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Certain Fixed-Point Results via DS-Weak Commutativity Condition in Neutrosophic Metric Spaces With Application to Non-linear Fractional Differential Equations

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ABSTRACT. This study demonstrates that, for the non-linear contractive conditions in Neutrosophic metric spaces, a common fixed-point theorem may be proved without requiring the continuity of any mappings. A novel commutativity requirement for mappings weaker than the compatibility of mappings is used to demonstrate the conclusion. We provide several examples to illustrate our major idea. Also, we provide an application to the non-linear fractional differential equation to show the validity of our main result.

1. INTRODUCTION AND PRELIMAINARIES

There is still one significant deficiency despite the development of the computer industry and its incredible achievement in changing several fields of research. Computers are not intended to process phrases with uncertainty. Logic must be developed to handle uncertainty using methods other than the classical ones. One approach to uncertainty is fuzzy set theory, where topological structures are

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fundamental building blocks for creating mathematical models that are applicable to real-world scenarios.

El Naschie's [1] established a relation between topology, algebra and geometry, he showed that the approach of dimension is based on topological structure. El Naschie [2] established a pure mathematical derivation of the structure of real spacetime from quantum set theory that is exceedingly straightforward and simple to understand. This is done by combining the von Neumann-Connes dimensional function of the Klein-Penrose modular holographic boundary of the E8E8 exceptional Lie group bulk of our universe with components of the Menger-Urysohn dimensional theory and the topological theory of cobordism. The end result of a paper is a clear, concise mental image: quantum spacetime is just the border or surface of the quantum wave empty set, while the quantum wave itself is an empty set that represents the surface, or boundary, of the zero-set quantum particle. The fundamental distinction between quantum spacetime and a quantum wave is that the latter is a multi-fractal type of infinitely many empty sets with varying degrees of emptiness, whilst the former is a simple empty set. The notion of intuitionistic fuzzy metric spaces (IFMSs), which introduced and discussed by Park in [3], is useful in modeling. It is based on the notion of the intuitionistic fuzzy set (IFS) given by Atanssov [4] and the concept of fuzzy metric space (FMS) given by George and Veeramani [5].

Using the concept of IFSs, Alaca et al. [6] established the concept of FMS as it was introduced by Kramosil and Michalek [7]. Additionally, they developed the concept of Cauchy sequences in IFMS and used the notion introduced by Grabiec [8] to demonstrate the extension of the Banach contraction principle given in [9] to FMS and Edelstein [10] to IFMS. The common fixed-point theorem of Jungck [11] was extended to IFMS by Turkoglu et al. in [12]. The idea of IFMS and their applications were researched by Gregory et al. [13], Sadati and Park [14], and Rodriguez-Lopez and Ramagurea [15]. Sessa [16] developed the idea of weak commuting(w-commuting) mappings in metric spaces. Broader commutativity, or compatibility, was introduced by Jungck [11]. Mishra et al. [17] established the notion of suitable maps in the context of FMSs. Vasuki [18] established the notion of an R-weak commutativity in the context of FMSs. R-w-commuting mappings of type (A) in metric spaces were established by Pathak et al. [19], who demonstrated that these mappings are incompatible and that R-w-commuting mappings are not always R-w-commuting of type (A). R-w- commuting mappings on IFMS were defined by Jesic and Babacev [20]. Sharma and Deshpande [21] described (DS)-weak commutativity in FMSs and established the notion of R-w-commuting mappings of type (A) in the context of FMSs. Recently, Boyd and Wong [22] and Jesic and Babacev [20] derived a common fixed-point theorem by utilizing nonlinear contractive conditions and assuming continuity for a pair of mappings on IFMS. The method of neutrosophic metric spaces (NMSs), which deals with membership, non-membership, and naturalness functions, was proposed by Kirişci and Simsek [23]. In the case of NMSs, Sowndrarajan, et al. [24] demonstrated some fixed-point results. Schweizer and Sklar worked on statistical metric spaces and Deshpande [26] derived several fixed-point results and under some interesting conditions in the context of an IFMS.

In this manuscript, we will study some interesting circumstances in which continuity may be not necessary to find an existence and uniqueness of a solution. Also, we establish an application to non-linear fractional differential equations to show the validity of our main result.

Now, we discuss some important notions that will be helpful for readers to understand main section.

Definition 1.1: [25] A binary operation $*: [0,1] \to [0,1]$ is said to be continuous t-norm (CTN) if $*$ is fulfill the aforementioned requirements:

(TN1) $\sigma * \varsigma = \varsigma * \sigma$,

(TN2) $\sigma * (\varsigma * \upsilon) = (\sigma * \varsigma) * \upsilon$,

(TN3) $*$ is continuous,

(TN4) $\sigma * 1 = 1$ for all $\sigma \in [0,1]$,

(TN5) $\sigma * \varsigma \leq \upsilon * \Delta$ whenever $\sigma \leq \upsilon$ and $\varsigma \leq \Delta$, for all $\sigma, \varsigma, \upsilon, \Delta \in [0,1]$.

Definition 1.2: [25] A binary operation $\Diamond: [0,1] \times [0,1] \to [0,1]$ is said to be continuous t-conorm (CTCN) if $\Diamond$ satisfies (TN1)-(TN3), (TN5) and the following condition:

(TN4)* $\sigma \Diamond 0 = \sigma$ for all $\sigma \in [0,1]$.

Definition 1.3: [6] A 5-tuple $(\Xi, \mathfrak{B}, \mathfrak{D}, *, \Diamond)$ is called an IFMS if Ξ is an arbitrary set $*$ is a CTN, $\Diamond$ is CTCN and $\mathfrak{B}, \mathfrak{D}$ are fuzzy sets on $\Xi^2 \times [0, \infty)$ verifies the below conditions:

(IFM1) $\mathfrak{B}(\varpi, \omega, \tau) + \mathfrak{D}(\varpi, \omega, \tau) \leq 1$ for all $\varpi, \omega \in \Xi$ and $\tau > 0$,

(IFM2) $\mathfrak{B}(\varpi, \omega, 0) = 0$ for all $\varpi, \omega \in \Xi$,

(IFM3) $\mathfrak{B}(\varpi, \omega, \tau) = 1$ for all $\varpi, \omega \in \Xi$ and $\tau > 0$ if and only if $\varpi = \omega$,

(IFM4) $\mathfrak{B}(\varpi, \omega, \tau) = \mathfrak{B}(\omega, \varpi, \tau)$ for all $\varpi, \omega \in \Xi, \tau > 0$,

(IFM5) $\mathfrak{B}(\varpi, \omega, \tau) * \mathfrak{B}(\omega, \mu, s) \leq \mathfrak{B}(\varpi, \mu, \tau + s)$ for all $\varpi, \omega, \mu \in \Xi$ and $s, \tau > 0$,

(IFM6) for all $\varpi, \omega \in \Xi, \mathfrak{B}(\varpi, \omega, .): [0, \infty) \to (0,1]$ is left-continuous, (IFM7) $\lim_{\tau\to\infty} \mathfrak{B}(\varpi,\omega,\tau) = 1$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$,

(IFM8) $\mathfrak{D}(\varpi,\omega,0) = 1$ for all $\varpi,\omega \in \Xi$,

(IFM9) $\mathfrak{D}(\varpi,\omega,\tau) = 0$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$ iff $\varpi = \omega$,

(IFM10) $\mathfrak{D}(\varpi,\omega,\tau) = \mathfrak{D}(\omega,\varpi,\tau)$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$,

(IFM11) $\mathfrak{D}(\varpi,\omega,\tau) \lozenge \mathfrak{D}(\omega,\mu,s) \geq \mathfrak{D}(\varpi,\mu,\tau+s)$ for all $\varpi,\omega,\mu \in \Xi$ and $s,\tau > 0$,

(IFM12) for all $\varpi,\omega \in \Xi, \mathfrak{D}(\varpi,\omega,.) : [0,\infty) \to [0,1]$ is a right-continuous,

(IFM13) $\lim_{\tau\to\infty} \mathfrak{D}(\varpi,\omega,\tau) = 0$ for all $\varpi,\omega \in \Xi$.

