method for finding the optimal solution to security assignment problems using projection onto a convex set. This study will help communities find the optimal number of assigned and reserved personnels in designating security officers to an area. This study is applicable as well to CCTV assignment problems. The main goal of this study is to give a new method that can be applied in solving security assignment problems. We used some of the known properties of the convex optimization in proving the properties of the optimal solution, such as the concept of proximity operator, projection onto the convex set, and primal and dual problem. In addition to that, we used some basic knowledge in graph theory to answer our real-life application of this study. The main results of this paper showed that we found instances when the optimal solution to a security problem exists and when the solutions exist, we can determine the answer to the problem explicitly. Also, we proved that there always exists a Pseudo-solution in security assignment problems, and if the solution exists, then the Pseudo-solution will coincide with it. The most important aspect of this paper is the introduction of the application of convex optimization in the security problem.

1. Introduction

There are many optimizations problem in optimization theory such as assignment problems, linear optimization problems, and convex optimization problems that are applicable in a real-life situation. For instance, linear programming is applicable in assignment problems [1–3] diet problems [4–6], and business problems in minimization of cost [7].

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The assignment problem is a problem so that there are n workers and m jobs where $m \leq n$. Any worker can be assigned to perform any job, with some costs that may vary depending on the workers-job assignment. It is required to perform a job by assigning at most one worker to each job and at most one job to each worker, in such a way that the total cost of the assignment is minimized. The assignment problem is a special type of $\{0, 1\}$ -linear programming [8, 9].

In this paper, we will propose a model and solution to security assignment problems, CCTV assignment problems, and any related problem.

In the security assignment problem, if we let G be the map of the city, E be the major road, V be the intersection of the major road, $w(e)$ be the criminal rate of the certain major road $e \in E$ and $\pi(e)$ be the time needed to travel the road e , then the goal is to assign a percentage of security personnel $\tau(e) \in [a, b] \subseteq [0, 1]$ in any major road $e \in E$ such that the crime will minimize. Furthermore, we want to maximize the reserved personnel that we will assigned to the strategic location of the city. The crime will minimize in the road $e \in E$ if the crime rate $w(e)$ and the percentage of assigned personnel $\tau(e)$ to that road has small distance. Thus, the goal is to find $\tau : E \rightarrow [a, b]$ such that the crime in G will minimize, that is, τ satisfy the following equation

$$\min_{\gamma: E \rightarrow [a, b]} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \tau(e))^2. \quad (1.1)$$

In addition, we want to have a reserved personnel that we will assign to the center of $G_\pi = (V, E, \pi)$ so that the total percentage of reserved personnel is maximized, and it is at least $1 - s$ but not more than $1 - r$ where $r, s \in [0, 1]$ and $r < s$. The quantity r and s are the lower bound and upper bound for the percentage of assigned personnel, respectively. For instance, if we want to avoid the case that the total percentage of reserved is at least 20% and it is at most the total percentage of assigned, then we set $r = 0.5$ and $s = 0.8$. This assumption can be formulated as

$$0 \leq r \leq \sum_{e \in E} \tau(e) \leq s \leq 1 \quad (1.2)$$

and

$$\max_{\gamma: E \rightarrow [a, b]} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \tau(e) \quad (1.3)$$

where $\sum_{e \in E} \tau(e)$ is the percentage of assigned personnel and $1 - \sum_{e \in E} \tau(e)$ is the percentage of reserved personnel. We will assigned the total reserved to the central point $u \in V$ of the graph G_π , that is, the vertex u where the greatest distance $d_{G_\pi}(u, v)$ to other vertices v is minimal. If there are more than one central points in the graph G_π , then we will divide the number of reserved personnel to the number of central points. Thus, we will have

$$\tau^*(v) = \begin{cases} \frac{1 - \sum_{e \in E} \tau(e)}{|C(G_\pi)|} & \text{if } v \in C(G_\pi) \\ 0 & \text{if } v \notin C(G_\pi). \end{cases} \quad (1.4)$$

Here, the set $C(G_\pi)$ is the set of all central points of the graph G_π , that is, the set of all vertices u where the greatest distance $d_{G_\pi}(u, v)$ to other vertices v is minimal.

Let $G_w = (V, E, w)$ and $G_\pi = (V, E, \pi)$ be a weighted graphs with weight function $w : E \rightarrow [0, 1]$ and $\pi : E \rightarrow \mathbb{R}$, respectively. Set

$$Y(E, [a, b], [r, s]) = \left\{ w : E \rightarrow [a, b] \mid r \leq \sum_{e \in E} w(e) \leq s \right\}.$$

Summarizing the equation in (1.1) to (1.4), then we will obtain the problem:

Find $\tau : E \rightarrow [a, b]$ satisfying the four condition: (1.5)

1. $0 \leq r \leq \sum_{e \in E} \tau(e) \leq s \leq 1$
2. $\max_{\gamma \in Y(E, [a, b], [r, s])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \tau(e)$
3. $\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \tau(e))^2$
4. $\tau^*(v) = \begin{cases} \frac{1 - \sum_{e \in E} \tau(e)}{|C(G_\pi)|} & \text{if } v \in C(G_\pi) \\ 0 & \text{if } v \notin C(G_\pi) \end{cases}$

where $r, s, a, b \in [0, 1]$ such that $a < b$ and $r < s$.

The following definition is taken from [14].

Definition 1.1. Let $G = (V, E, w)$ be a weighted graph where V is the vertex set and E is the edge set of G . Then, the center of the graph G , denoted by $C(G)$, is the set of all vertices $u \in V$ where the greatest distance $d(u, v)$ to other vertices v is minimal.

The main objective of this study is to address the problem in definition 1.2.

