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On Existence and Attractivity of Ψ-Hilfer Hybrid Fractional-order Langevin Differential Equations

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Abstract. The work reported in this article studies the equivalence relationship between fractional integral equation and Ψ-Hilfer Hybrid Langevin Differential Equations of fractional order with nonlocal initial conditions, and then we use this relationship to establish the existence of the results by means of Banach algebra and Schauder's fixed point theorem. We then demonstrate the uniform local attractiveness of all the solutions.

1. Introduction

ODEs are extended to include fractional differential equations (FDEs), where the order of the derivative can be any positive number. For this reason, approaching the problem as an FDE typically allows us to model an experimental dynamic more effectively. Which fractional derivative (FD) is most appropriate at this point? The solution to this question typically depends on the problem and hence on the collected information. Consider using a definition of fractional operators that is more broad to get around the multitude of definitions for FDs. For better and more accurate simulations, we use Ψ-Hilfer fractional derivative(Ψ-HFD) and fixed point theory as an important tool to derive existence criterion of solutions. Kilbas et al. [1] introduced the notion of FD with respect to another function in the context of the RL FD. Similar to this, Almeida [15] proposed the Ψ- Caputo FD and looked at a variety of intriguing aspects of this operator. FD operator with two parameters was presented by Hilfer [16]. The Hilfer derivative unifies the RL FD and Caputo FD-based theories of FDEs. Sousa and Oliveira

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presented the Hilfer FD with respect to another function in [9]; known as the Ψ-HFD. The Ψ-HFD's significance stems from the fact that it uses a number of well-known FD operators as its specific cases, for example, RL [1], Caputo [1], Hadmard [1], Riesz [1], Erdeyl-Kober [1], Ψ-Caputo [15], Katugampola [21], Hilfer [16, 17] and so on. With this approach, it is possible to examine a wide range of properties of FDE solutions that employ several FD operators using a single FD operator. A number of researches have been conducted using Ψ-HFD [6, 7, 10, 12–14, 17–20].

"In 2021, Bachir et al. [8] proved the existence and attractivity of solutions for Ψ-Hilfer hybrid FDEs:

$$D^{\lambda,\sigma;\Psi}\frac{u(t)}{v(t,u(t))} = w(t,u(t));\ a.e.\ t\in\mathbb{R}_+, \tag{1.1}$$

$$(\Psi(t)-\Psi(0))^{1-\zeta}u(t)|_{t=0} = u_0;\ u_0\in\mathbb{R}, \tag{1.2}$$

where $\mathbb{R}_+ = [0,\infty)$, $0<\lambda<1$, $0\le\sigma\le 1$, $\zeta=\lambda+\sigma(1-\lambda)$, $D^{\lambda,\sigma;\Psi}$ is the Ψ-HFD of order λ and type σ, $v:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}^*$ and $w:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ are given functions."

"In 2022, Kucche et al. [11] established the existence of solutions in the weighted space for the following Ψ- Hilfer Hybrid FDE:

$$D^{\mu,\nu;\Psi}\frac{y(t)}{f(t,y(t))} = g(t,y(t));\ a.e.\ t\in(0,T], \tag{1.3}$$

$$(\Psi(t)-\Psi(0))^{1-\xi}y(t)|_{t=0} = y_0;\ y_0\in\mathbb{R}, \tag{1.4}$$

where $0<\mu<1$, $0\le\nu\le 1$, $\xi=\mu+\nu(1-\mu)$, $D^{\mu,\nu;\Psi}$ is the Ψ-HFD of order μ and type ν, $f\in\mathcal{C}(\mathfrak{I}\times\mathbb{R},\mathbb{R}-\{0\})$ is bounded, $\mathfrak{I}=[0,T]$ and $g\in\mathcal{C}(\mathfrak{I}\times\mathbb{R},\mathbb{R})$={h|the map w → h(t,w) is continuous for each t and the map $t\to h(t,w)$ is measurable for each w}."

Motivated by [11], [8], the following IVP of the Ψ-Hilfer type fractional-order langevin equation with nonlocal initial conditions is explored, and the existence and attractivity results are obtained:

$$\mathcal{D}_{0^+}^{\nu_1,\beta_1;\Psi}\left(\mathcal{D}_{0^+}^{\nu_2,\beta_2;\Psi}\frac{\varkappa(t)}{\mathcal{G}(t,\varkappa(t))}+p\varkappa(t)\right) = \mathcal{F}(t,\varkappa(t)) \tag{1.5}$$

$$\varkappa(t)|_{t=0} = 0,\ (\Psi(t)-\Psi(0))^{1-\gamma_1-\nu_2}\varkappa(t)|_{t=0} = \varkappa_0 \tag{1.6}$$

where $\mathcal{D}_{0^+}^{\nu_i,\beta_i;\Psi}$, $i=1,2$ is the Ψ-HFD of order ν_i, $0<\nu_i<1$ and type β_i, $0\le\beta_i\le 1$; $1<\nu_1+\nu_2\le 2$, $\mathcal{F}:\mathfrak{I}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $p\in\mathbb{R}$, $t\in[0,\varepsilon]$ and $\gamma_i=\nu_i+\beta_i(1-\nu_i)$.

Special Cases:

(a) For $\beta_1,\beta_2=0$, $\Psi(t)=t$; we get nonlinear hybrid RL Langevin FDE for a.e. $t\in(0,\varepsilon)$ of the form

$$^{RL}\mathcal{D}^{\nu_1}\left(^{RL}\mathcal{D}^{\nu_2}\frac{\varkappa(t)}{\mathcal{G}(t,\varkappa(t))}+p\varkappa(t)\right) = \mathcal{F}(t,\varkappa(t))$$

$$\varkappa(t)|_{t=0} = 0$$

For $\nu_1 = 1, \nu_2 = 1$, we generate hybrid differential equations of integer order

$$\frac{d}{d\mathfrak{t}}\left\{\frac{d}{d\mathfrak{t}}\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))} + p\varkappa(\mathfrak{t})\right\} = \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$$
$$\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = 0$$

(b) For $\nu_1 = 0,\ \beta_2 = 0,\ p = 0,\ \Psi(\mathfrak{t}) = \mathfrak{t},\ \varkappa_0 = 0$ and for a.e. $\mathfrak{t} \in (0,\varepsilon)$ we obtain the nonlocal hybrid FDEs of the form

$$^{RL}\mathcal{D}^{\nu_2}\left[\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))}\right] = \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$$
$$\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = 0$$

the existence results for which are obtained in [24].