Then $(\mathfrak{B},\mathfrak{D})$ is called an intuitionistic fuzzy metric on Ξ.

Example 1.1: Suppose (Ξ,Δ) be a metric space. Define CTN by $\sigma * \varsigma = \min\{\sigma,\varsigma\}$ and CTCN by $\sigma \lozenge \varsigma = \max\{\sigma,\varsigma\}$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$,

$$\mathfrak{B}_{\Delta}(\varpi,\omega,\tau) = \frac{\tau}{\tau + \Delta(\varpi,\omega)}, \qquad \mathfrak{D}_{\Delta}(\varpi,\omega,\tau) = \frac{\Delta(\varpi,\omega)}{\tau + \Delta(\varpi,\omega)}.$$

Then $(\Xi,\mathfrak{B},\mathfrak{D},*,\lozenge)$ is an IFMS.

Definition 1.4: [23] A 6-tuple $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\lozenge)$ is called an NMS if Ξ is an arbitrary set, $*$ is a CTN, $\lozenge$ is CTCN and $\mathfrak{B},\mathfrak{D},\mathfrak{S}$ are neutrosophic sets on $\Xi^2 \times [0,\infty)$ verifies the following conditions:

(NMS1) $\mathfrak{B}(\varpi,\omega,\tau) + \mathfrak{D}(\varpi,\omega,\tau) + \mathfrak{S}(\varpi,\omega,\tau) \leq 3$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$,

(NMS2) $\mathfrak{B}(\varpi,\omega,0) = 0$ for all $\varpi,\omega \in \Xi$,

(NMS3) $\mathfrak{B}(\varpi,\omega,\tau) = 1$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$ if and only if $\varpi = \omega$,

(NMS4) $\mathfrak{B}(\varpi,\omega,\tau) = \mathfrak{B}(\omega,\varpi,\tau)$ for all $\varpi,\omega \in \Xi, \tau > 0$,

(NMS5) $\mathfrak{B}(\varpi,\omega,\tau) * \mathfrak{B}(\omega,\mu,s) \leq \mathfrak{B}(\varpi,\mu,\tau+s)$ for all $\varpi,\omega,\mu \in \Xi$ and $s,\tau > 0$,

(NMS6) for all $\varpi,\omega \in \Xi, \mathfrak{B}(\varpi,\omega,.) : [0,\infty) \to (0,1]$ is left-continuous,

(NMS7) $\lim_{\tau\to\infty} \mathfrak{B}(\varpi,\omega,\tau) = 1$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$,

(NMS8) $\mathfrak{D}(\varpi,\omega,0) = 1$ for all $\varpi,\omega \in \Xi$,

(NMS9) $\mathfrak{D}(\varpi,\omega,\tau) = 0$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$ iff $\varpi = \omega$,

(NMS10) $\mathfrak{D}(\varpi,\omega,\tau) = \mathfrak{D}(\omega,\varpi,\tau)$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$,

(NMS11) $\mathfrak{D}(\varpi,\omega,\tau) \lozenge \mathfrak{D}(\omega,\mu,s) \geq \mathfrak{D}(\varpi,\mu,\tau+s)$ for all $\varpi,\omega,\mu \in \Xi$ and $s,\tau > 0$,

(NMS12) for all $\varpi,\omega \in \Xi, \mathfrak{D}(\varpi,\omega,.) : [0,\infty) \to [0,1]$ is a right-continuous,

(NMS13) $\lim_{\tau\to\infty} \mathfrak{D}(\varpi,\omega,\tau) = 0$ for all $\varpi,\omega \in \Xi$.

(NMS14) $\mathfrak{S}(\varpi,\omega,0) = 1$ for all $\varpi,\omega \in \Xi$,

(NMS15) $\mathfrak{S}(\varpi,\omega,\tau) = 0$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$ iff $\varpi = \omega$,

(NMS16) $\mathfrak{S}(\varpi,\omega,\tau) = \mathfrak{S}(\omega,\varpi,\tau)$ for all $\varpi,\omega \in \Xi$ and $\tau > 0$, (NMS17) $\mathfrak{S}(\varpi,\omega,\tau)\Diamond\,\mathfrak{S}(\omega,\mu,s) \geq \mathfrak{S}(\varpi,\mu,\tau+s)$ for all $\varpi,\omega,\mu \in \Xi$ and $s,\tau > 0$,

(NMS18) for all $\varpi,\omega \in \Xi, \mathfrak{S}(\varpi,\omega,\ .) : [0,\infty) \to [0,1]$ is a right-continuous,

(NMS19) $\lim_{\tau\to\infty} \mathfrak{S}(\varpi,\omega,\tau) = 0$ for all $\varpi,\omega \in \Xi$.

Then $(\mathfrak{B},\mathfrak{D},\mathfrak{S})$ is called a neutrosophic metric on Ξ.

Definition 1.5: [23] A sequence $\{\varpi_n\}$ in NMS $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ is called a convergent to a point $\varpi \in \Xi$ if and only if $\lim_{n\to\infty}\mathfrak{B}(\varpi_n,\varpi,\tau) = 1$, $\lim_{n\to\infty}\mathfrak{D}(\varpi_n,\varpi,\tau) = 0$ and $\lim_{n\to\infty}\mathfrak{S}(\varpi_n,\varpi,\tau) = 0$ for each $\tau > 0$.

Definition 1.6: [23] A sequence in NMS $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ is called a Cauchy if for each $\varepsilon > 0$ and for each $\tau > 0$,there exist $n_\circ \in \mathbb{N}$ such that $\mathfrak{B}(\varpi_n,\varpi_m,\tau) > 1-\varepsilon, \mathfrak{D}(\varpi_n,\varpi_m,\tau) < \varepsilon$ and $\mathfrak{S}(\varpi_n,\varpi_m,\tau) < \varepsilon$ for all $n,m \geq n_o$, $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ is considered to be complete if and only if each Cauchy sequence is convergent.

2. MAIN RESULTS

In this section, we show the existence and uniqueness of a fixed point by utilizing some interesting conditions with contractions.

Lemma 2.1: If $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S}\,*,\Diamond)$ is an NMS and $\lim_{n\to\infty}\varpi_n = \varpi$, $\lim_{n\to\infty}\omega_n = \omega$ then

$$\lim_{n\to\infty}\mathfrak{B}(\varpi_n,\omega_n,\tau) = \mathfrak{B}(\varpi,\omega,\tau)\,,$$
$$\lim_{n\to\infty}\mathfrak{D}(\varpi_n,\omega_n,\tau) = \mathfrak{D}(\varpi,\omega,\tau),$$
$$\lim_{n\to\infty}\mathfrak{S}(\varpi_n,\omega_n,\tau) = \mathfrak{S}(\varpi,\omega,\tau).$$

Definition 2.1: Let $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ be an NMS and $Q \subseteq \Xi$. The smallest closed set that contains Q is known as closure of the set Q and is indicated by $\bar{Q}$.

Remark 2.1: An element $\varpi \in \bar{Q}$ if and only if there exist a sequence $\{\varpi_n\}$ in Q such that $\varpi_n \to \varpi$.

Definition 2.2: Let $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ be an NMS. Let for all $r \in (0,1)$ then a collection $\{F_n\}_{n\in\mathbb{N}}$ have neutrosophic diameter zero (ND-zero) if for each $\tau > 0$ there exists $n_o \in \mathbb{N}$ such that $\mathfrak{B}(\varpi,\omega,\tau) > 1-r$, $\mathfrak{D}(\varpi,\omega,\tau) < r$ and $\mathfrak{S}(\varpi,\omega,\tau) < r$ for all $\varpi,\omega \in F_{n_o}$.

Theorem 2.1: An NMS $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ is said to be complete if and only if each nested sequence $\{F_n\}_{n\in\mathbb{N}}$ of non-empty closed sets with (ND-zero) have non-empty intersection.

Remark 2.2: An element ϖ is unique if $\varpi \in \cap_{n\in\mathbb{N}} F_n$.