Definition 1.2. Let $G_w = (V, E, w)$ and $G_\pi = (V, E, \pi)$ be a weighted graph with weight function $w : E \rightarrow [0, 1]$ and $\pi : E \rightarrow \mathbb{R}$, respectively. Let $r, s, a, b \in [0, 1]$ such that $r < s$ and $a < b$. Set

$$Y(E, [a, b], [r, s]) = \left\{ w : E \rightarrow [a, b] \mid r \leq \sum_{e \in E} w(e) \leq s \right\}.$$

The (r, s, a, b) -security assignment problem of G is a problem of finding (τ, τ^*) where $\tau : E \rightarrow [a, b]$ and $\tau^* : V \rightarrow [0, 1]$ such that

1. $r \leq \sum_{e \in E} \tau(e) \leq s \leq 1$
2. $\max_{\gamma \in Y(E, [a, b], [r, s])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \tau(e)$
3. $\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \tau(e))^2$
4. $\tau^*(v) = \begin{cases} \frac{1 - \sum_{e \in E} \tau(e)}{|C(G_\pi)|} & \text{if } v \in C(G_\pi) \\ 0 & \text{if } v \notin C(G_\pi) \end{cases}$

where $C(G_\pi)$ is the center of the graph G_π . The solution to the security assignment problem is the ordered pair (τ, τ^*) .

The paper is organized as follows. In section 2, we proved a necessary and sufficient condition when the security assignment problem has a solution. Furthermore, we give an explicit formula for the solution to the security assignment problem in terms of projection onto the convex set if it has a solution. In section 3, we record some important properties of the pseudo-solution or solution to the security assignment problem. Finally, in section 4 we discussed the real-life application of security assignment problems such as security, CCTV assignment problems, and military assignment problems.

In this study, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are the standard norm and standard inner product in \mathbb{R}^n , respectively.

2. Existence of Solution to the Security Assignment Problem

First, we recall some terminology in convex analysis. The function $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is convex if and only if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in \mathbb{R}^n$ and for all $\alpha \in [0, 1]$. We say that f is proper if and only if $f \not\equiv +\infty$. In this paper, the function $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ is always convex, continuous and proper function.

The following definition taken from [10, 11] is vital in this paper.

Definition 2.1. Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a convex, continuous and proper function and $u \in \mathbb{R}^n$. The proximity of u in f is the unique point $\text{Prox}_f(u) \in \mathbb{R}^n$ such that

$$\text{Prox}_f(u) = \arg \min_{x \in \mathbb{R}^n} \left(\frac{1}{2} \|x - u\|^2 + f(x) \right). \quad (2.1)$$

In particular, if C is closed and convex set and $f = \iota_C$ where $\iota_C(x) = 0$ if $x \in C$ and $+\infty$, otherwise, then $\text{Prox}_f(u) = P_C(u)$ where $x_0 = P_C(u)$ is a unique point in C such that $\inf_{x \in C} \|x - u\| = \|x_0 - u\|$.

It is well-known that if $g = f + \frac{\alpha}{2} \|\cdot - u\|^2 + \langle \cdot, v \rangle + \beta$ where $\alpha \geq 0$ and $\beta \in \mathbb{R}$, then $\text{Prox}_g(x) = \text{Prox}_{(\alpha+1)^{-1}f}((\alpha+1)^{-1}(x + (\alpha u - v)))$ (see Proposition 24.8 of [10]). Consequently, if $\alpha = 0$, $f = \iota_C$, then $\text{Prox}_g(x) = P_C(x - v)$ where C is closed and convex set.

The following lemma solves a minimization problem that can be apply in the security assignment problem.

Lemma 2.1. Let $u_1, u_2, \dots, u_n \in \mathbb{R}$ and let $r, s, a, b \in [0, 1]$ such that $r < s$ and $a < b$. Consider the problem:

$$\min_{x_1, x_2, \dots, x_n \in \mathbb{R}} \left(\sum_{i=1}^n (u_i - x_i)^2 + \sum_{i=1}^n x_i \right) \quad (2.2)$$

subject to

1. $a \leq x_i \leq b$
2. $r \leq \sum_{i=1}^n x_i \leq s$

If $C = \{(x_1, x_2, \dots, x_n) : a \leq x_i \leq b \text{ and } 0 \leq r \leq \sum_{i=1}^n x_i \leq s \leq 1\}$ is nonempty, then the problem above has exactly one solution. Furthermore, $P_C(u - \frac{1}{2}v) = (x_1^*, x_2^*, \dots, x_n^*)$ is the solution to the problem in (2.2).

Proof. Let $x = (x_1, x_2, \dots, x_n)$, $u \in (u_1, u_2, \dots, u_n)$ and $v = (1, 1, 1, \dots, 1)$. Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ be the solution to the problem in (2.2). Set

$$C = \left\{ (x_1, x_2, \dots, x_n) : a \leq x_i \leq b \text{ and } 0 \leq r \leq \sum_{i=1}^n x_i \leq s \leq 1 \right\}. \tag{2.3}$$

Then, the problem in (2.2) can be written as

$$\min_{x \in \mathbb{R}^n} \left(\frac{1}{2} \|x - u\|^2 + \frac{1}{2} \langle x, v \rangle + \iota_C(x) \right). \tag{2.4}$$

Since $C = [a, b]^n \cap \{(x_1, x_2, \dots, x_n) : 0 \leq r \leq \sum_{i=1}^n x_i \leq s \leq 1\}$ is the intersection of two closed and convex sets, then C is closed and convex. Therefore, the definition 2.1 guaranty that the problem in (2.2) has a unique solution

$$\text{Prox}_g(u) = \arg \min_{x \in \mathbb{R}^n} \left(\frac{1}{2} \|x - u\|^2 + \langle x, \frac{1}{2}v \rangle + \iota_C(x) \right) \tag{2.5}$$

where $g(x) = \frac{1}{2} \langle x, v \rangle + \iota_C(x)$. Furthermore, applying proposition 24.8 of [10] by setting $\alpha = 0$ and $f = \iota_C$, then we have

$$\text{Prox}_g(u) = \text{Prox}_{\iota_C} \left(u - \frac{1}{2}v \right) = P_C \left(u - \frac{1}{2}v \right)$$

as desired. □

The following theorem gives an explicit form of the solution to the security assignment problem in terms of projection onto a convex set if it is existing.