For $\nu_2 = 1$ and $\mathfrak{t} \in (0,\varepsilon)$ a.e., the investigations of [4] regarding the hybrid differential equation of integer order are incorporated into the results of the current work

$$\frac{d}{d\mathfrak{t}}\left[\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))}\right] = \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$$
$$\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = 0$$

(c) The outcomes are relevant to the below mentioned nonlinear Ψ-HFD for $\mathfrak{t} \in (0,\varepsilon)$ and $\mathcal{G} = 1$

$$\mathcal{D}_{0^+}^{\nu_1,\beta_1;\Psi}\left(\mathcal{D}_{0^+}^{\nu_2,\beta_2;\Psi} + p\right)\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$$
$$\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = 0, (\Psi(\mathfrak{t}) - \Psi(0))^{1-\gamma_1-\nu_2}\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = \varkappa_0$$

When $\mathcal{G}$ is nothing but constant function with value equal to 1, $\beta_1, \beta_2 = 0, \nu_1 = 0, \Psi(\mathfrak{t}) = \mathfrak{t}$, i.e. $\gamma_1 = 0$ the investigation of nonlinear FDEs involving RL FD is among the obtained outcomes of [22] for a.e. $\mathfrak{t} \in (0,\varepsilon)$

$$^{RL}\mathcal{D}^{\nu_2}\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$$
$$\mathfrak{t}^{1-\nu_2}\varkappa(\mathfrak{t})|_{\mathfrak{t}_0} = \varkappa_0 \in \mathbb{R}.$$

With $\beta_1 = 0, \beta_2 = 1, \nu_1 = 0, \Psi(\mathfrak{t}) = \mathfrak{t}$, i.e.$\gamma_1 = 0$, $\mathcal{G} = 1$ the investigation of nonlinear FDEs utilising the Caputo FD is among the obtained outcomes [23] for a.e. $\mathfrak{t} \in (0,\varepsilon)$

$$^{C}\mathcal{D}^{\nu_2}\varkappa(\mathfrak{t}) = \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$$
$$\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = \varkappa_0$$

The following is how the paper is organised: Section 2 goes through a few crucial foundational concepts from fractional calculus. Finding corresponding integral equations for the Hybrid-HFDE is the main topic of Section 3, along with the existence of solutions to the IVP (1.5)-(1.6) using FPT and uniform local attractiveness of the solution. Section 4 provides a summary of the results.

2. Auxiliary results

Let $L_p([0,\varepsilon],\mathbb{R})$ be the Banach space of all Lebesgue measurable functions from $[0,\varepsilon]$ to $\mathbb{R}$ with $\|f\|_{L_p[0,\varepsilon]} < \infty$. Consider a function which is differentiable and increasing for all $\mathfrak{t} \in [0,\varepsilon] = \mathfrak{I}$, say $\Psi \in \mathcal{C}^1(\mathfrak{I},\mathbb{R})$, where $(0 < \varepsilon < \infty)$ and $\Psi'(\mathfrak{t}) \neq 0$. We will be using, $\mathcal{P}^{\sigma}_{\Psi}(\mathfrak{t},\mathfrak{s}) = (\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\sigma}$, $\mathcal{W}^{\nu}_{\Psi}(\mathfrak{t},\mathfrak{s}) = \Psi'(s)(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\nu}$ and $\gamma_1 + \nu_2 = \sigma$ throughout this paper to reduce the length of the equations. Here, we've listed some of the spaces used in this article:

(a) $\mathcal{C}_{1-\sigma;\Psi}([0,\varepsilon],\mathbb{R})$, the weighted Banach Space of all partially ordered functions with $||.||_{\mathcal{C}_{1-\sigma;\Psi}[0,\varepsilon]}$ defined as:

$$\mathcal{C}_{1-\sigma;\Psi}[0,\varepsilon] = \{h : (0,\varepsilon] \to \mathbb{R} | \mathcal{P}^{1-\sigma}_{\Psi}(\mathfrak{t},0)h(\mathfrak{t}) \in \mathcal{C}[0,\varepsilon]\} \; where,$$

$$||h||_{\mathcal{C}_{1-\sigma;\Psi}[0,\varepsilon]} = \max_{t\in[0,\varepsilon]} |\mathcal{P}^{1-\sigma}_{\Psi}(\mathfrak{t},0)h(\mathfrak{t})|.$$

Let $\Xi = \left(\mathcal{C}_{1-\sigma;\Psi}(\mathfrak{I},\mathbb{R}), \|.\|_{\mathcal{C}_{1-\sigma;\Psi}(\mathfrak{I},\mathbb{R})}\right)$ be a Banach algebra where $(\varkappa y)(\mathfrak{t}) = \varkappa(\mathfrak{t})y(\mathfrak{t}), \mathfrak{t} \in \mathfrak{I}$ is how the product of vectors is defined.

(b) Let $\mathcal{B}_{\mathcal{C}} = \mathcal{B}_{\mathcal{C}}(\mathbb{R}_+)$ be the Banach space of all functions $\Phi : \mathbb{R}_+$ to $\mathbb{R}$ which are bounded as well as continuous.

(c) $\mathcal{B}_{\mathcal{C}1-\sigma} = \mathcal{B}_{\mathcal{C}1-\sigma}(\mathbb{R}_+)$, denote the weighted space defined by $\mathcal{B}_{\mathcal{C}1-\sigma} = \{\phi : \mathbb{R}_+ \to \mathbb{R} : \mathcal{P}^{1-\sigma}_{\Psi}(\mathfrak{t},0)\Phi(\mathfrak{t}) \in \mathcal{B}_{\mathcal{C}}\}$ of all bounded and continuous functions with the norm

$$\|\Phi\|_{\mathcal{B}_{\mathcal{C}1-\sigma}} = \sup_{\mathfrak{t}\in\mathbb{R}_+} |\mathcal{P}^{1-\sigma}_{\Psi}(\mathfrak{t},0)\Phi(\mathfrak{t})|.$$

Let's revisit some fractional calculus definitions and characteristics.

Definition 2.1. *[1] "Let $\nu > 0$, $\nu \in \mathbb{R}$, and $g \in L^1([0,\varepsilon],\mathbb{R})$. The Ψ-R-L fractional integral of a function g with respect to Ψ is defined by*

$$I^{\nu;\Psi}_{a^+} = \frac{1}{\Gamma(\nu)} \int_a^t \Psi'(s)(\Psi(\mathfrak{t}) - \Psi(s))^{\nu-1} g(s)ds."$$

Definition 2.2. *[9] "Let $n-1 < \nu < n$, $n \in \mathbb{N}$ and $g \in C^n([0,\varepsilon],\mathbb{R})$. The Ψ-HFD ${}^{H}\mathcal{D}^{\nu,\beta;\Psi}_{a^+}(.)$ of a function g of order ν and type $0 \le \beta \le 1$ is defined by*

$${}^{H}\mathcal{D}^{\nu,\beta;\Psi}_{a^+} g(\mathfrak{t}) = I^{\beta(n-\nu);\Psi}_{a^+} \left(\frac{1}{\Psi'(s)}\frac{d}{d\mathfrak{t}}\right)^n I^{(1-\beta)(n-\nu);\Psi}_{a^+} g(\mathfrak{t})."$$

Lemma 2.1. *[9] "Let $\nu > 0$ and $\delta > 0$. Then*

(i) $I^{\nu;\Psi}_{a^+} I^{\nu;\Psi}_{a^+} h(\mathfrak{t}) = I^{\nu+\nu;\Psi}_{a^+} h(\mathfrak{t})$;

(ii) $I^{\nu;\Psi}_{a^+} (\Psi(\mathfrak{t}) - \Psi(a))^{\delta-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\delta)} (\Psi(\mathfrak{t}) - \Psi(a))^{\nu+\delta-1}$."

And we observe that ${}^{H}\mathcal{D}^{\nu,\beta;\Psi}_{a^+} (\Psi(\mathfrak{t}) - \Psi(a))^{(\gamma-1)} = 0$.