Definition 2.3: Let $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\Diamond)$ be an NMS. A subset Q of Ξ is called a neutrosophic bounded if there exists $\tau > 0$ and $r \in (0,1)$ such that $\mathfrak{B}(\varpi,\omega,\tau) > 1-r$, $\mathfrak{D}(\varpi,\omega,\tau) < r$ and $\mathfrak{S}(\varpi,\omega,\tau) < r$ for all $\varpi,\omega \in Q$.

Definition 2.4: Let $(\Xi, \mathfrak{B}, \mathfrak{O}, \mathfrak{S}, *, \Diamond)$ be an NMS. Let the mapping $\delta_Q(\tau) : (0, \infty) \to [0,1]$, $\rho_Q(\tau): (0, \infty) \to [0,1]$ and $\gamma_Q(\tau): (0, \infty) \to [0,1]$ be define as

$$\delta_Q(\tau) = \inf_{\varpi, \omega \in Q} \sup_{\varepsilon < \tau} \mathfrak{B}(\varpi, \omega, \varepsilon),$$

$$\rho_Q(\tau) = \sup_{\varpi, \omega \in Q} \inf_{\varepsilon < \tau} \mathfrak{O}(\varpi, \omega, \varepsilon),$$

$$\gamma_Q(\tau) = \sup_{\varpi, \omega \in Q} \inf_{\varepsilon < \tau} \mathfrak{S}(\varpi, \omega, \varepsilon).$$

The constants $\delta_Q = \sup_{\tau > 0} \delta_Q(\tau), \rho_Q = \inf_{\tau > 0} \rho_Q(\tau)$ and $\gamma_Q = \inf_{\tau > 0} \gamma_Q(\tau)$, we will call neutrosophic diameter of nearness, non-nearness and naturalness of set Q.

Remark 2.3: The inequalities $\delta_Q \geq 1 - r, \rho_Q \leq r$ and $\gamma_Q \leq r$ are fulfilled, if Q is a neutrosophic bounded set.

Definition 2.5: The set Q is called neutrosophic strongly bounded set (NSBS), if $\delta_Q = 1$, $\rho_Q = 0$ and $\gamma_Q = 0$.

Lemma 2.2: A mapping $\phi: (0, \infty) \to (0, \infty)$ is continuous, and non-decreasing, if it verifies $\phi(\tau) < \tau$ for all $\tau > 0$. Then $\lim_{n \to \infty} \phi^n(\tau) = 0$, for all $\tau > 0$, where $\phi^n(\tau)$ shows the nth iteration of ϕ.

Lemma 2.3: Suppose $(\Xi, \mathfrak{B}, \mathfrak{O}, \mathfrak{S}, *, \Diamond)$ be an NMS. A mapping $\phi: (0, \infty) \to (0, \infty)$ be a continuous, non-decreasing, if it verifies the condition $\phi(\tau) < \tau$ for all $\tau > 0$. Then the below circumstances are satisfied:

(a) If for all $\varpi, \omega \in \Xi$ it satisfies that $\mathfrak{B}(\varpi, \omega, \phi(\tau)) \geq \mathfrak{B}(\varpi, \omega, \tau)$ then $\varpi = \omega$.

(b) If for all $\varpi, \omega \in \Xi$ it holds that $\mathfrak{O}(\varpi, \omega, \phi(\tau)) \leq \mathfrak{O}(\varpi, \omega, \tau)$ then $\varpi = \omega$.

(c) If for all $\varpi, \omega \in \Xi$ it holds that $\mathfrak{S}(\varpi, \omega, \phi(\tau)) \leq \mathfrak{S}(\varpi, \omega, \tau)$ then $\varpi = \omega$.

Definition 2.6: Let Q and U are the self-mappings of Ξ in NMS $(\Xi, \mathfrak{B}, \mathfrak{O}, \mathfrak{S}, *, \Diamond)$. These maps are said to be compatible if for all $\tau > 0$,

$$\lim_{n \to \infty} \mathfrak{B}(QU\varpi_n, UQ\varpi_n, \tau) = 1,$$

$$\lim_{n \to \infty} \mathfrak{O}(QU\varpi_n, UQ\varpi_n, \tau) = 0,$$

$$\lim_{n \to \infty} \mathfrak{S}(QU\varpi_n, UQ\varpi_n, \tau) = 0,$$

whenever $\{\varpi_n\}$ is a sequence in Ξ such that, $\lim_{n \to \infty} Q\varpi_n = \lim_{n \to \infty} U\varpi_n = \mu$ for some $\mu \in \Xi$.

Definition 2.7: Let Q and U are the self-mappings of Ξ in NMS $(\Xi, \mathfrak{B}, \mathfrak{O}, \mathfrak{S}, *, \Diamond)$. If there exist a positive real number R then the mappings are called (DS_Q)-w-commuting at $\varpi \in \Xi$ such that

$$\mathfrak{B}(QU\varpi, UU\varpi, \tau) \geq \mathfrak{B}\left(Q\varpi, U\varpi, \frac{\tau}{R}\right),$$

$$\mathfrak{O}(QU\varpi, UU\varpi, \tau) \leq \mathfrak{O}\left(Q\varpi, U\varpi, \frac{\tau}{R}\right),$$

$$\mathfrak{S}(QU\varpi, UU\varpi, \tau) \leq \mathfrak{S}\left(Q\varpi, U\varpi, \frac{\tau}{R}\right).$$

Then, Q and U are (DS_Q)-w-commuting on Ξ for all $\varpi \in \Xi$.

Definition 2.8: Suppose $(\Xi, \mathfrak{B}, \mathfrak{D}, \mathfrak{S}, *, \diamondsuit)$ be an NMS and Q and U self-mappings of Ξ. If there exists a positive real number R, then the mappings Q and U are called (DS_U)-w-commuting at $\varpi \in \Xi$ such that

$$\mathfrak{B}(UQ\varpi, QQ\varpi, \tau) \geq \mathfrak{B}\left(Q\varpi, U\varpi, \frac{\tau}{R}\right),$$

$$\mathfrak{D}(UQ\varpi, QQ\varpi, \tau) \leq \mathfrak{D}\left(Q\varpi, U\varpi, \frac{\tau}{R}\right),$$

$$\mathfrak{S}(UQ\varpi, QQ\varpi, \tau) \leq \mathfrak{S}\left(Q\varpi, U\varpi, \frac{\tau}{R}\right).$$

If these inequalities are fulfilled then Q and U are (DS_U)-w-commuting on Ξ for all $\varpi \in \Xi$. If the self–mappings Q and U holds Definitions 2.7 and 2.8 then Q and U are called (DS)-w-commuting mappings.

Example 2.1: Suppose $\Xi = [1,5]$ with the usual metric space $\Delta(\varpi, \omega) = |\varpi - \omega|$. For each $\tau \in (0, \infty)$. Define

$$\mathfrak{B}(\varpi, \omega, \tau) = \frac{\tau}{\tau + \Delta(\varpi, \omega)}, \qquad \varpi, \omega \in \Xi,$$

$$\mathfrak{D}(\varpi, \omega, \tau) = \frac{\Delta(\varpi, \omega)}{\tau + \Delta(\varpi, \omega)}, \quad \varpi, \omega \in \Xi,$$

$$\mathfrak{S}(\varpi, \omega, \tau) = \frac{\Delta(\varpi, \omega)}{\tau}, \quad \varpi, \omega \in \Xi.$$

Clearly, $(\Xi, \mathfrak{B}, \mathfrak{D}, \mathfrak{S}, *, \diamondsuit)$ be an NMS, where $\sigma * \varsigma = \sigma\varsigma$ and $\sigma \diamondsuit \varsigma = \min\{1, \sigma + \varsigma\}$. Define $Q, U: \Xi \to \Xi$ by

$$Q(\varpi) = \varpi \text{ if } \varpi \in [0,1),\ Q(\varpi) = \tfrac{1}{4} \text{ if } \varpi \in [1,5),$$

$$U(\varpi) = \frac{1}{1+\varpi} \text{ for all } \varpi \in [0,5].$$

Suppose the sequence $\left\{\varpi_n = 3 + \frac{1}{n} : n \geq 1\right\}$ in Ξ. Then

$$\lim_{n\to\infty} Q\varpi_n = \frac{1}{4},\ \ \lim_{n\to\infty} U\varpi_n = \frac{1}{4}$$

but

$$\lim_{n\to\infty} \mathfrak{B}(QU\varpi_n, UQ\varpi_n, \tau) = \frac{\tau}{\tau + \left|\frac{1}{4} - \frac{4}{5}\right|} \neq 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(QU\varpi_n, UQ\varpi_n, \tau) = \frac{\left|\frac{1}{4} + \frac{4}{5}\right|}{\tau + \left|\frac{1}{4} - \frac{4}{5}\right|} \neq 0,$$

$$\lim_{n\to\infty} \mathfrak{S}(QU\varpi_n, UQ\varpi_n, \tau) = \frac{\left|\frac{1}{4}+\frac{4}{5}\right|}{\tau} \neq 0.$$

Which is shows that Q and U are non-compatible. But we can obtain for $R \geq \frac{2}{3}$ and Q and U are (DS_Q)-w-commuting at $\varpi = 1$.