Theorem 2.1. *The (r, s, a, b) -security assignment problem of G has at most one solution (τ, τ^*) . Furthermore, if it has a solution (τ, τ^*) , then $\tau(e_i) = \langle P_D(u - \frac{1}{2}v), u_i \rangle$ where $u_i = (0, 0, 0, \dots, 0, 1, 0, \dots, 0)$ is the vector where the i th coordinate is 1, and 0 elsewhere where $i \in \{1, 2, \dots, |E|\}$ and*

$$D = \left\{ (x_1, x_2, \dots, x_{|E|}) \in \mathbb{R}^{|E|} : 0 \leq x_i \leq 1 \text{ and } r \leq \sum_{i=1}^{|E|} x_i \leq s \right\}. \tag{2.6}$$

Proof. If the (r, s, a, b) -security assignment problem of G has a solution (τ, τ^*) and (γ, γ^*) , then $(\tau(e_1), \tau(e_2), \dots, \tau(e_{|E|}))$ and $(\gamma(e_1), \gamma(e_2), \dots, \gamma(e_{|E|}))$ are the solution to the problem

$$\min_{x_1, x_2, \dots, x_{|E|} \in \mathbb{R}} \left(\sum_{i=1}^{|E|} (w(e_i) - x_i)^2 + \sum_{i=1}^{|E|} x_i \right) \quad (2.7)$$

subject to

1. $a \leq x_i \leq b$
2. $r \leq \sum_{i=1}^{|E|} x_i \leq s$

Applying lemma 2.1, then $(\tau(e_1), \tau(e_2), \dots, \tau(e_{|E|})) = (\gamma(e_1), \gamma(e_2), \dots, \gamma(e_{|E|}))$, and therefore, $\tau = \gamma$ and $\tau^* = \gamma^*$. Furthermore, if the solution (τ, τ^*) to the security assignment problem exists, then lemma 2.1 implies that $(\tau(e_1), \tau(e_2), \dots, \tau(e_{|E|})) = P_D(u - \frac{1}{2}v)$ where $u = (w(e_1), w(e_2), \dots, w(e_{|E|}))$ and $v = (1, 1, 1, \dots, 1)$. Thus, for all $i = 1, 2, \dots, |E|$, we obtain $\tau(e_i) = \langle P_D(u - \frac{1}{2}v), u_i \rangle$, as desired. \square

Lemma 2.1 gives the following definition.

Definition 2.2. Suppose $D = \{(x_1, x_2, \dots, x_{|V|}) \in \mathbb{R}^{|E|} : 0 \leq x_i \leq 1 \text{ and } 0 \leq r \leq \sum_{i=1}^{|E|} x_i \leq s \leq 1\} \neq \emptyset$. The pseudo-solution of the (r, s, a, b) -security assignment problem of G is the unique ordered pair (θ, θ^*) where $\theta : E \rightarrow [a, b]$ and $\theta^* : V \rightarrow [0, 1]$ such that

1. $0 \leq r \leq \sum_{e \in E} \theta(e) \leq s \leq 1$.
2. $\min_{\gamma \in \mathcal{Y}(E, [a, b], [r, s])} \left(\sum_{e \in E} (w(e) - \gamma(e))^2 + \sum_{e \in E} \gamma(e) \right) = \sum_{e \in E} (w(e) - \theta(e))^2 + \sum_{e \in E} \theta(e)$.
3. $\theta^*(v) = \begin{cases} \frac{1 - \sum_{e \in E} \theta(e)}{|C(G_\pi)|} & \text{if } v \in C(G_\pi) \\ 0 & \text{if } v \notin C(G_\pi) \end{cases}$

The vector solution to the (r, s, a, b) -security assignment problem is the vector $(\theta(e_1), \theta(e_2), \dots, \theta(e_{|E|}))$.

Remark 2.1. If the (r, s, a, b) -security assignment problem of G has a solution (τ, τ^*) , then it is unique, and its solution is its pseudo-solution (θ, θ^*) .

Corollary 2.1. Let (θ, θ^*) be the pseudo-solution of (r, s, a, b) -security assignment problem of G . Then, the (r, s, a, b) -security assignment problem of G has a solution if and only if

1. $\max_{\gamma \in \mathcal{Y}(E, [a, b], [r, s])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \theta(e)$.
2. $\min_{\gamma \in \mathcal{Y}(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \theta(e))^2$.

Example 2.1: Given the crime rate and distance of the road e_i where $i = 1, 2, 3, \dots, 8$ in the graph G below (see figure 2.1), determine the percentage of the number of assigned personnel for each road so that the crime will minimize. Furthermore, we want that the total percentage of assigned personnel is

at most 90% and at least 85%, while the percentage of assigned personnel for each road is at most 25% and at least 5%.