Lemma 2.2. *[9] "Let $f \in L(0,\varepsilon), n-1<\nu \leq n, n \in \mathbb{N}, 0 \leq \beta \leq 1, \gamma=\nu+\beta(1-\nu)$, $I_{a^+}^{(1-\beta)(n-\nu)} f \in AC^k[0,\varepsilon]$. Then*

$$\left(I_{a^+}^{\nu;\Psi}\, {}^{H}\mathcal{D}_{a^+}^{\nu,\beta;\Psi} f\right)(\mathfrak{t})=f(\mathfrak{t})-\sum_{k=1}^{n} \frac{(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\gamma-k}}{\Gamma(\gamma-k+1)}\left(\frac{1}{\Psi'(s)} \frac{d}{dt}\right)^{n} \lim_{t \rightarrow a^{+}}(I_{a^+}^{(1-\beta)(n-\nu)} f)(\mathfrak{t})."$$

Also, note that $\mathcal{D}_{a^+}^{\nu_1,\beta;\Psi} I_{a^+}^{\nu_2;\Psi} f(\mathfrak{t})=I_{a^+}^{\nu_2-\nu_1} f(\mathfrak{t})$, if $\nu_2>\nu_1$ and $\mathcal{D}_{a^+}^{\nu_1-\nu_2} f(\mathfrak{t})$, if $\nu_1>\nu_2$.

For the readers' convenience, we have included some of the Fixed Point Theorems (FPTs) that were utilised in this article.

Lemma 2.3. *[3]"Let S be a non-empty closed, convex and bounded subset of the Banach algebra Ξ and let $A: \Xi \rightarrow \Xi$ and $B: S \rightarrow \Xi$ be two operators such that*

(i) A is Lipschitzian with a Lipschitz constant α;
(ii) B is completely continuous;
(iii) $y=Ay B\varkappa \implies y \in S$ for all $\varkappa \in S$ and
(iv) $\alpha M<1$ where $M=\sup\{\|B\varkappa\|: \varkappa \in S\}$.

Then, the operator equation $y=AyBy$ has a solution in S."

Lemma 2.4. *[8] "Solution of equation $(\mathcal{K}(\varkappa))(\mathfrak{t})=\mathfrak{t}$ are locally attractive if there exists a ball $B(\varkappa_0,\mu)$ in the space $\mathcal{B}_{\mathcal{C}}$ such that, for any solutions $y=y(\mathfrak{t})$ and $\sigma=\sigma(\mathfrak{t})$ of above equations that belong to $B(\varkappa_0,\mu) \cap \Lambda$, we can write*

$$\lim_{\mathfrak{t} \rightarrow \infty}(y(\mathfrak{t})-\sigma(\mathfrak{t}))=0. \tag{2.1}$$

If the limit (2.1) is uniform with respect to $B(\varkappa_0,\mu) \cap \Lambda$, where $\phi \neq \Lambda \subset \mathcal{B}_{\mathcal{C}}$, then the solutions are said to be uniformly locally attractive (or, equivalently, that the solutions are locally asymptotically stable)."

Lemma 2.5. *[5] "Let $M \subset \mathcal{B}_{\mathcal{C}}$. Then M is relatively compact in $\mathcal{B}_{\mathcal{C}}$ if the following conditions are satisfied:*

(i) M is uniformly bounded in $\mathcal{B}_{\mathcal{C}}$;
(ii) the functions belonging to M are almost equicontinuous in $\mathbb{R}_+$, i.e., equicontinuous on every compact set in $\mathbb{R}_+$;
(iii) the functions from M are equiconvergent, i.e. given $\epsilon>0$, there exists $L(\epsilon)>0$ such that

$$|\varkappa(\mathfrak{t})-\lim_{t \rightarrow \infty} \varkappa(\mathfrak{t})|<\epsilon,$$

for any $\mathfrak{t} \geq L(\epsilon)$ and $\varkappa \in M$."

Theorem 2.1. *[2] (Schauder Fixed-Point Theorem). "Let F be a Banach space, let U be a nonempty bounded convex and closed subset of F, and let $K: U \rightarrow U$ be a compact and continuous map. Then, K has at least one fixed point in U."*

3. Main Results

Here, we develop an auxiliary lemma showing the relationship between the fractional IVP (1.5)-(1.6) and a corresponding fractional IE.

Lemma 3.1. *The hybrid fractional IVP (1.5)-(1.6) for* $\mathfrak{t} \in [0, \varepsilon]$ *is equivalent to the hybrid fractional IE*

$$\varkappa(\mathfrak{t}) = \mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))\left\{\frac{\varkappa_0}{\mathcal{G}(0, \varkappa(0))}\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t}, 0) + I_{0^+}^{\nu_1+\nu_2;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) - pI_{0^+}^{\nu_2;\Psi}\varkappa(\mathfrak{t})\right\} \tag{3.1}$$

and thus a function $\varkappa \in \mathcal{C}_{1-\sigma}(\mathfrak{J}, \mathbb{R})$ *is a solution of (1.5)-(1.6) iff it is a solution of (3.1).*

Proof. We shall establish that a solution of the IVP (1.5)-(1.6) is a solution of the fractional IE (3.1). Using Lemma (2.2) and the Ψ-R-L FI of order ν_1 on equation (1.5), we obtain

$$\mathcal{D}_{0^+}^{\nu_2,\beta_2;\Psi}\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))} + p\varkappa(\mathfrak{t}) = I_{0^+}^{\nu_1;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) + \frac{c_0}{\Gamma(\gamma_1)}\mathcal{P}_{\Psi}^{\gamma_1-1}(\mathfrak{t}, 0). \tag{3.2}$$

Using Lemma (2.2) and the Ψ-R-L FI of order ν_2 on equation (3.2), we get

$$\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))} = I_{0^+}^{\nu_1+\nu_2;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) - pI_{0^+}^{\nu_2;\Psi}\varkappa(\mathfrak{t}) + \frac{c_0}{\Gamma(\sigma)}\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t}, 0) + \frac{c_1}{\Gamma(\gamma_2)}\mathcal{P}_{\Psi}^{\gamma_2-1}(\mathfrak{t}, 0). \tag{3.3}$$

Using $\varkappa(\mathfrak{t})|_{\mathfrak{t}=0} = 0$, we get $c_1 = 0$ for $\mathcal{G}(0, \varkappa(0)) = 0$ Thus,

$$\varkappa(\mathfrak{t}) = \mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))\left\{\frac{c_0}{\Gamma(\sigma)}\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t}, 0) + I_{0^+}^{\nu_1+\nu_2;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) - pI_{0^+}^{\nu_2;\Psi}\varkappa(\mathfrak{t})\right\}. \tag{3.4}$$

Multiplying $\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}, 0)$ on both sides of above equation, we get

$$\begin{aligned}\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}, 0)\varkappa(\mathfrak{t}) =& \frac{c_0}{\Gamma(\sigma)}\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t})) + \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}, 0)\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))I_{0^+}^{\nu_1+\nu_2;\Psi}F(t, \varkappa(\mathfrak{t})) \\ &- p\,\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}, 0)I_{0^+}^{\nu_2;\Psi}\varkappa(\mathfrak{t}).\end{aligned}$$

Applying initial condition (1.6) and substituting $\mathfrak{t} = 0$, we obtain

$$c_0 = \frac{\varkappa_0\Gamma(\sigma)}{\mathcal{G}(0, \varkappa(0))}.$$

Replacing c_0 in eqn. (3.4), we get

$$\varkappa(\mathfrak{t}) = \mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))\left\{\frac{\varkappa_0}{\mathcal{G}(0, \varkappa(0))}\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t}, 0) + I_{0^+}^{\nu_1+\nu_2;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) - pI_{0^+}^{\nu_2;\Psi}\varkappa(\mathfrak{t})\right\}.$$