Example 2.2: Suppose $\Xi = [1, 10]$ with the usual metric space $\Delta(\varpi, \omega) = |\varpi - \omega|$ for each $\tau \in (0, \infty)$. Define

$$\mathfrak{B}(\varpi, \omega, \tau) = \frac{\tau}{\tau + \Delta(\varpi, \omega)}, \quad \varpi, \omega \in \Xi$$

$$\mathfrak{D}(\varpi, \omega, \tau) = \frac{\Delta(\varpi, \omega)}{\tau + \Delta(\varpi, \omega)}, \quad \varpi, \omega \in \Xi,$$

$$\mathfrak{S}(\varpi, \omega, \tau) = \frac{\Delta(\varpi, \omega)}{\tau}, \quad \varpi, \omega \in \Xi,$$

Clearly, $(\Xi, \mathfrak{B}, \mathfrak{D}, \mathfrak{S}, *, \diamondsuit)$ be an NMS where $\sigma * \varsigma = \sigma\varsigma$ and $\sigma \diamondsuit \varsigma = \min\{1, \sigma + \varsigma\}$. Define $Q, U: \Xi \to \Xi$ by

$$Q\varpi = \varpi \text{ if } 1 \leq \varpi \leq 5, Q\varpi = \varpi - 3 \text{ if } 5 < \varpi \leq 10,$$

$$U\varpi = 2 \text{ if } 1 \leq \varpi \leq 5, U\varpi = \frac{\varpi - 1}{2} \text{ if } 5 < \varpi \leq 1,$$

Consider the sequence $\{\varpi_n = 3 + \frac{1}{n} : n \geq 1\}$ in Ξ. Then

$$\lim_{n\to\infty} QU\varpi = 2, \ \lim_{n\to\infty} U\varpi_n = 2.$$

$$\lim_{n\to\infty} \mathfrak{B}(QU\varpi_n, UQ\varpi_n, \tau) = \lim \frac{\tau}{\tau + \left|2 + \frac{1}{2n} - 2\right|} = 1,$$

$$\lim_{n\to\infty} \mathfrak{D}(QU\varpi_n, UQ\varpi_n, \tau) = \lim \frac{\left|2 + \frac{1}{2n} - 2\right|}{\tau + \left|2 + \frac{1}{2n} - 2\right|} = 0,$$

$$\lim_{n\to\infty} \mathfrak{S}(QU\varpi_n, UQ\varpi_n, \tau) = \lim \frac{\left|2 + \frac{1}{2n} - 2\right|}{\tau} = 0.$$

Which shows that Q and U are compatible and (DS_Q)-w-commuting for $R \geq 1$, at $\varpi = 7$.

Theorem 2.2: Let $(\Xi, \mathfrak{B}, \mathfrak{D}, \mathfrak{S}, *, \diamondsuit)$ be a complete NMS. Let $\xi, \Upsilon: \Xi \to \Xi$. Suppose $\Upsilon(\Xi)$ is a NSBS and $\Upsilon(\Xi) \subseteq \xi(\varpi)$ satisfying the following conditions:

$$\left.\begin{aligned} \mathfrak{B}(\Upsilon(\varpi), \Upsilon(\omega), \phi(\tau), \geq \mathfrak{B}(\xi(\varpi), \xi(\omega), \tau) \\ \mathfrak{D}(\Upsilon(\varpi), \Upsilon(\omega), \phi(\tau) \leq \mathfrak{D}(\xi(\varpi), \xi(\omega), \tau) \\ \mathfrak{S}(\Upsilon(\varpi), \Upsilon(\omega), \phi(\tau) \leq \mathfrak{S}(\xi(\varpi), \xi(\omega), \tau) \end{aligned}\right\}. \qquad (2.1)$$

A mapping $\phi: (0, \infty) \to (0, \infty)$, is non-decreasing that verifies $\phi(\tau) < \tau$ for all $\tau > 0$. Then ξ and Υ have a coincidence point. If ξ and Υ are DS-w-commuting at coincidence point then both mapping ξ and Υ have a unique common fixed point.

Proof: Suppose $\varpi \in \Xi$ be any element of Ξ. Because $\Upsilon(\Xi) \subseteq \xi(\Xi)$, so there exist an element $\varpi \in \Xi$ such that $\Upsilon(\varpi_o) = \xi(\varpi_1)$. By induction $\{\varpi_n\}$ be a sequence such that $\Upsilon(\varpi_n) = \xi(\varpi_{n+1})$. Assume a nested sequence of non-empty closed sets defined by $F = \{\overline{\Upsilon\varpi_n, \Upsilon\varpi_{n+_1}, \dots}\}$, $n \in \mathbb{N}$. We shall prove that the ND- zero of the family $\{F_n\}_{n\in\mathbb{N}}$. Moreover, suppose $r \in (0,1)$ and $\tau > 0$ be randomly picked. By $F_\varrho \in \overline{\mathrm{g}(\varpi)}$ it follows that F_ϱ is a NSBS for any $\varrho \in \mathbb{N}$. This implies that there exists $\tau_o > 0$ such that

$$\mathfrak{B}(\varpi, \omega, \tau_o) > 1 - r, \qquad \mathfrak{D}(\varpi, \omega, \tau_o) < r \text{ and } \mathfrak{S}(\varpi, \omega, \tau_o) < r \text{ for all } \varpi, \omega \in F_\varrho. \tag{1}$$

From $\lim_{n\to\infty} \phi^n(\tau_o) = 0$ we deduce that there exists $m \in \mathbb{N}$ such that $\phi^m(\tau_o) < \tau$. Let $n = m + \varrho$ and $\varpi, \omega \in F_n$ be arbitrary. There exists sequence

$$\{\Upsilon\varpi_{n(i)}\}, \{\Upsilon\varpi_{n(j)}\} \text{ in } F_n(n(i), n(j) \geq n, i, j \in \mathbb{N}) \text{ such that}$$

$$\lim_{i\to\infty} \Upsilon\varpi_{n(i)} = \varpi \text{ and } \lim_{j\to\infty} \Upsilon\varpi_{n(j)} = \omega.$$

From (1), we have

$$\mathfrak{B}\left(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi(\tau)\right) \geq \mathfrak{B}\big(\xi\varpi_{n(i)}, \xi\varpi_{n(j)}, \tau\big) = \mathfrak{B}(\Upsilon\varpi_{n(i)-1}, \tau)$$

$$\mathfrak{D}\left(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi(\tau)\right) \leq \mathfrak{D}\big(\xi\varpi_{n(i)}, \xi\varpi_{n(j),}\tau\big) = \mathfrak{D}\big(\Upsilon\varpi_{n(i)-1}, \Upsilon\varpi_{n(j)-1}, \tau\big),$$

and

$$\mathfrak{S}\left(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi(\tau)\right) \leq \mathfrak{S}\big(\xi\varpi_{n(i)}, \xi\varpi_{n(j),}\tau\big) = \mathfrak{S}\big(\Upsilon\varpi_{n(i)-1}, \Upsilon\varpi_{n(j)-1}, \tau\big).$$