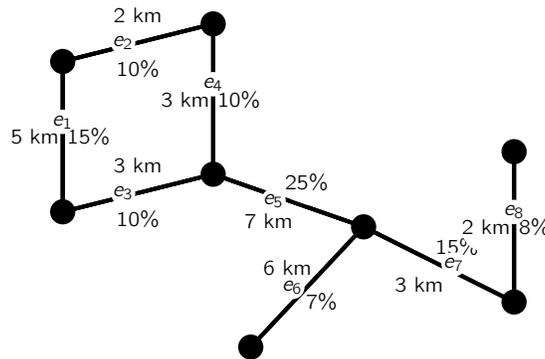


Figure 2.1: The Graph G

Solution: We want to solve the $(0.85, 0.9, 0.05, 0.25)$ – security assignment problem, that is, we want to find $\tau : E \rightarrow [0.05, 0.25]$ satisfying the four condition:

1. $0.85 \leq \sum_{e \in E} \tau(e) \leq 0.90 \leq 1$
2. $\max_{\gamma \in Y(E, [0.05, 0.25], [0.85, 0.9])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \tau(e)$
3. $\min_{\gamma \in Y(E, [0.05, 0.25], [0.85, 0.9])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \tau(e))^2$
4. $\tau^*(v) = \begin{cases} \frac{1 - \sum_{e \in E} \tau(e)}{|C(G_\pi)|} & \text{if } v \in C(G_\pi) \\ 0 & \text{if } v \notin C(G_\pi) \end{cases}$

where $\pi(e_i)$ is the length of the road e_i (in km). Let $x_i = \tau(e_i)$. Then, from lemma 2.1 and theorem 2.1, $(x_1, x_2, x_3, \dots, x_8)$ is the solution to the minimization problem:

$$\min_{x_1, x_2, \dots, x_8 \in \mathbb{R}} \left(\sum_{i=1}^8 (w(e_i) - x_i)^2 + \sum_{i=1}^8 x_i \right)$$

subject to

1. $0.05 \leq x_i \leq 0.25$
2. $0.85 \leq \sum_{i=1}^8 x_i \leq 0.9$.

Applying Lagrange Multiplier using MATLAB, then we obtain the solution $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \approx (0.135, 0.08, 0.08, 0.08, 0.23, 0.05, 0.135, 0.06)$. Thus, the total percentage of assigned security personnel is $x_1 + x_2 + \dots + x_8 = 85\%$ and the total percentage of reserved personnel is 15%. The following table discussed the optimal percentage of assigned security personnel for each road.

Table 2.1: Percentage of Assigned Number of Personnel per Road

Road	Crime Rate	Percentage of assigned number of Personnel
e_1	15%	13.5%
e_2	10%	8%
e_3	10%	8%
e_4	10%	8%
e_5	25%	23%
e_6	7%	5%
e_7	15%	13.5%
e_8	8%	6%

We assigned 15% of the total security personnel to the center of the graph G_π where the center of G_π is the intersection of the road e_5 , e_6 and e_7 .

3. Properties of the Pseudo-Solution

In this section, we record some important properties of the pseudo-solution that can be help in solving the security assignment problem. In this section, $u = (w(e_1), w(e_2), \dots, w(e_{|E|}))$, $v = (1, 1, 1, \dots, 1)$ and

$$D = \left\{ (x_1, x_2, \dots, x_{|E|}) \in \mathbb{R}^{|E|} : 0 \leq x_i \leq 1 \text{ and } 0 \leq r \leq \sum_{i=1}^{|E|} x_i \leq s \leq 1 \right\}. \quad (3.1)$$

Theorem 3.1. Suppose $D \neq \emptyset$. The pseudo-solution (θ, θ^*) to the (r, s, a, b) -security assignment problem is a solution to the minimization problem: Find $\theta : E \rightarrow [a, b]$ such that

$$\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e) + 0.5)^2 = \sum_{e \in E} (w(e) - \theta(e) + 0.5)^2 \quad (3.2)$$

subject to

$$0 \leq r \leq \sum_{e \in E} \theta(e) \leq s \leq 1. \quad (3.3)$$

Thus, if $(x_1, x_2, \dots, x_{|E|})$ is the solution to the minimization problem, then we have

$$\min_{y_1, y_2, \dots, y_{|E|} \in \mathbb{R}} \sum_{i=1}^{|E|} (w(e_i) - y_i + 0.5)^2 = \sum_{i=1}^{|E|} (w(e_i) - x_i + 0.5)^2 \quad (3.4)$$

and

1. $a \leq x_i \leq b$
2. $0 \leq r \leq \sum_{i=1}^{|E|} x_i \leq s \leq 1$.

Proof. By lemma 2.1, x_0 is a solution to the problem

$$\min_{x \in \mathbb{R}^{|E|}} \left(\frac{1}{2} \|x - u\|^2 + \langle x, \frac{1}{2}(1, 1, 1, \dots, 1) + \nu_D(x) \rangle \right) \quad (3.5)$$

if and only if $P_D(u - \frac{1}{2}v) = x_0$ if and only if

$$\inf_{x \in D} \frac{1}{2} \left\| x - \left(u - \frac{1}{2}v \right) \right\|^2 = \frac{1}{2} \left\| x_0 - \left(u - \frac{1}{2}v \right) \right\|^2. \tag{3.6}$$

Thus, $x_0 = (\theta(e_1), \theta(e_2), \dots, \theta(e_{|E|}))$ if and only if

$$\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e) + 0.5)^2 = \sum_{e \in E} (w(e) - \theta(e) + 0.5)^2 \tag{3.7}$$

subject to

$$0 \leq r \leq \sum_{e \in E} \theta(e) \leq s \leq 1. \tag{3.8}$$

as desired. □

The projection theorem state that $x_0 = P_C(x)$ if and only if $x_0 \in C$ and $\langle p - y, p - x \rangle \leq 0$ for all $y \in C$ (See theorem 3.16 in [10]). Thus, the following corollary directly holds.

Corollary 3.1. *Suppose $D \neq \emptyset$. The point $x_0 \in \mathbb{R}^{|E|}$ is the solution to*

$$\min_{x \in \mathbb{R}^{|E|}} \left\{ \frac{1}{2} \|x - u\|^2 + \langle x, \frac{1}{2}v \rangle + \iota_D(x) \right\}. \tag{3.9}$$

if and only if $x_0 \in D$ and for all $y \in D$, we have

$$\langle x_0 - y, x_0 - u + \frac{1}{2}v \rangle \leq 0. \tag{3.10}$$

Proof. The point x_0 is a solution to

$$\min_{x \in \mathbb{R}^{|E|}} \left\{ \frac{1}{2} \|x - u\|^2 + \langle x, \frac{1}{2}v \rangle + \iota_D(x) \right\} \tag{3.11}$$

if and only if $P_D(u - \frac{1}{2}v) = x_0$. Thus, the projection theorem implies that $P_D(u - \frac{1}{2}v) = x_0$ if and only if $x_0 \in D$ and for all $y \in D$, we have

$$\langle x_0 - u + \frac{1}{2}v, x_0 - y \rangle \leq 0 \tag{3.12}$$

as desired. □

The following definition taken from [10] state the notion of Fenchel conjugate and Gateaux-differentiable of a function which is important in the next result.