Conversely, a solution of the fractional IE (3.1) is also a solution of the IVP (1.5)-(1.6). Then, The aforementioned equation may be expressed as

$$\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))} = \frac{\varkappa_0}{\mathcal{G}(0, \varkappa(0))}\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t}, 0) + I_{0^+}^{\nu_1+\nu_2;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) - pI_{0^+}^{\nu_2;\Psi}\varkappa(\mathfrak{t}). \tag{3.5}$$

Operating the Ψ-HD, $D^{\nu_2,\beta_2;\Psi}$ on both sides and using the Lemma (2.2) , we obtain

$$D^{\nu_2,\beta_2;\Psi}\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t}, \varkappa(\mathfrak{t}))} = \frac{\varkappa_0}{\mathcal{G}(0, \varkappa(0))}\mathcal{P}_{\Psi}^{\gamma_1-1}(\mathfrak{t}, 0) + I_{0^+}^{\nu_1;\Psi}\mathcal{F}(\mathfrak{t}, \varkappa(\mathfrak{t})) - p\varkappa(\mathfrak{t}).$$

Again applying $D^{\nu_1,\beta_1;\Psi}$ on above equation , we get

$$D^{\nu_1,\beta_1;\Psi}\left(D^{\nu_2,\beta_2;\Psi}\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))}+p\varkappa(\mathfrak{t})\right)=\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}D^{\nu_1,\beta_1;\Psi}\mathcal{P}_\Psi^{\gamma_1-1}(\mathfrak{t},0)+\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t})).$$

Now, using the Lemma (2.1) $D^{\nu_1,\beta_1;\Psi}\mathcal{P}_\Psi^{\gamma_1-1}(\mathfrak{t},0)=0$, we get

$$D^{\nu_1,\beta_1;\Psi}\left(D^{\nu_2,\beta_2;\Psi}\frac{\varkappa(\mathfrak{t})}{\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))}+p\varkappa(\mathfrak{t})\right)=\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t})).$$

At $\mathfrak{t}=0$ and $\mathcal{F}(0,\varkappa(0))=0$, the given equation simplifies to $\varkappa(\mathfrak{t})|_{\mathfrak{t}=0}=0$ and from equation (3.5) and Lemma 2.1(ii), we get $\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)\varkappa(\mathfrak{t})|_{\mathfrak{t}=0}=\varkappa_0$. □

In the next theorem, we utilise Banach algebra to demonstrate the existence of solution for (1.5)-(1.6). We require the following hypotheses on $\mathcal{G}$ and $\mathcal{F}$ in order to establish our conclusion:

(A) $\mathcal{G}$ is a bounded function in $\mathcal{C}\big(\mathfrak{J}\times\mathbb{R},\mathbb{R}-\{0\}\big)$ such that:
   (i) $\varkappa\to\frac{\varkappa}{\mathcal{G}(\mathfrak{t},\varkappa)}$ for $\mathfrak{t}\in\mathfrak{J}$ a.e. is an increasing map in $\mathbb{R}$;
   (ii) For all $\varkappa,y\in\mathbb{R}$, $\mathfrak{t}\in\mathfrak{J}$, such that $\mathcal{G}$ satisfies Lipchitz condition for second variable.

(B) For all $\varkappa\in\mathbb{R}$ and $\mathfrak{t}\in\mathfrak{J}$ a.e. $\exists h_1,h_2\in\mathcal{C}(\mathfrak{J},\mathbb{R})$, such that $|\mathcal{F}(\mathfrak{t},\varkappa)|\le h_1(\mathfrak{t})$ and $\varkappa(\mathfrak{t})\le h_2(\mathfrak{t})$.

Theorem 3.1. *If (A)-(B) holds. Then, $\exists$ a solution $\varkappa\in\mathcal{C}_{1-\sigma;\Psi}(\mathfrak{J},\mathbb{R})$ of the hybrid FDE (1.5)-(1.6) provided*

$$\mathcal{L}\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+\frac{\|h_1\|_\infty\mathcal{P}_\Psi^{1-\gamma_1+\nu_1}(\varepsilon,0)}{\Gamma(\nu_1+\nu_2+1)}+p\frac{\|h_2\|_\infty\mathcal{P}_\Psi^{1-\gamma_1}(\varepsilon,0)}{\Gamma(\nu_2+1)}\right\}<1. \tag{3.6}$$

Proof. Define,

$$\mathcal{S}=\{\varkappa\in\Xi:\|\varkappa\|_{\mathcal{C}_{1-\sigma};\Psi}(\mathfrak{J},\mathbb{R})\le\mathfrak{R}\}$$

where

$$\mathfrak{R}=K\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+\frac{\|h_1\|_\infty\mathcal{P}_\Psi^{1-\gamma_1+\nu_1}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2+1)}+p\frac{\|h_2\|_\infty\mathcal{P}_\Psi^{1-\gamma_1}(\mathfrak{t},0)}{\Gamma(\nu_2+1)}\right\}$$

and K is bound on $\mathcal{G}$. It is evident that $\mathcal{S}$ is a bounded subset of Ξ which is closed and convex. Define $A:\Xi\to\Xi$ and $B:\mathcal{S}\to\Xi$ as

$$\begin{aligned}\mathcal{A}\varkappa(\mathfrak{t})=&\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t})),\mathfrak{t}\in\mathfrak{J},\\ \mathcal{B}\varkappa(\mathfrak{t})=&\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\mathcal{P}_\Psi^{\sigma-1}(\mathfrak{t},0)+\frac{1}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}\\ &-p\frac{1}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\end{aligned}$$

Thus, equation (3.1) is nothing but $\varkappa=\mathcal{A}\varkappa\mathcal{B}\varkappa,\ \varkappa\in\Xi$. We shall demonstrate that $\mathcal{A}$ and $\mathcal{B}$ meet all of the criteria of Lemma(2.3):

Firstly, we shall prove that

$$\|\mathcal{A}\varkappa-\mathcal{A}y\|_{\mathcal{C}_{1-\sigma},\Psi(\mathfrak{J},\mathbb{R})}\le\mathcal{L}\|\varkappa-y\|_{\mathcal{C}_{1-\sigma},\Psi(\mathfrak{J},\mathbb{R})} \tag{3.7}$$

i.e. $\mathcal{A}$ is an operator satisfying Lipschitz condition.

From assumption (A)(ii), we observe that

$$\begin{aligned}|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)(\mathcal{A}\varkappa(\mathfrak{t})-\mathcal{A}y(\mathfrak{t}))| &= \left|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)\Big(\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))-g(\mathfrak{t},y(\mathfrak{t}))\Big)\right| \\ &\le \mathcal{L}|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)(\varkappa(\mathfrak{t})-y(\mathfrak{t}))| \\ &\le \mathcal{L}\|\varkappa-y\|_{\mathcal{C}_{1-\sigma},\Psi(\mathfrak{I},\mathbb{R})}\end{aligned}$$

Next, we need to prove that $\mathcal{B}:\mathcal{S}\to\Xi$ is completely continuous.