Thus, by induction, we get

$$\mathfrak{B}\left(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi^m(\tau)\right) \geq \mathfrak{B}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau\big),$$

$$\mathfrak{D}(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi^m(\tau) \leq \mathfrak{D}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau\big),$$

and

$$\mathfrak{S}(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi^m(\tau) \leq \mathfrak{S}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau\big).$$

Since $\phi^m(\tau_o) < \tau$ and because $\mathfrak{B}(\varpi, \omega, .)$ is non-decreasing and $\mathfrak{D}(\varpi, \omega, .)$ is non-increasing function,

$$\left.\begin{aligned} \mathfrak{B}\big(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \tau\big) \geq \mathfrak{B}\big(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi^m(\tau_o)\ \big) \geq \mathfrak{B}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau_o\big),\\ \mathfrak{D}(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \tau) \leq \mathfrak{D}\left(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi^m(\tau_o)\right) \leq \mathfrak{D}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau_o\big),\\ \mathfrak{S}(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \tau) \leq \mathfrak{S}\left(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \phi^m(\tau_o)\right) \leq \mathfrak{S}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau_o\big) \end{aligned}\right\}. \tag{2}$$

As $\{\Upsilon\varpi_{n(i)-m}\}$ and $\{\Upsilon\varpi_{n(j)-m}\}$ are sequence in F_ϱ from (1), it follows that

$$\left.\begin{aligned} \mathfrak{B}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau_o\big) > 1 - r\ \ \forall\, i, j \in \mathbb{N},\\ \mathfrak{D}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau_o\big) < r\ \ \forall\, i, j \in \mathbb{N},\\ \mathfrak{S}\big(\Upsilon\varpi_{n(i)-m}, \Upsilon\varpi_{n(j)-m}, \tau_o\big) < r\ \forall\, i, j \in \mathbb{N} \end{aligned}\right\}. \tag{3}$$

By implying (1) to (3), we examine

$$\mathfrak{B}\big(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \tau\big) > 1 - r,$$
$$\mathfrak{O}\big(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \tau\big) < r,$$
$$\mathfrak{S}\big(\Upsilon\varpi_{n(i)}, \Upsilon\varpi_{n(j)}, \tau\big) < r$$

$\forall\, i, j \in \mathbb{N}$. By applying the limits as $i, j \to \infty$ and the Lemma 2.1. We conclude

$$\mathfrak{B}(\varpi, \omega, \tau) > 1 - r, \ \ \mathfrak{O}(\varpi, \omega, \tau) < r \text{ and } \mathfrak{S}(\varpi, \omega, \tau) < r \text{ for all } \varpi, \omega \in F_n.$$

Then, $\{F_n\}_{n\in\mathbb{N}}$ has ND- zero. By using the Theorem 2.1 we examine that $\{F_n\}_{n\in\mathbb{N}}$ has non-empty intersection, which consists of exactly one point μ. Since $\{F_n\}_{n\in\mathbb{N}}$ has ND- zero and $\mu \in F_n$ for all $n \in \mathbb{N}$ then for each $r \in (0,1)$ and each $\tau > 0$ there exists $n_o \in \mathbb{N}$ such that for all $n \geq n_o$

$$\mathfrak{B}(\Upsilon\varpi_n, \mu, \tau) > 1 - r, \ \mathfrak{O}(\Upsilon\varpi_n, \mu, \tau) < r \text{ and } \mathfrak{S}(\Upsilon\varpi_n, \mu, \tau) < r.$$

For all $r \in (0,1)$, it's obviously satisfied

$$\lim_{n\to\infty} \mathfrak{B}(\Upsilon\varpi_n, \mu, \tau) > 1 - r \text{ and } \lim_{n\to\infty} \mathfrak{O}(\Upsilon\varpi_n, \mu, \tau) < r.$$

By using the limit that $r \to 0$, we get

$$\lim_{n\to\infty} \mathfrak{B}(\Upsilon\varpi_n, \mu, \tau) = 1, \lim_{n\to\infty} \mathfrak{O}(\Upsilon\varpi_n, \mu, \tau) = 0 \text{ and } \lim_{n\to\infty} \mathfrak{S}(\Upsilon\varpi_n, \mu, \tau) = 0.$$

That is, $\lim_{n\to\infty} \Upsilon\varpi_n = \mu$. Sequence $\{\xi\varpi_n\}$ follows the condition that $\lim_{n\to\infty} \xi\varpi_n = \mu$.

Since $\Upsilon(\Xi) \subseteq \xi(\Xi)$, there exists $u \in \Xi$ such that $\mu = \xi(u)$. Then utilizing (1), we get

$$\mathfrak{B}\big(\Upsilon(u), \Upsilon(\varpi_n), \phi(\tau)\big) \geq \mathfrak{B}(\xi(u), \xi(\varpi_n), \tau),$$
$$\mathfrak{O}\big(\Upsilon(u), \Upsilon(\varpi_n), \phi(\tau)\big) \leq \mathfrak{O}(\xi(u), \xi(\varpi_n), \tau),$$
$$\mathfrak{S}\big(\Upsilon(u), \Upsilon(\varpi_n), \phi(\tau)\big) \leq \mathfrak{S}(\xi(u), \xi(\varpi_n), \tau).$$

Letting $n \to \infty$, we get

$$\mathfrak{B}\big(\Upsilon(u), \mu, \phi(\tau)\big) \geq \mathfrak{B}(\mu, \mu, \tau) = 1,$$
$$\mathfrak{O}\big(\Upsilon(u), \mu, \phi(\tau)\big) \leq \mathfrak{O}(\mu, \mu, \tau) = 0,$$
$$\mathfrak{S}\big(\Upsilon(u), \mu, \phi(\tau)\big) \leq \mathfrak{S}(\mu, \mu, \tau) = 0.$$

Since

$$\mathfrak{B}(\Upsilon(u), \mu, \tau\,) \geq \mathfrak{B}\big(\Upsilon(u), \mu, \phi(\tau)\big),$$
$$\mathfrak{O}(\Upsilon(u), \mu, \tau) \leq \mathfrak{O}\big(\Upsilon(u), \mu, \phi(\tau)\big),$$
$$\mathfrak{S}(\Upsilon(u), \mu, \tau) \leq \mathfrak{S}\big(\Upsilon(u), \mu, \phi(\tau)\big).$$

We get, $\mathfrak{B}(\Upsilon(u), \mu, \tau) = 1$, $\mathfrak{O}(\Upsilon(u), \mu, \tau) = 0$ and $\mathfrak{S}(\Upsilon(u), \mu, \tau) = 0$ for all $\tau > 0$. Thus $\Upsilon(u) = \mu$. Therefore $\xi(u) = \Upsilon(u) = \mu$ which is shows that u is coincidence point of ξ and Υ. Since ξ and Υ are (DS)-w-commuting at coincidence point, so ξ and Υ are $\big(DS_\xi\big)$-w-commuting at coincidence points, such that

$$\mathfrak{B}(\xi\Upsilon u,\Upsilon\Upsilon u,\tau)\geq\xi\left((u),\Upsilon(u),\frac{\tau}{R}\right),$$
$$\mathfrak{D}(\xi\Upsilon u,\Upsilon\Upsilon u,\tau)\leq\mathfrak{D}\left(\xi(u),\Upsilon(u),\frac{\tau}{R}\right),$$
$$\mathfrak{S}(\xi\Upsilon u,\Upsilon\Upsilon u,\tau)\leq\mathfrak{S}\left(\xi(u),\Upsilon(u),\frac{\tau}{R}\right).$$

Thus, $\mathfrak{B}(\xi\Upsilon u,\Upsilon\Upsilon u,\tau)=1$, $\mathfrak{D}(\xi\Upsilon u,\Upsilon\Upsilon u,\tau)=0$ and $\mathfrak{S}(\xi\Upsilon u,\Upsilon\Upsilon u,\tau)=0.$ So, $\xi\Upsilon u=\Upsilon\Upsilon u$ that is $\xi\mu=\Upsilon\mu$.