Definition 3.1. *Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a convex, continuous and proper function. The Fenchel conjugate f^* of f is a function defined by*

$$f^* = \sup_{x \in \mathbb{R}^n} (\langle x, \cdot \rangle - f(x)).$$

Definition 3.2. *Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a convex, continuous and proper function. The function f is said to be Gateaux differentiable at x_0 if and only if the limit below exists for all $u \in \mathbb{R}^n$.*

$$D_u f(x_0) = \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} (f(x_0 + \alpha u) - f(x_0)).$$

The function $D : \mathbb{R}^n \rightarrow \mathbb{R} : u \mapsto D_u f(x_0)$ is called the Gateaux derivative of f . The Gateaux gradient of f is the unique vector $\nabla f(x)$ satisfying the equation $D_u f(x) = \langle u, \nabla f(x) \rangle$.

The next results give a characterization of the pseudo-solution in terms of the solution to the dual problem of the primal problem

$$\min_{x \in \mathbb{R}^{|E|}} \left\{ \frac{1}{2} \|x - u\|^2 + \langle x, \frac{1}{2}v \rangle + \iota_D(x) \right\}. \quad (3.13)$$

Theorem 3.2. *Suppose $D \neq \emptyset$. Let $x_0 = (\tau(e_1), \tau(e_2), \dots, \tau(e_{|E|}))$ be the vector solution to the (r, s, a, b) -security assignment problem, let $u = (w(e_1), w(e_2), \dots, w(e_{|E|}))$ and $v = (1, 1, 1, \dots, 1)$. Then, the problem*

$$\min_{x \in \mathbb{R}^{|E|}} \left\{ \frac{1}{2} \|y\|^2 - \langle y, u \rangle + \sup_{x \in D} \langle x, y - \frac{1}{2}v \rangle \right\} \quad (3.14)$$

has a unique solution $u - x_0$. In addition, $\text{Prox}_h(u) = x_0$ where $h(y) = \sup_{x \in D} \langle x, y - \frac{1}{2}v \rangle$.

Proof. Consider the primal problem

$$\min_{x \in \mathbb{R}^{|E|}} \left\{ \frac{1}{2} \|x - u\|^2 + \langle x, \frac{1}{2}v \rangle + \iota_D(x) \right\}. \quad (3.15)$$

Then, x_0 is the solution to the primal problem above. Let $f(x) = \frac{1}{2} \|x - u\|^2$ and $g(x) = \langle x, \frac{1}{2}v \rangle + \iota_D(x)$.

Then, $f^*(a) = \frac{1}{2} \|a\|^2 + \langle a, u \rangle$ and

$$g^*(a) = \sup_{x \in \mathbb{R}^{|E|}} \{ \langle x, a - \frac{1}{2}v \rangle - \iota_D(x) \} = \sup_{x \in D} \langle x, a - \frac{1}{2}v \rangle. \quad (3.16)$$

Thus, the dual problem of the primal problem in (3.15) is

$$\min_{x \in \mathbb{R}^{|E|}} \left\{ \frac{1}{2} \|y\|^2 + \langle y, u \rangle + \sup_{x \in D} \langle x, y - \frac{1}{2}v \rangle \right\}. \quad (3.17)$$

Since y is the solution to the dual problem, x_0 is the solution to the primal problem and f^* is Gateaux differentiable everywhere, then by proposition 19.4 of [10], $x_0 = \nabla f^*(-y)$, and thus, $x_0 = \nabla f^*(-y) = -y + u$, as desired. \square

4. Some Real-Life Application

In this section, we present some real-life application of security assignment problem.

a. Security Assignment Problem for a Road

Let G represent the whole map of a city, E be the set of all roads (or major roads) of the city, V be the set of the intersection of roads. Let $w(e)$ be the criminal rate (or population rate, number of houses, etc.) of the road $e \in E$ and $\pi(e)$ be the time needed to travel the road e (or length of the road, the volume of a vehicle in the road, etc.). Then, the (r, s, a, b) -security assignment problem is a problem of finding the percentage of security personnel $\tau(e)$ that will assign in road e and finding the percentage of number of reserved security personnel $\tau^*(e)$ that will assign to the center of the city $C(G_\pi)$ so that the total percentage of reserved security personnel is at most $1 - r$ and at least $1 - s$ where $r, s \in [0, 1]$ and $r < s$. We want to find $\tau : E \rightarrow [a, b]$ where $a, b \in [0, 1]$ and $a < b$ such

that if we assign $100\tau(e)$ percent of security personnel to road e , for all $e \in E$, then the crime will minimize, that is, we want to solve the problem of finding $\tau : E \rightarrow [a, b]$ such that

$$\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \tau(e))^2. \tag{4.1}$$

In addition to that, we want to maximize the total percentage of reserved security personnel, that is, we have

$$\max_{\gamma \in Y(E, [a, b], [r, s])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \tau(e). \tag{4.2}$$

Since the total percentage of reserved security personnel is at-most $1 - r$ and at least $1 - s$ where $r, s \in [0, 1]$ and $r < s$, then we have

$$0 \leq r \leq \sum_{e \in E} \tau(e) \leq s \leq 1. \tag{4.3}$$

Summarizing equation (4.1) to (4.3), then we recover the problem in definition 1.2.