For this we shall prove that $\mathcal{B}$ is continuous, uniformly bounded and equicontinuous. For continuity of $\mathcal{B}$, consider a sequence $\varkappa_n\to\varkappa$ in $\mathcal{S}$. Then,

$$\begin{aligned}\|\mathcal{B}\varkappa_n-\mathcal{B}\varkappa\|_{\mathcal{C}_{1-\sigma;\Psi(\mathfrak{I},\mathbb{R})}} &= \max_{\mathfrak{t}\in\mathfrak{I}}|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)\big(\mathcal{B}\varkappa_n(\mathfrak{t})-\mathcal{B}\varkappa(\mathfrak{t})\big)| \\ &\le \max_{t\in\mathfrak{I}}\left\{\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa_n(\mathfrak{s}))-\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s}\right. \\ &\quad \left.-p\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa_n(\mathfrak{s})-\varkappa(\mathfrak{s})|d\mathfrak{s}\right\}.\end{aligned}$$

As $n\to\infty$, $\|\mathcal{B}\varkappa_n-\mathcal{B}\varkappa\|_{\mathcal{C}_{1-\sigma;\Psi(\mathfrak{I},\mathbb{R})}}\to 0$ by virtue of continuity of $\mathcal{F}$ and Lebesgue dominated convergence theorem.

For any $\mathfrak{t}\in\mathfrak{I}$ and $\varkappa\in\mathcal{S}$, we shall exhibit that $\mathcal{B}(\mathcal{S})=\{\mathcal{B}\varkappa:\varkappa\in\mathcal{S}\}$ is uniformly bounded.

$$\begin{aligned}|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)\mathcal{B}\varkappa(\mathfrak{t})| &\le \left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s} \\ &\quad +p\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s} \\ &\le \left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+\|h_1\|_\infty\frac{\mathcal{P}_\Psi^{1-\gamma_1+\nu_1}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2+1)}+p\|h_2\|_\infty\frac{\mathcal{P}_\Psi^{1-\gamma_1}(\mathfrak{t},0)}{\Gamma(\nu_2+1)}.\end{aligned}$$

Therefore,

$$\|\mathcal{B}\varkappa\|_{\mathcal{C}_{1-\sigma;\Psi(\mathfrak{I},\mathbb{R})}} \le \left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+\|h_1\|_\infty\frac{\mathcal{P}_\Psi^{1-\gamma_1+\nu_1}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2+1)}+p\|h_2\|_\infty\frac{\mathcal{P}_\Psi^{1-\gamma_1}(\mathfrak{t},0)}{\Gamma(\nu_2+1)}. \tag{3.8}$$

Now, for any $\varkappa\in\mathcal{S}$ and $\mathfrak{t}_1,\mathfrak{t}_2\in\mathfrak{I}$ with $\mathfrak{t}_1<\mathfrak{t}_2$ we shall prove the equicontinuity of $\mathcal{B}(\mathcal{S})$.

Making use of assumption (B), we have

$$\begin{aligned}&|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t}_2,0)\mathcal{B}\varkappa(\mathfrak{t}_2)-\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t}_1,0)\mathcal{B}\varkappa(\mathfrak{t}_1)| \\ =&\Bigg|\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}+\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t}_2,0)\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}\right. \\ &\left.-p\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s}\right\} \\ &-\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}+\frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}\right.\end{aligned}$$

$$
\begin{aligned}
&- p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s}\Big\}\Big|\\
\leq&\Big|\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\|h_1\|_\infty\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,0)d\mathfrak{s}-\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)}\|h_1\|_\infty\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})d\mathfrak{s}\\
&- p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\|h_2\|_\infty\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})d\mathfrak{s}+p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_2)}\|h_2\|_\infty\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})d\mathfrak{s}\Big|\\
\leq&\frac{\|h_1\|_\infty}{\Gamma(\nu_1+\nu_2)}\{\mathcal{P}_{\Psi}^{1-\gamma_1+\nu_1}(\mathfrak{t}_2,0)-\mathcal{P}_{\Psi}^{1-\gamma_1+\nu_1}(\mathfrak{t}_1,0)\}+p\frac{\|h_2\|_\infty}{\Gamma(\nu_2)}\{\mathcal{P}_{\Psi}^{1-\gamma_1}(\mathfrak{t}_1,0)-\mathcal{P}_{\Psi}^{1-\gamma_1}(\mathfrak{t}_2,0)\}.
\end{aligned}
$$

Thus, the continuity of Ψ implies $|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)\mathcal{B}\varkappa(\mathfrak{t}_2)-\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)\mathcal{B}\varkappa(\mathfrak{t}_1)|\to 0$ as $|\mathfrak{t}_1-\mathfrak{t}_2|\to 0$. Thus, Arzela-Ascoli theorem implies $\mathcal{B}(\mathcal{S})$ is relatively compact and hence a compact operator as a result. It is completely continuous from the continuity and compactness of $\mathcal{B}:\mathcal{S}\to\Xi$.

Now, we shall show that for any $\mathfrak{u}\in\Xi$, $\mathfrak{u}=\mathcal{A}\mathfrak{u}\mathcal{B}\varkappa \implies \mathfrak{u}\in\mathcal{S}$ for all $\varkappa\in\mathcal{S}$.
Let any $\mathfrak{u}\in\Xi$ and $\varkappa\in\mathcal{S}$ such that $\mathfrak{u}=\mathcal{A}\mathfrak{u}\mathcal{B}\varkappa$. The function $\mathcal{G}$ being bounded and using the hypothesis (B), for any $\mathfrak{t}\in\mathfrak{J}$, we have

$$
\begin{aligned}
|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathfrak{u}(\mathfrak{t})|=&|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathcal{A}\mathfrak{u}(\mathfrak{t})\mathcal{B}\varkappa(\mathfrak{t})|\\
\leq&\Big|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathcal{G}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))\Big\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t},0)\\
&+\frac{1}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}-p\frac{1}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\Big\}\Big|\\
\leq&|\mathcal{G}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))|\Big\{\Big|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Big|+\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s}\\
&+p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s}\Big\}\\
\leq&K\Big\{\Big|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Big|+\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2+1)}\|h_1\|_\infty\mathcal{P}_{\Psi}^{\nu_1+\nu_2}(\mathfrak{t},0)+p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2+1)}\|h_2\|_\infty\mathcal{P}_{\Psi}^{\nu_2}(\mathfrak{t},0)\Big\}.
\end{aligned}
$$

i.e.,

$$
\|\mathfrak{u}\|_{\mathcal{C}_{1-\sigma;\Psi}(\mathfrak{J},\mathbb{R})}\leq K\Big\{\Big|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Big|+\|h_1\|_\infty\frac{\mathcal{P}_{\Psi}^{1-\gamma_1+\nu_1}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2+1)}+p\|h_2\|_\infty\frac{\mathcal{P}_{\Psi}^{1-\gamma_1}(\mathfrak{t},0)}{\Gamma(\nu_2+1)}\Big\}=\mathfrak{R}
$$

$\implies \mathfrak{u}\in\mathcal{S}$.
In the end, we shall show that for $M=\sup\{\|B\mathfrak{u}\|_{\mathcal{C}_{1-\sigma;\Psi}(\mathfrak{J},\mathbb{R})}:\mathfrak{u}\in\mathcal{S}\}$, we have $\alpha M<1$. Utilising inequality (3.8), we obtain

$$
\begin{aligned}
M=&\sup\Big\{\|B\varkappa\|_{\mathcal{C}_{1-\sigma;\Psi}(\mathfrak{J},\mathbb{R})}:\varkappa\in\mathcal{S}\Big\}\\
\leq&\Big\{\Big|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Big|+\frac{\|h_1\|_\infty\mathcal{P}_{\Psi}^{1-\gamma_1+\nu_1}(\varepsilon,0)}{\Gamma(\nu_1+\nu_2+1)}+p\frac{\|h_2\|_\infty\mathcal{P}_{\Psi}^{1-\gamma_1}(\varepsilon,0)}{\Gamma(\nu_2+1)}\Big\}
\end{aligned}
$$