Again using (1), we have

$$\mathfrak{B}\big(\Upsilon(\varpi_n),\Upsilon(\mu),\phi(\tau)\big)\geq\mathfrak{B}(\xi(\varpi_n),\xi(\mu),\tau),$$
$$\mathfrak{D}\big((\Upsilon\varpi_n),\Upsilon(\mu),\phi(\tau)\big)\leq\mathfrak{D}(\xi(\varpi_n),\xi(\mu),\tau),$$
$$\mathfrak{S}\big((\Upsilon\varpi_n),\Upsilon(\mu),\phi(\tau)\big)\leq\mathfrak{S}(\xi(\varpi_n),\xi(\mu),\tau).$$

Letting $n\to\infty$, we have

$$\mathfrak{B}\big(\mu,\Upsilon(\mu),\phi(\tau)\big)\geq\mathfrak{B}(\mu,\Upsilon(\tau),\tau),$$
$$\mathfrak{D}\big(\mu,\Upsilon(\tau),\phi(\tau)\big)\leq\mathfrak{D}(\mu,\Upsilon(\mu),\tau),$$
$$\mathfrak{S}\big(\mu,\Upsilon(\tau),\phi(\tau)\big)\leq\mathfrak{S}(\mu,\Upsilon(\mu),\tau)$$

for all $\tau>0.$ Utilizing the Lemma 2.3, we have $\Upsilon(\mu)=\mu.$ Thus $\xi(\mu)=\Upsilon(\mu)=\mu.$ Now, we show the uniqueness of μ. Suppose $\mu\neq w$ be another fixed point, then utilizing (2.1), we get

$$\mathfrak{B}\big(\Upsilon(\mu),\Upsilon(w),\phi(\tau)\big)\geq\mathfrak{B}(\xi(\mu),\xi(w),\tau),$$
$$\mathfrak{D}\big(\Upsilon(\mu),\Upsilon(w),\phi(\tau)\big)\leq\mathfrak{D}(\xi(\mu),\xi(w),\tau),$$
$$\mathfrak{S}\big(\Upsilon(\mu),\Upsilon(w),\phi(\tau)\big)\leq\mathfrak{S}(\xi(\mu),\xi(w),\tau)$$

$\forall\ \tau>0$. That is

$$\mathfrak{B}\big(\mu,u,\phi(\tau)\big)\geq\mathfrak{B}(\mu,u,\tau),$$
$$\mathfrak{D}\big(\mu,u,\phi(\tau)\big)\leq\mathfrak{D}(\mu,u,\tau),$$
$$\mathfrak{S}\big(\mu,u,\phi(\tau)\big)\leq\mathfrak{S}(\mu,u,\tau)$$

for all $\tau>0.$ By using the Lemma 2.3, it follows that $\mu=w.$

Example 2.3: Consider $\Xi=[0,2]$ with the usual metric space $\Delta(\varpi,\omega)=|\varpi-\omega|$ and for each $\tau\in[0,1]$, define

$$\mathfrak{B}(\varpi,\omega,\tau)=\frac{\tau}{\tau+|\varpi-\omega|},\varpi,\omega\in\Xi,$$
$$\mathfrak{D}(\varpi,\omega,\tau)=\frac{|\varpi-\omega|}{\tau+|\varpi-\omega|},\varpi,\omega\in\Xi,$$
$$\mathfrak{S}(\varpi,\omega,\tau)=\frac{|\varpi-\omega|}{\tau},\varpi,\omega\in\Xi.$$

Clearly, $(\Xi,\mathfrak{B},\mathfrak{D},\mathfrak{S},*,\diamondsuit)$ be a complete NMS where $\sigma*\varsigma=\sigma\varsigma$ and $\sigma\diamondsuit\varsigma=\min\{\sigma,\varsigma\}$.

Define $\xi, \Upsilon: \Xi \to \Xi$ by

$$\xi(\varpi) = \begin{cases} 1 & \text{if } \varpi = 1 \\ \dfrac{\varpi}{2} + 1 & \text{if } \varpi \neq 1, \end{cases}$$

$$\Upsilon(\varpi) = \begin{cases} 1 & \text{if } \varpi = 1 \\ \dfrac{1}{1+\varpi} & \text{if } \varpi \neq 1, \end{cases}$$

$$\phi(\tau) = \frac{\tau}{2}, \tau > 0.$$

Then $\Upsilon(\Xi) \subseteq \xi(\Xi)$. We see that $\{\varpi_n\}$ is decreasing sequence, so

$$\lim_{n\to\infty} \xi(\varpi_n) = \lim_{n\to\infty} \Upsilon(\varpi_n).$$

Thus, mappings are not compatible. But both are (DS)-w-commuting at coincidence point. We will demonstrate that axiom (1) is also met. Clearly, $\forall\, \varpi, \omega \in \Xi$. We get

$$\frac{1}{(1+\varpi)(1+\omega)} \leq 1.$$

We get

$$\mathfrak{B}\big(\Upsilon(\varpi), \Upsilon(\omega), \phi(\tau)\big) = \frac{\tau}{\tau + 2\left(\frac{|\varpi-\omega|}{(1+\varpi)(1+\omega)}\right)} \geq \mathfrak{B}(\xi(\varpi), \xi(\omega), \tau),$$

as easy to see in figure 1.

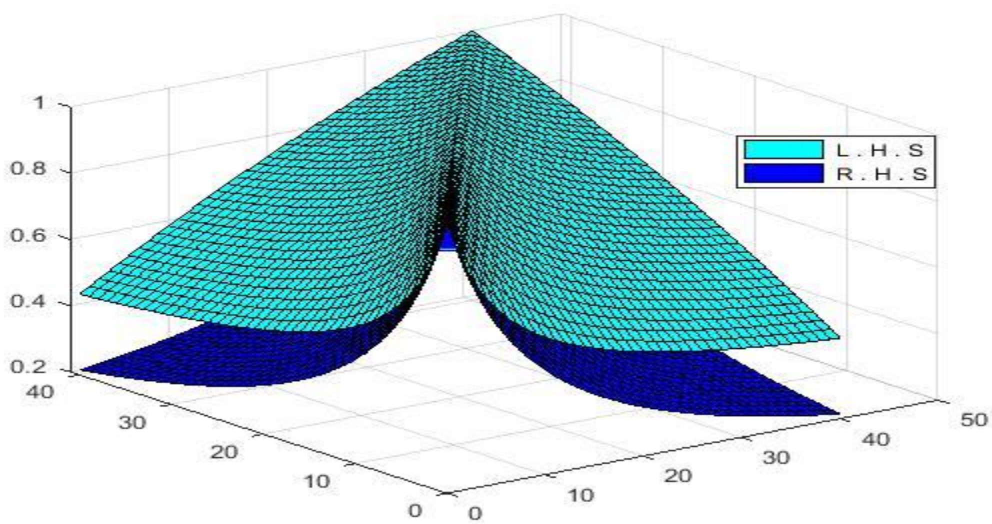

Figure 1 Shows the graphical behavior of the contraction $\mathfrak{B}\big(\Upsilon(\varpi), \Upsilon(\omega), \phi(\tau)\big) \geq \mathfrak{B}(\xi(\varpi), \xi(\omega), \tau)$ when $\tau = 1$.

Now for second $\mathfrak{D}$, we have

$$\mathfrak{D}\big(\Upsilon(\varpi), \Upsilon(\omega), \phi(\tau)\big) = \frac{2\left(\frac{|\varpi-\omega|}{(1+\varpi)(1+\omega)}\right)}{\tau + 2\left(\frac{|\varpi-\omega|}{(1+\varpi)(1+\omega)}\right)} \leq \mathfrak{D}(\xi(\varpi), \xi(\omega), \tau),$$

as easy to see in figure 2.