Note that the security assignment problem can be applied in assigning security personnel for the checkpoint or any related problem. If the (r, s, a, b) -security assignment problem has no solution, then the strategic solution is the pseudo-solution of the (r, s, a, b) -security assignment problem.

b. CCTV Assignment Problem for a Road

Let G represent the whole map of a certain town, E be the set of all roads (or major roads) of the town, V be the set of the intersection of roads. Let $w(e)$ be the criminal rate (or population rate, number of houses, etc.) of the road $e \in E$ and $\pi(e)$ be the length the road e (or population, number of establishments, the volume of vehicle in the road, etc.). Then, the (r, s, a, b) -security assignment problem for CCTV installation is a problem of finding the percentage of number of CCTV $\tau(e) \in [a, b]$ that will install in road e and finding the percentage of number of CCTV $\tau^*(e)$ that will install to the center of the city $C(G_\pi)$ so that the total percentage of CCTV that will assign to the center of the city $C(G)$ is at-most $1 - r$ and at least $1 - s$ where $r, s \in [0, 1]$ and $r < s$. We want to find $\tau : E \rightarrow [a, b]$ where $a, b \in [0, 1]$ and $a < b$ such that if we install $100\tau(e)$ percent of CCTV to road e , for all $e \in E$, then this is the best strategic installation, that is, we want to solve the problem of finding $\tau : E \rightarrow [a, b]$ such that

$$\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \tau(e))^2. \tag{4.4}$$

In addition to that, we want to maximize the total percentage of CCTV that will assign to the center of the city $C(G_\pi)$, that is, we have

$$\max_{\gamma \in Y(E, [a, b], [r, s])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \tau(e). \tag{4.5}$$

Since the total percentage of CCTV that will assign to the center of the city $C(G)$ is at most $1 - r$ and at least $1 - s$ where $r, s \in [0, 1]$ and $r < s$, then we have

$$0 \leq r \leq \sum_{e \in E} \tau(e) \leq s \leq 1. \quad (4.6)$$

Summarizing the equation (4.4) to (4.6), then we obtain the problem in definition 1.2. If the (r, s, a, b) -security assignment problem has no solution, then the strategic solution is the pseudo-solution of (r, s, a, b) -security assignment problem.

c. Security Assignment Problem for a Planar Map

Before we discuss the application of this study in planar graph, we recall first the definition of line graph $L(G)$ of a graph G and planar graph.

Definition 4.1. Let G be a graph. The line graph $L(G)$ of G is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G .

Definition 4.2. Let H be a map with region U_1, U_2, \dots, U_n . The planar graph of H is the graph G where the vertex of G represent the region of H and two vertices in G are adjacent if and only if their corresponding region are adjacent.

Consider the map with regions U_1, U_2, \dots, U_n and let $G = (V, E)$ be a planar graph of the map where $V = \{u_1, u_2, \dots, u_n\}$. Let $\mu(U_i)$ be the criminal rate (or population rate, number of houses, etc.) of the region U_i . Let $L(G) = (L(V), L(E))$ be the line graph of G and set $w : L(E) \rightarrow [0, 1]$ such that $w(u_{i,j}, u_{i,k}) = \frac{\mu(U_i)}{\binom{\deg u_i}{2}}$ where

$$\binom{\deg u_i}{2} = \frac{\deg u_i!}{(\deg u_i! - 2)2!} = \frac{\deg u_i(\deg u_i - 1)}{2}. \quad (4.7)$$

Set $\pi = \tau$. The (r, s, a, b) -security assignment problem is a problem of finding the percentage of number of security personnel $\tau(e)$ that will assign in the edge e and finding the percentage of number of reserved security personnel $\tau^*(e)$ that will assign to the center of the graph $L(G)$ so that the total percentage of reserved security personnel is at most $1 - r$ and at least $1 - s$ where $r, s \in [0, 1]$ and $r < s$. We want to find $\tau : E \rightarrow [a, b]$ such that if we assign $100\tau(e)$ percent of security personnel to edge e , for all $e \in L(E)$, then the crime will minimize, that is, we want to solve the problem of finding $\tau : E \rightarrow [0, 1]$ such that

$$\min_{\gamma \in Y(L(E), [a, b], [r, s])} \sum_{e \in L(E)} (w(e) - \gamma(e))^2 = \sum_{e \in L(E)} (w(e) - \tau(e))^2. \quad (4.8)$$

In addition to that, we want to maximize the total percentage of reserved security personnel, that is, we have

$$\max_{\gamma \in Y(L(E), [a, b], [r, s])} \left(1 - \sum_{e \in L(E)} \gamma(e) \right) = 1 - \sum_{e \in L(E)} \tau(e). \tag{4.9}$$

Since the total percentage of reserved security personnel is at most $1 - r$ and at least $1 - s$ where $r, s \in [0, 1]$ and $r < s$, then we have

$$0 \leq r \leq \sum_{e \in L(E)} \tau(e) \leq s \leq 1. \tag{4.10}$$

The percentage of assigned security personnel in region U_i is

$$\sum_{\substack{j \in 1, 2, \dots, |V| \\ u_{i,j} u_{i,k} \in L(E)}} \tau(u_{i,j} u_{i,k}). \tag{4.11}$$

If the (r, s, a, b) -security assignment problem has no solution, then the strategic solution is the pseudo-solution of (r, s, a, b) -security assignment problem.

Example 4.1: Consider the map below, which shows the crime rate of each region. We need to determine the best strategic assignment of security personnel from region 1 to region 5 in order to minimize crime. However, we must ensure that we assign a maximum of 30% of security personnel for each $e \in L(G)$. Additionally, we want to ensure that the reserve security personnel make up at least 5%.