Making use of inequality (3.7), we get $\alpha = \mathcal{L}$. Therefore, as a consequence of the condition (3.6), we get the required

$$\alpha M \leq \mathcal{L}\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right| + \frac{\|h_1\|_\infty \mathcal{P}_\Psi^{1-\gamma_1+\nu_1}(\varepsilon,0)}{\Gamma(\nu_1+\nu_2+1)} + p\frac{\|h_2\|_\infty \mathcal{P}_\Psi^{1-\gamma_1}(\varepsilon,0)}{\Gamma(\nu_2+1)}\right\} < 1.$$

On applying Lemma (2.3), the solution for equation $\varkappa = \mathcal{A}\varkappa\mathcal{B}\varkappa$ in $\mathcal{S}$ is obtained and thus for hybrid FDE (1.5)-(1.6). □

Using Schauder's FPT, we can now exhibit the existence and attractiveness of solutions. Assume the following:

(C) For each $\varkappa \in \mathcal{B}_{\mathcal{C}1-\sigma}$, $\mathfrak{t} \to \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$ is measurable on $\mathbb{R}_+$; for a.e. $\mathfrak{t} \in \mathbb{R}_+$ the mapping $\varkappa \to \mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t}))$ is continuous on $\mathcal{B}_{\mathcal{C}1-\sigma}$ and $\varkappa \to \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))$ is continuous and bounded.

(D) For each $\varkappa \in \mathbb{R}$ and a.e. $\mathfrak{t} \in \mathbb{R}_+$, $\exists\, T : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is a continuous function and

$$\mathcal{F}(\mathfrak{t},\varkappa(\mathfrak{t})) \leq \frac{T(\mathfrak{t})}{1+p|\varkappa|},$$

$$\lim_{\mathfrak{t}\to\infty} \mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi} + pI_{0^+}^{\nu_2;\Psi})T(\mathfrak{t}) = 0.$$

Set

$$T^* = \sup_{t\in\mathbb{R}_+} \mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi} + pI_{0^+}^{\nu_2;\Psi})T(\mathfrak{t}) < \infty.$$

Theorem 3.2. *If (C)- (D) holds, then, $\exists$ at least one solution for problem (1.5) defined on $\mathbb{R}_+$ which is uniformly locally attractive.*

Proof. Define $\mathcal{K}$ for $\varkappa \in \mathcal{B}_{\mathcal{C}1-\sigma}$

$$(\mathcal{K}\varkappa)(\mathfrak{t}) = \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\left\{\frac{\varkappa_0\mathcal{P}_\Psi^{\sigma-1}(\mathfrak{t},0)}{\mathcal{G}(0,\varkappa(0))} + \frac{1}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} \right.$$
$$\left. - p\frac{1}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right\}$$

Let function $\mathcal{G}$ be bounded by $\mathcal{N}$. Now, for $\varkappa \in \mathcal{B}_{\mathcal{C}1-\sigma}, \mathfrak{t} \in \mathbb{R}_+$

$$\begin{aligned}|\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)(\mathcal{K}\varkappa)(\mathfrak{t})| \leq& |\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))|\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right| + \frac{\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s}\right.\\ &\left. + \frac{p\mathcal{P}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s}\right\},\\ \leq& \mathcal{N}\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right| + T^*\right\} = R^*.\\ \|\mathcal{K}(\varkappa)\|_{\mathcal{B}_\mathcal{C}} \leq& R_*, \end{aligned} \tag{3.9}$$

implies, $\mathcal{K}(\varkappa) \in \mathcal{B}_{\mathcal{C}1-\sigma}$ and $\mathcal{K}(\mathcal{B}_{\mathcal{C}1-\sigma}) \subset \mathcal{B}_{\mathcal{C}1-\sigma}$ as a result of the continuity of $\mathcal{K}(\varkappa)$ on $\mathbb{R}_+$; for any $\varkappa \in \mathcal{B}_{\mathcal{C}1-\sigma}$.

Consider $B_{R^*} = B(0,R_*) = \{\mathcal{G} \in \mathcal{B}_{\mathcal{C}1-\sigma} : \|\mathcal{G}\|_{\mathcal{B}_{\mathcal{C}_{1-\sigma}}} \leq R_*\}$.

Equation (3.9) implies $\mathcal{K}$ transforms the ball B_{R^*} into itself. From Lemma (3.1) the solutions of problem (1.5)-(1.6) are nothing but the fixed points of $\mathcal{K}(\varkappa)$. We shall show that the operator $\mathcal{K}$ satisfies all the assumptions of Theorem (2.1).

Step 1. Firstly, we shall prove the continuity of $\mathcal{K}$.

Consider a convergent sequence $\{\varkappa_n\}_{n\in\mathbb{N}}$ in B_{R_*} such that $\varkappa_n \to \varkappa$. Then, for each $\mathfrak{t} \in \mathbb{R}_+$, we have

$$\begin{aligned}
&|\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)(\mathcal{K}\varkappa_n)(\mathfrak{t}) - \mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)(\mathcal{K}\varkappa)(\mathfrak{t})| \\
\leq&\Bigg|\mathcal{G}(\mathfrak{t},\varkappa_n(\mathfrak{t}))\Bigg\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa_n(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)} \\
&\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa_n(\mathfrak{s})d\mathfrak{s}\Bigg\} - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\Bigg\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} \\
&- p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\Bigg\}\Bigg| \\
\leq&\Bigg|\mathcal{G}(t,\varkappa_n(\mathfrak{t}))\Bigg\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(s,\varkappa_n(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)} \\
&\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa_n(\mathfrak{s})d\mathfrak{s}\Bigg\} - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\Bigg\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(s,\varkappa_n(\mathfrak{s}))d\mathfrak{s} \\
&- p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa_n(\mathfrak{s})d\mathfrak{s}\Bigg\} + \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)} \\
&\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(s,\varkappa_n(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa_n(\mathfrak{s})d\mathfrak{s}\Bigg\} - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\Bigg\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} \\
&+ \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\Bigg\}\Bigg| \\
\leq&\Bigg|\mathcal{G}(\mathfrak{t},\varkappa_n(\mathfrak{t})) - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\Bigg|\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(s,\varkappa_n(\mathfrak{s}))d\mathfrak{s} \\
&- p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa_n(\mathfrak{s})d\mathfrak{s}\Bigg|\Bigg\} + |\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))|\Bigg\{\Bigg|\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s}) \\
&\Big(\mathcal{F}(s,\varkappa_n(\mathfrak{s})) - \mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))\Big)d\mathfrak{s} - p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})(\varkappa_n(\mathfrak{s}) - \varkappa(\mathfrak{s}))d\mathfrak{s}\Bigg|\Bigg\} \\
\leq&\Bigg|\mathcal{G}(t,\varkappa_n(\mathfrak{t})) - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\Bigg|\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg| + \frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa_n(\mathfrak{s}))|d\mathfrak{s} \\
&+ p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa_n(\mathfrak{s})|d\mathfrak{s}\Bigg\} + \mathcal{N}\Bigg\{\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s}) \\
&|\mathcal{F}(\mathfrak{s},\varkappa_n(\mathfrak{s})) - \mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s} + p\frac{\mathcal{Q}_\Psi^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}} \mathcal{W}_\Psi^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa_n(\mathfrak{s}) - \varkappa(\mathfrak{s})|d\mathfrak{s}\Bigg\}.
\end{aligned}$$

Case I. For $\mathfrak{t} \in [0,\varepsilon]$, by applying Lebesgue dominated convergence theorem and $\varkappa_n \to \varkappa$ as $n \to \infty$ on above equation along with the continuity of $\mathcal{G}$ and $\mathcal{F}$, we get $\|\mathcal{K}(\varkappa_n) - \mathcal{K}(\varkappa)\|_{\mathcal{B}_{\mathcal{C}_{1-\sigma}}} \to 0$ *as* $n \to \infty$.