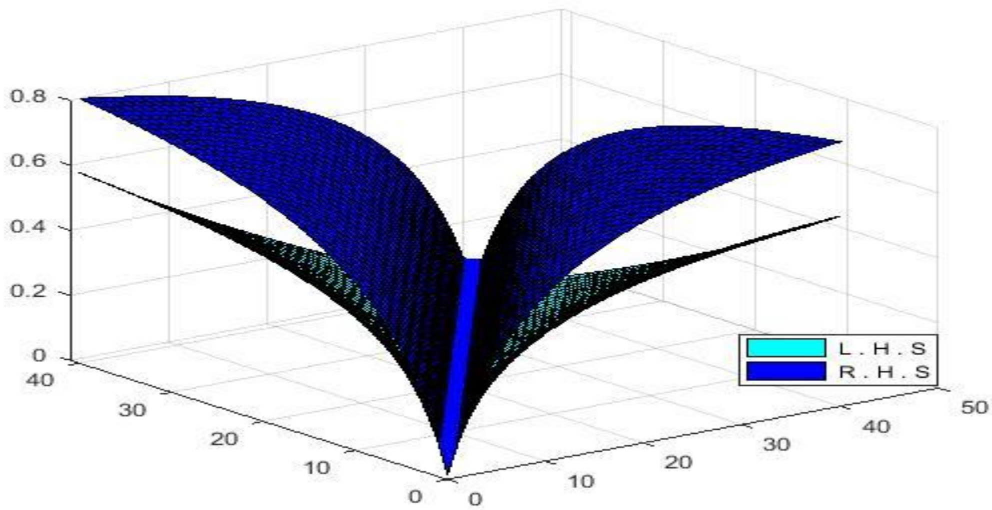

Figure 2 Shows the graphical behavior of the contraction $\mathfrak{O}\big(\Upsilon(\varpi),\Upsilon(\omega),\phi(\tau)\big)\leq\mathfrak{O}(\xi(\varpi),\xi(\omega),\tau)$ when $\tau=1$.

Now for third function $\mathfrak{S}$, we have

$$\mathfrak{S}\big(\Upsilon(\varpi),\Upsilon(\omega),\phi(\tau)\big)=\frac{2\left(\frac{|\varpi-\omega|}{(1+\varpi)(1+\omega)}\right)}{\tau}\leq\mathfrak{S}(\xi(\varpi),\xi(\omega),\tau),$$

as easy to see in figure 3.

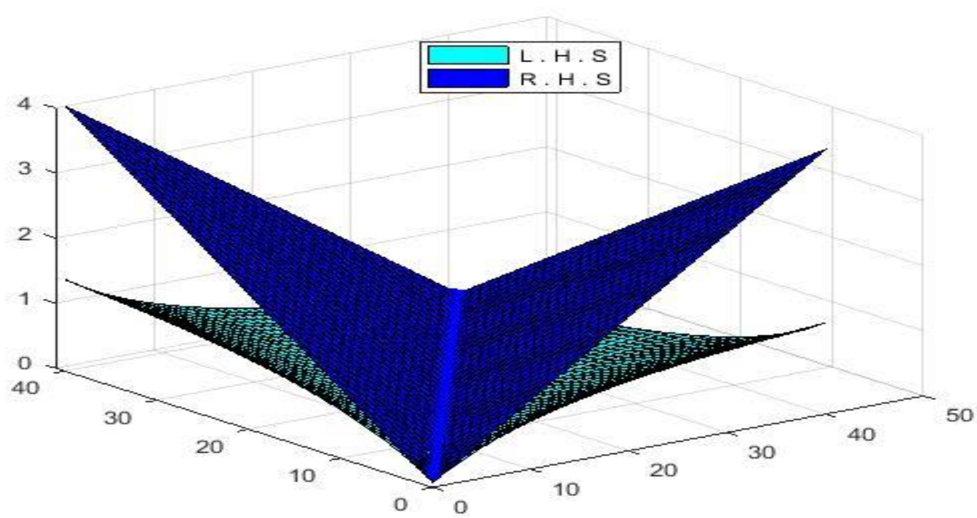

Figure 3 Shows the graphical behavior of the contraction $\mathfrak{S}\big(\Upsilon(\varpi),\Upsilon(\omega),\phi(\tau)\big)\leq\mathfrak{S}(\xi(\varpi),\xi(\omega),\tau)$ when $\tau=1$.

All the conditions of theorem 2.2 are fulfill. So, both mappings $\xi(\varpi)$ and $\Upsilon(\varpi)$ have a unique common fixed point, which is $\varpi = 1$.

Theorem 2.3: Let $(\Xi, \mathfrak{B}, \mathfrak{D}, \mathfrak{S}, *, \diamond)$ be a complete NMS. Let $\Upsilon: \Xi \to \Xi$. Suppose $\Upsilon(\Xi)$ is NSBS verifying the axiom (2) with

$$\mathfrak{B}(\Upsilon(\varpi), \Upsilon(\omega), \varrho\tau) \geq \mathfrak{B}(\varpi, \omega, \tau),$$
$$\mathfrak{D}(\Upsilon(\varpi), \Upsilon(\omega), \varrho\tau) \leq \mathfrak{D}(\varpi, \omega, \tau),$$
$$\mathfrak{S}(\Upsilon(\varpi), \Upsilon(\omega), \varrho\tau) \leq \mathfrak{S}(\varpi, \omega, \tau)$$

for some $\varrho \in (0,1)$. Then Υ has a unique fixed point.

Remark 2.4: The hypothesis of Theorem 2.2 that is $\Upsilon(\Xi)$ is a NSBS, can be exchanged with the following one: there exists an element $\varpi_o \in \Xi$ such that the orbit of an element ϖ_o described by $\phi(\varpi_o, \Upsilon) = \{\varpi_o, \Upsilon\varpi_o, \Upsilon^2\varpi_o, \dots\}$ is a NSBS. This tells us the sequence used in the start of the proof of the Theorem 2.2 for $\xi = 1$ is a sequence of Picard iterates defined by $\varpi_{n+1} = \Upsilon\varpi_n$.

3. APPLICATION

Now, we aim to apply Theorem 2.3 to obtain the existence of solution to a nonlinear fractional differential equation (NFDE)

$$D_{0+}^{p} h(\varpi) = \xi\big(\varpi, h(\varpi)\big), \qquad 0 < \varpi < 1 \qquad (4)$$

with the boundary conditions

$$h(0) + h'(0) = 0, h(1) + h'(1) = 0,$$

where $1 < p \leq 2$ is a number, D_{0+}^{p} is the Caputo fractional derivative and $\xi : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. Let $\Xi = C([0,1],(0,1])$ denote the space of all continuous functions defined on $[0,1]$. Let $\sigma * \varsigma = \min\{\sigma, \varsigma\}$ for all $\sigma, \varsigma \in [0,1]$ and define an NMS as follows:

$$\mathfrak{B}(h(\mathrm{t}), \delta(\varpi), \tau) = \begin{cases} 1 & \text{if} \quad h(\tau) = \delta(\varpi), \\ \dfrac{\alpha\tau}{\alpha\tau + \gamma \max\{\sup_{\varpi\epsilon[0,1]} h(\varpi), \sup_{\varpi\epsilon[0,1]} \delta(\varpi)\}}, & \text{otherwise}, \end{cases}$$

$$\mathfrak{D}(h(\mathrm{t}), \delta(\varpi), \tau) = \begin{cases} 0 & \text{if} \quad h(\tau) = \delta(\varpi), \\ \dfrac{\gamma \max\{\sup_{\varpi\epsilon[0,1]} h(\varpi), \sup_{\varpi\epsilon[0,1]} \delta(\varpi)\}}{\alpha\tau + \gamma \max\{\sup_{\varpi\epsilon[0,1]} h(\varpi), \sup_{\varpi\epsilon[0,1]} \delta(\varpi)\}}, & \text{otherwise}, \end{cases}$$

and

$$\mathfrak{S}(h(\mathrm{t}), \delta(\varpi), \tau) = \begin{cases} 0 & \text{if} \quad h(\tau) = \delta(\varpi), \\ \dfrac{\gamma \max\{\sup_{\varpi\epsilon[0,1]} h(\varpi), \sup_{\varpi\epsilon[0,1]} \delta(\varpi)\}}{\alpha\tau}, & \text{otherwise}. \end{cases}$$

Observe that $h \in \Xi$ solves (4) whenever $h \in \Xi$ solves the below integral equation:

$$h(\varpi) = \frac{1}{\Gamma(p)}\int_0^1 (1-s)^{p-1}(1-\varpi)\xi\big(s,h(s)\big)\Delta s + \frac{1}{\Gamma(p-1)}\int_0^1 (1-s)^{p-2}(1-\varpi)\xi\big(s,h(s)\big)\Delta s + \frac{1}{\Gamma(p)}\int_0^{\varpi} (\varpi-s)^{p-1}\xi\big(s,h(s)\big)\Delta s.$$

The graphical behavior of $h(\varpi)$ for different values of p is shown in figure 4.