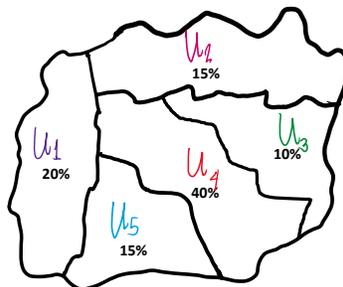


Figure 4.1 The Planar Region

Constructing the planar graph G of the map, then we obtain

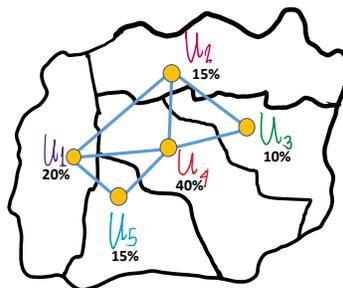


Figure 4.2: The Planar Graph and Planar Region

By constructing the line graph of the planar graph G , then we obtain $L(G)$

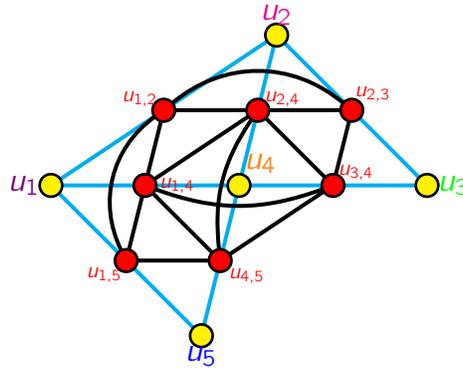


Figure 4.3: The Line Graph and Planar Graph

Set $w : L(E) \rightarrow [0, 1]$ such that $w(u_{i,j}u_{i,k}) = \frac{\mu(U_i)}{\binom{\deg u_i}{2}}$ where $\mu(U_i)$ is the crime rate of region U_i and

$$\binom{\deg u_i}{2} = \frac{\deg u_i!}{(\deg u_i - 2)!2!} = \frac{\deg u_i(\deg u_i - 1)}{2}. \tag{4.12}$$

Thus, we obtain

$$w(u_{1,2}u_{1,4}) = w(u_{1,2}u_{1,5}) = w(u_{1,4}u_{1,5}) = \frac{20\%}{3} = \frac{1}{15}$$

$$w(u_{2,4}u_{1,2}) = w(u_{2,4}u_{2,3}) = w(u_{1,2}u_{2,3}) = \frac{15\%}{3} = \frac{1}{15}$$

$$w(u_{2,3}u_{3,4}) = \frac{10\%}{1} = \frac{1}{10}$$

$$\begin{aligned} w(u_{1,4}u_{2,4}) &= w(u_{2,4}u_{3,4}) = w(u_{3,4}u_{4,5}) = w(u_{1,4}u_{4,5}) \\ &= w(u_{2,4}u_{4,5}) = w(u_{1,4}u_{3,4}) = \frac{40\%}{6} = \frac{1}{15} \end{aligned}$$

$$w(u_{1,5}u_{4,5}) = \frac{15\%}{1} = \frac{3}{20}$$

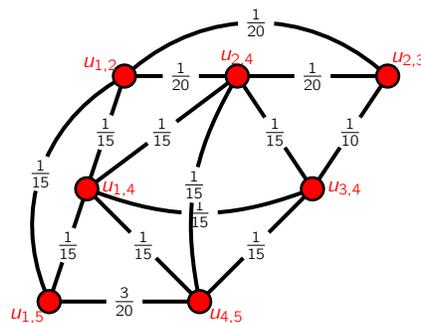


Figure 4.4: Weighted Line Graph

The problem wants to solve the (r, s, a, b) -security assignment problem of $L(G)$ where $r = 0$, $s = 0.95$, $a = 0$ and $b = 0.3$. The $(0, 0.95, 0, 0.3)$ -security assignment problem of $L(G)$ is a problem of finding $\tau : L(E) \rightarrow [0, 0.3]$ such that

$$1. \min_{\gamma: L(E) \rightarrow [0, 0.3]} \sum_{e \in L(E)} (w(e) - \gamma(e))^2 = \sum_{e \in L(E)} (w(e) - \tau(e))^2.$$

2. $\max_{\gamma: L(E) \rightarrow [0,0.3]} \left(1 - \sum_{e \in L(E)} \gamma(e)\right) = 1 - \sum_{e \in L(E)} \tau(e).$
3. $0 \leq r \leq \sum_{e \in L(E)} \tau(e) \leq s \leq 1.$

From theorem 3.1, we can obtain the vector solution of the (r, s, a, b) -security assignment problem of $L(G)$ by finding the solution $(x_1^*, x_2^*, \dots, x_{14}^*)$ of the problem

$$\min_{x_1, x_2, \dots, x_{14} \in \mathbb{R}} \left(\sum_{i=1}^{14} (x_i - w(e_i) - 0.5) \right)^2 \tag{4.13}$$

subject to

1. $0 \leq x_i \leq 0.3.$
2. $0 \leq \sum_{i=1}^{|L(E)|} x_i \leq 0.95.$

Thus, we have

$$\begin{aligned} \min_{x_1, x_2, \dots, x_{14} \in \mathbb{R}} & \left(\left(x_1 - \frac{17}{30}\right)^2 + \left(x_2 - \frac{17}{30}\right)^2 + \left(x_3 - \frac{17}{30}\right)^2 + \left(x_4 - \frac{11}{20}\right)^2 + \left(x_5 - \frac{11}{20}\right)^2 \right. \\ & + \left(x_6 - \frac{11}{20}\right)^2 + \left(x_7 - \frac{3}{5}\right)^2 + \left(x_8 - \frac{17}{30}\right)^2 + \left(x_9 - \frac{17}{30}\right)^2 + \left(x_{10} - \frac{17}{30}\right)^2 + \left(x_{11} - \frac{17}{30}\right)^2 \\ & \left. + \left(x_{12} - \frac{17}{30}\right)^2 + \left(x_{13} - \frac{17}{30}\right)^2 + \left(x_{14} - \frac{13}{20}\right)^2 \right) \end{aligned}$$

subject to

1. $0 \leq x_i \leq 0.3.$
2. $0 \leq \sum_{i=1}^{|L(E)|} x_i \leq 0.95.$

Applying Lagrange multiplier, then we obtain the approximate solution $(0.0631, 0.0631, 0.0631, 0.0464, 0.0464, 0.0464, 0.0964, 0.0631, 0.0631, 0.0631, 0.0631, 0.0631, 0.0631, 0.1464)$. Thus, we have the following:

Percentage of Assigned Number of Personnel per Region		
Table 5.1: Region	Crime Rate	Percentage of assigned number of Personnel
1	20%	$0.0631 + 0.0631 + 0.0631 = 0.1893$ (18.93%)
2	15%	$0.0464 + 0.0464, +0.0464 = 0.13.92$ (13.92%)
3	10%	0.0964 (9.64%)
4	40%	$0.0631 + 0.0631 + 0.0631 + 0.0631 + 0.0631 + 0.0631 = 0.3786$ (37.86%)
5	15%	0.1464 (14.64%)

Therefore, by considering the graph and weight of the region, the optimal percentage of total assigned security personnel is 95% and the percentage of reserved security personnel is 5%.

5. Conclusions

This study proved that the (r, s, a, b) -security assignment problem of G has at most one solution (τ, τ^*) . Furthermore, if it has a solution (τ, τ^*) , then $\tau(e_i) = \langle P_D(u - \frac{1}{2}v), u_i \rangle$ where

$u_i = (0, 0, 0, \dots, 0, 1, 0, \dots, 0)$ is the vector where the i th coordinate is 1, and 0 elsewhere where $i \in \{1, 2, \dots, |E|\}$ and

$$D = \left\{ (x_1, x_2, \dots, x_{|E|}) \in \mathbb{R}^{|E|} : 0 \leq x_i \leq 1 \text{ and } 0 \leq r \leq \sum_{i=1}^{|E|} x_i \leq s \leq 1 \right\}$$

In addition, we proved that if (θ, θ^*) is the pseudo-solution of (r, s, a, b) -security assignment problem of G , then the (r, s, a, b) -security assignment problem of G has a solution if and only if

1. $\max_{\gamma \in Y(E, [a, b], [r, s])} \left(1 - \sum_{e \in E} \gamma(e) \right) = 1 - \sum_{e \in E} \theta(e).$
2. $\min_{\gamma \in Y(E, [a, b], [r, s])} \sum_{e \in E} (w(e) - \gamma(e))^2 = \sum_{e \in E} (w(e) - \theta(e))^2.$

The model is applicable in security assignment problem and CCTV assignment problem.

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References

- [1] Z. Ghazali, M.A.A. Majid, M. Shazwani, Optimal Solution of Transportation Problem Using Linear Programming: A Case of a Malaysian Trading Company, *J. Appl. Sci.* 12 (2012), 2430-2435. <https://doi.org/10.3923/jas.2012.2430.2435>.
- [2] A.J. Mehta, A.K. Rifai, Goal Programming Application to Assignment Problem in Marketing, *J. Acad. Market. Sci.* 7 (1979), 108-116. <https://doi.org/10.1007/bf02721918>.
- [3] J.A. Breslaw, A Linear Programming Solution to the Faculty Assignment Problem, *Socio-Econ. Plan. Sci.* 10 (1976), 227-230. [https://doi.org/10.1016/0038-0121\(76\)90008-2](https://doi.org/10.1016/0038-0121(76)90008-2).
- [4] C. van Dooren, A Review of the Use of Linear Programming to Optimize Diets, Nutritiously, Economically and Environmentally, *Front. Nutr.* 5 (2018), 48. <https://doi.org/10.3389/fnut.2018.00048>.
- [5] J. Foytik, Very Low-Cost Nutritious Diet Plans Designed by Linear Programming, *J. Nutr. Educ.* 13 (1981), 63-66. [https://doi.org/10.1016/s0022-3182\(81\)80098-2](https://doi.org/10.1016/s0022-3182(81)80098-2).
- [6] A. Briend, N. Darmon, Determining Limiting Nutrients by Linear Programming: A New Approach to Predict Insufficient Intakes From Complementary Foods, *Pediatrics.* 106 (2000), 1288-1289. <https://doi.org/10.1542/peds.106.s4.1288>.
- [7] A. Briend, E. Ferguson, N. Darmon, Local Food Price Analysis by Linear Programming: A New Approach to Assess the Economic Value of Fortified Food Supplements, *Food Nutr. Bull.* 22 (2001), 184-189. <https://doi.org/10.1177/156482650102200210>.
- [8] D.H. Stimson, R.P. Thompson, Linear Programming, Busing and Educational Administration, *Socio-Econ. Plan. Sci.* 8 (1974), 195-206. [https://doi.org/10.1016/0038-0121\(74\)90043-3](https://doi.org/10.1016/0038-0121(74)90043-3).
- [9] J. Reeb, S. Leavengood, Using Duality and Sensitivity Analysis to Interpret Linear Programming Solutions, Oregon State University, 2000. <http://hdl.handle.net/1957/20129>.
- [10] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, Cham, 2017. <https://doi.org/10.1007/978-3-319-48311-5>.
- [11] Y.L. Yu, The Proximity Operators, (2014). https://www.cs.cmu.edu/~suvrit/teach/yaoliang_proximity.pdf.
- [12] J. Dattorro, *Convex Optimization in Euclidean Distance Geometry*, (2005). <https://ccrma.stanford.edu/~dattorro/0976401304.pdf>.
- [13] S.P. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.

-
- [14] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier, New York, (1976).
[15] B. Nag, Business Applications of Operations Research, Business Expert Press, New York, 2014.