Case II. For $\mathfrak{t} \in (\varepsilon, \infty)$, then, from the hypotheses and above equation, we have

$$|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(\mathcal{K}\varkappa_n)(\mathfrak{t}) - \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(\mathcal{K}\varkappa)(\mathfrak{t})| \leq \left|\mathcal{G}(t,\varkappa_n(\mathfrak{t})) - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\right|\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|\right.$$
$$+ \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\left(I_{0^+}^{\nu_1+\nu_2;\Psi}\mathcal{F}(s,\varkappa_n(\mathfrak{s})) + pI_{0^+}^{\nu_2;\Psi}\varkappa_n(\mathfrak{s})\right) + \mathcal{N}\left\{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\left(I_{0^+}^{\nu_1+\nu_2;\Psi}|\mathcal{F}(s,\varkappa_n(\mathfrak{s}))\right.\right.$$
$$\left.\left.- \mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))| + pI_{0^+}^{\nu_2;\Psi}|\varkappa_n(\mathfrak{s}) - \varkappa(\mathfrak{s})|\right)\right\}$$
$$\leq \left|\mathcal{G}(t,\varkappa_n(\mathfrak{t})) - \mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))\right|\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right| + \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\left(I_{0^+}^{\nu_1+\nu_2;\Psi} + pI_{0^+}^{\nu_2;\Psi}\right)T(\mathfrak{t})\right\}$$
$$+ 2\mathcal{N}\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\left(I_{0^+}^{\nu_1+\nu_2;\Psi} + pI_{0^+}^{\nu_2;\Psi}\right)T(\mathfrak{t}).$$

Since, $\varkappa_n \to \varkappa$ as $n \to \infty$, $\mathcal{G}$ is continuous and $\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi} + pI_{0^+}^{\nu_2;\Psi})T(\mathfrak{t}) \to 0$ as $\mathfrak{t} \to \infty$, it follows from above equation that $\|\mathcal{K}(\varkappa_n) - \mathcal{K}(\varkappa)\|_{\mathcal{B}_{\mathcal{C}1-\sigma}} \to 0$ *as* $n \to \infty$.

Step 2. On every compact subset $[0,\varepsilon]$ of $\mathbb{R}_+, \varepsilon > 0$, we need to prove the uniform boundedness and equi-continuity of $L(B_{R_*})$. Since B_{R_*} is bounded and $L(B_{R_*}) \subset B_{R_*}$, so $L(B_{R_*})$ is uniformly bounded. For each $\varkappa \in B_{R_*}$ and $\mathfrak{t}_1, \mathfrak{t}_2 \in [0,\varepsilon], \mathfrak{t}_1 < \mathfrak{t}_2$, we have

$$|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)(\mathcal{K}\varkappa)(\mathfrak{t}_2) - \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)(\mathcal{K}\varkappa)(\mathfrak{t}_1)|$$
$$\leq\left|\mathcal{G}(\mathfrak{t}_2,\varkappa(\mathfrak{t}_2))\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\right.\right.$$
$$\left.\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right\} - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\right.$$
$$\left.\left.\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right\}\right|$$
$$\leq\left|\mathcal{G}(\mathfrak{t}_2,\varkappa(\mathfrak{t}_2))\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\right.\right.$$
$$\left.\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,s)\varkappa(\mathfrak{s})d\mathfrak{s}\right\} - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\right.$$
$$\left.\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right\} + \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\right.$$
$$\left.\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right\} - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))$$
$$\left.\left\{\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right\}\right|$$
$$\leq|\mathcal{G}(\mathfrak{t}_2,\varkappa(\mathfrak{t}_2)) - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))|\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}\right.\right.$$
$$\left.\left.+ p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s}\right|\right\} + |\mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))|\left\{\left|\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}\right.\right.$$
$$\left.+ \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s} - \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}_1}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}\right\}$$

$$+ p\Big\{ \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)} \int_0^{\mathfrak{t}_1} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s} - \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_2)} \int_0^{\mathfrak{t}_1} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s} \Big| \Big\}$$

$$\leq |\mathcal{G}(\mathfrak{t}_2,\varkappa(\mathfrak{t}_2)) - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))| \Big\{ \Big| \frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} \Big| + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \int_0^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s}$$
$$+ p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)} \int_0^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s} \Big\} + \mathcal{L}\Big\{ \Big(\int_0^{\mathfrak{t}_1} \Big| \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})$$
$$- \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s}) \Big| |\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s} \Big)$$
$$+ p\Big(\int_0^{\mathfrak{t}_1} \Big| \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s}) - \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_2)} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_1,\mathfrak{s}) \Big| |\varkappa(\mathfrak{s})|d\mathfrak{s}$$
$$+ \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_2)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t}_2,\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s} \Big) \Big\}$$
$$\leq |\mathcal{G}(\mathfrak{t}_2,\varkappa(\mathfrak{t}_2)) - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))| \Big\{ \Big| \frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} \Big| + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \int_0^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})T(\mathfrak{s})d\mathfrak{s} \Big\}$$
$$+ \mathcal{L}\Big\{ \Big(\int_0^{\mathfrak{t}_1} \Big| \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_2,\mathfrak{s}) - \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s}) \Big| T(\mathfrak{s})d\mathfrak{s}$$
$$+ \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})T(\mathfrak{s})d\mathfrak{s} \Big) \Big\}.$$

Given that T, $\mathcal{G}$ are continuous and setting $T_* = \sup_{t\in[0,\varepsilon]} T(\mathfrak{t})$, we obtain

$$|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)(\mathcal{K}\varkappa)(\mathfrak{t}_2) - \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)(\mathcal{K}\varkappa)(\mathfrak{t}_1)|$$
$$\leq |\mathcal{G}(\mathfrak{t}_2,\varkappa(\mathfrak{t}_2)) - \mathcal{G}(\mathfrak{t}_1,\varkappa(\mathfrak{t}_1))| \Big\{ \Big| \frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} \Big| + T_* \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \int_0^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})d\mathfrak{s} \Big\} + \mathcal{L}T_*$$
$$\Big(\int_0^{\mathfrak{t}_1} \Big| \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s}) - \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)}{\Gamma(\nu_1+\nu_2)} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s}) \Big| d\mathfrak{s} + \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)}{\Gamma(\nu_1+\nu_2)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t}_1,\mathfrak{s})d\mathfrak{s} \Big).$$

As $\mathfrak{t}_1 \to \mathfrak{t}_2$, we have $|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_2,0)(\mathcal{K}\varkappa)(\mathfrak{t}_2) - \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t}_1,0)(\mathcal{K}\varkappa)(\mathfrak{t}_1)| \to 0$.

Step 3. To prove the equiconvergence of $L(B_R)$. For any $\varkappa \in L(B_{R_*})$,

$$|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(\mathcal{K}\varkappa)(\mathfrak{t})| \leq |\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))| \Big\{ \Big| \frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} \Big| + \Big| \frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)} \int_0^{\mathfrak{t}} \mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))d\mathfrak{s}$$
$$- p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)} \int_0^{\mathfrak{t}} \mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t},\mathfrak{s})\varkappa(\mathfrak{s})d\mathfrak{s} \Big| \Big\}, \ \mathfrak{t} \in \mathbb{R}_+$$
$$\leq \mathcal{L}\Big\{ \Big| \frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))} \Big| + \mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi} + pI_{0^+}^{\nu_2;\Psi})T(\mathfrak{t}) \Big\}.$$

Since $\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi}+I_{0^+}^{\nu_2;\Psi})T(\mathfrak{t})\to 0$ as $t\to\infty$, we find

$$|(\mathcal{K}\varkappa(\mathfrak{t})|\leq\mathcal{L}\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathcal{G}(0,\varkappa(0))}\Bigg|+\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi}T)(\mathfrak{t})}{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}+p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_2;\Psi}T)(\mathfrak{t})}{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}\Bigg\}.$$

Hence, $|(\mathcal{K}\varkappa)(\mathfrak{t})-(\mathcal{K}\varkappa)(\infty)|\to 0$ as $\mathfrak{t}\to\infty$. Thus, $\mathcal{K}:B_{R_*}\to B_{R_*}$ is compact and continuous using Lemma (2.5). Applying Schauder FPT (2.1),the fixed point of $\mathcal{K}$ is a solution of (1.5) on $\mathbb{R}_+$.

Step 4. Uniform local attractivity.

Let $\varkappa_*$ be a solution of hybrid FDE (1.5) and $\varkappa\in B\left(\varkappa_*,2\mathcal{L}\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+2T^*\right\}\right)$, we have

$$\begin{aligned}
&|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathcal{K}(\varkappa)(\mathfrak{t})-\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(\varkappa_*)(\mathfrak{t})|\\
\leq&|\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathcal{K}(\varkappa)(\mathfrak{t})-\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)\mathcal{K}(\varkappa_*)(\mathfrak{t})|\\
\leq&\Big|\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))-\mathcal{G}(\mathfrak{t},\varkappa_*(\mathfrak{t}))\Big|\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg|+\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))|d\mathfrak{s}\\
&+p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa(\mathfrak{s})|d\mathfrak{s}\Bigg\}+\mathcal{L}\Bigg\{\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))\\
&-\mathcal{F}(\mathfrak{s},\varkappa_*(\mathfrak{s}))|d\mathfrak{s}+p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa(\mathfrak{s})-\varkappa_*(\mathfrak{s})|d\mathfrak{s}\Bigg\}\\
\leq&2\mathcal{L}\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg|+\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|T(\mathfrak{s})|d\mathfrak{s}\Bigg\}\\
&+2\mathcal{L}\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}{\Gamma(\nu_1+\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|T(\mathfrak{s})|d\mathfrak{s}\\
\leq&2\mathcal{L}\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg|+2T^*\Bigg\}.
\end{aligned}$$

Thus, we get

$$\|\mathcal{K}(\varkappa)-\varkappa_*\|_{\mathcal{B}_{\mathcal{C}_{1-\sigma}}}\leq 2\mathcal{L}\Bigg\{\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg|+2T^*\Bigg\}.$$

This implies the continuity of $\mathcal{K}$ such that

$$\mathcal{K}\left(B\left(\varkappa_*,2\mathcal{L}\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+2T^*\right\}\right)\right)\subset\left(B\left(\varkappa_*,2\mathcal{L}\left\{\left|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\right|+2T^*\right\}\right)\right).$$

Moreover, if $\varkappa$ is a solution of problem (1.5), then

$$\begin{aligned}
&|\varkappa(\mathfrak{t})-\varkappa_*(\mathfrak{t})|\\
&=|\mathcal{K}\varkappa(\mathfrak{t})-\mathcal{K}\varkappa_*(\mathfrak{t})|\\
&\leq|\mathcal{G}(\mathfrak{t},\varkappa(\mathfrak{t}))-\mathcal{G}(\mathfrak{t},\varkappa_*(\mathfrak{t}))|\Bigg\{\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t},0)\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg|+\Bigg\{\frac{1}{\Gamma(\nu_1+\nu_2)}\\
&\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_1+\nu_2-1}(\mathfrak{t},\mathfrak{s})|\mathcal{F}(\mathfrak{s},\varkappa(\mathfrak{s}))-\mathcal{F}(\mathfrak{s},\varkappa_*(\mathfrak{s}))|d\mathfrak{s}-\frac{p}{\Gamma(\nu_2)}\int_0^{\mathfrak{t}}\mathcal{W}_{\Psi}^{\nu_2-1}(\mathfrak{t},\mathfrak{s})|\varkappa(\mathfrak{s})-\varkappa_*(s)|d\mathfrak{s}\Bigg\}\\
&\leq 2\mathcal{L}\Bigg\{\mathcal{P}_{\Psi}^{\sigma-1}(\mathfrak{t},0)\Bigg|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Bigg|+2\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_1+\nu_2;\Psi}T)(\mathfrak{t})}{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}+2p\frac{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)(I_{0^+}^{\nu_2;\Psi}T)(\mathfrak{t})}{\mathcal{P}_{\Psi}^{1-\sigma}(\mathfrak{t},0)}\Bigg\}.
\end{aligned}$$

Therefore,

$$|\varkappa(t)-\varkappa_*(t)| = |(\mathcal{K}(\varkappa(t))-(\mathcal{K}(\varkappa_*(t))| \leq 2\mathcal{L}\Big\{\mathcal{P}_\Psi^{\sigma-1}(t,0)\Big|\frac{\varkappa_0}{\mathcal{G}(0,\varkappa(0))}\Big| + 2\frac{\mathcal{P}_\Psi^{1-\sigma}(t,0)(I_{0^+}^{\nu_1+\nu_2;\Psi}T)(t)}{\mathcal{P}_\Psi^{1-\sigma}(t,0)} + 2p\frac{\mathcal{P}_\Psi^{1-\sigma}(t,0)(I_{0^+}^{\nu_2;\Psi}T)(t)}{\mathcal{P}_\Psi^{1-\sigma}(t,0)}\Big\}. \tag{3.10}$$

By using (3.10) and

$$\lim_{t\to\infty}\mathcal{P}_\Psi^{1-\sigma}(t,0)(I_{0^+}^{\nu_1+\nu_2;\Psi}+I_{0^+}^{\nu_2;\Psi})T(t)=0,$$

we conclude

$$\lim_{t\to\infty}|\varkappa(t)-\varkappa_*(t)|=0.$$

The Lemma (2.4) indicates that solutions of IVP (1.5) has uniform local attractiveness. □

4. Conclusion

The criteria presented in this work ensured the existence and uniform local attractivity of solutions for some Hybrid FDEs with Ψ-Hilfer FD. The methodology is predicated on Banach algebras and Schauder's FPT.

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