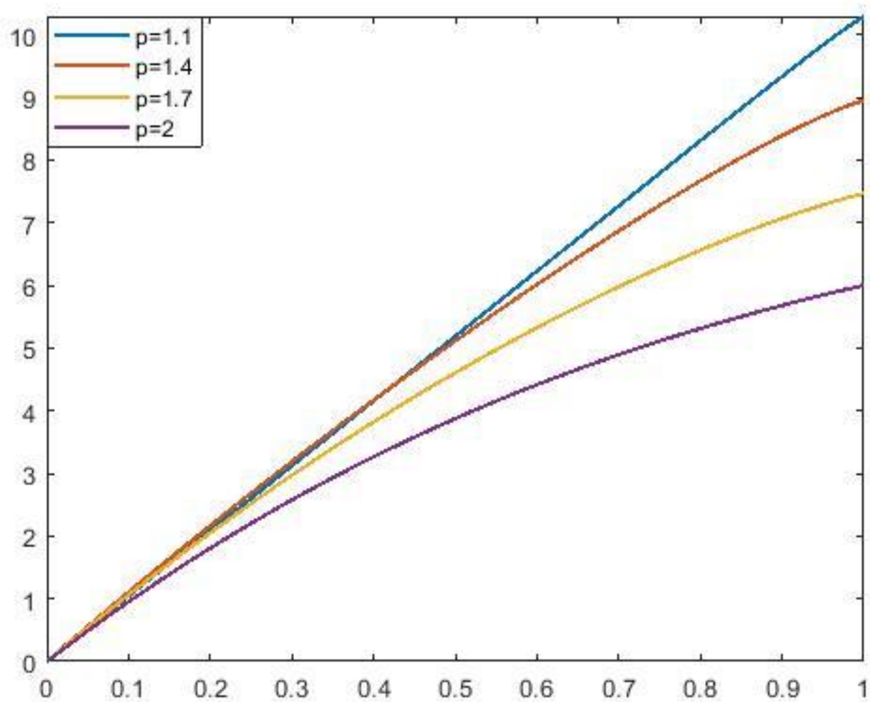

Figure 4 Shows the graphical behavior of $h(\varpi)$ for $p = 1.1,$ $p = 1.4,$ $p = 1.7$ and $p = 2.$

Theorem 4.2 The integral operator $\Upsilon: \Xi \to \Xi$ is given by

$$\Upsilon h(\varpi) = \frac{1}{\Gamma(p)}\int_0^1 (1-s)^{p-1}(1-\varpi)\xi\big(s,h(s)\big)\Delta s + \frac{1}{\Gamma(p-1)}\int_0^1 (1-s)^{p-2}(1-\varpi)\xi\big(s,h(s)\big)\Delta s + \frac{1}{\Gamma(p)}\int_0^{\varpi} (\varpi-s)^{p-1}\xi\big(s,h(s)\big)\Delta s,$$

where $\xi: [0,1] \times [0,\infty) \to [0,\infty)$ fulfilling the following criteria:

$$\max\{\sup_{s\in[0,1]} \xi\,\big(s,h(s)\big),\ \sup_{s\in[0,1]} \xi\big(s,\delta(s)\big)\} \le \frac{1}{4}\max\{\sup_{s\in[0,1]} h(s),\ \sup_{s\in[0,1]} \delta(s)\}, \text{for all } h,\delta \in \Xi.$$

Also, suppose that

$$\sup_{\varpi\in(0,1)} \frac{1}{4}\left[\frac{1-\varpi}{\Gamma(p+1)} + \frac{1-\varpi}{\Gamma(p)} + \frac{\varpi^p}{\Gamma(p+1)}\right] \le \varrho < 1.$$

Then NFDE has a unique solution in Ξ.

Proof: Let

$$\begin{aligned}
&\max\{\Upsilon h(\varpi), \Upsilon\delta(\varpi)\}\\
&\quad= \frac{1-\varpi}{\Gamma(p)}\int_0^1 (1-s)^{p-1} \max\{\sup_{s\in[0,1]}\xi(s,h(s)), \sup_{s\in[0,1]} \xi(s,\delta(s))\}\Delta s\\
&\quad+ \frac{1-\varpi}{\Gamma(p-1)}\int_0^1 (1-s)^{p-2} \max\{\sup_{s\in[0,1]}\xi(s,h(s)), \sup_{s\in[0,1]} \xi(s,\delta(s))\}\Delta s\\
&\quad+ \frac{1}{\Gamma(p)}\int_0^\varpi (\varpi-s)^{p-1} \max\{\sup_{s\in[0,1]}\xi(s,h(s)), \sup_{s\in[0,1]} \xi(s,\delta(s))\}\\
&\le \frac{1-\varpi}{\Gamma(p)}\int_0^1 (1-s)^{p-1} \frac{\max\{\sup_{s\in[0,1]} h(s), \sup_{s\in[0,1]} \delta(s)\}}{4}\Delta s\\
&\quad+ \frac{1-\varpi}{\Gamma(p-1)}\int_0^1 (1-s)^{p-2} \frac{\max\{\sup_{s\in[0,1]} h(s), \sup_{s\in[0,1]} \delta(s)\}}{4}\Delta s\\
&\quad+ \frac{1}{\Gamma(p)}\int_0^\varpi (\varpi-s)^{p-1} \frac{\max\{\sup_{s\in[0,1]} h(s), \sup_{s\in[0,1]} \delta(s)\}}{4}\Delta s\\
&\le \frac{1}{4}\max\{\sup_{s\in[0,1]} h(s), \sup_{s\in[0,1]} \delta(s)\}\left(\frac{1-\varpi}{\Gamma(p)}\int_0^1 (1-s)^{p-1}\Delta s + \frac{1-\varpi}{\Gamma(p-1)}\int_0^1 (1-s)^{p-2}\Delta s\right.\\
&\quad\left.+ \frac{1}{\Gamma(p)}\int_0^\varpi (\varpi-s)^{p-1}\Delta s\right)\\
&\le \frac{1}{4}\max\{\sup_{s\in[0,1]} h(s), \sup_{s\in[0,1]} \delta(s)\} \sup_{\varpi\in[0,1]}\left[\frac{1-\varpi}{\Gamma(p+1)} + \frac{1-\varpi}{\Gamma(p)} + \frac{\varpi^p}{\Gamma(p+1)}\right]\\
&\quad= \varrho \max\{\sup_{s\in[0,1]} h(s), \sup_{s\in[0,1]} \delta(s)\},
\end{aligned}$$

where,

$$\varrho = \sup_{\varpi\epsilon(0,1)}\frac{1}{4}\left[\frac{1-\varpi}{\Gamma(p+1)} + \frac{1-\varpi}{\Gamma(p)} + \frac{\varpi^p}{\Gamma(p+1)}\right].$$

Therefore, the above equation

$$\begin{aligned}
&\max\{\sup_{\varpi\in[0,1]} \Upsilon h(\varpi), \sup_{\varpi\in[0,1]} \Upsilon\delta(\varpi)\} \le \varrho \max\{\sup_{\varpi\in[0,1]} h(\varpi), \sup_{\varpi\in[0,1]} \delta(\varpi)\}\\
&\Rightarrow \alpha\tau + \frac{\gamma}{\varrho}\max\{\sup_{\varpi\in[0,1]} \Upsilon h(\varpi), \sup_{\varpi\in[0,1]} \Upsilon\delta(\varpi)\}\\
&\qquad\le \alpha\tau + \gamma\max\{\sup_{\varpi\in[0,1]} h(\varpi), \sup_{\varpi\in[0,1]} \delta(\varpi)\}\\
&\Rightarrow \frac{\alpha(\varrho\tau)}{\alpha(\varrho\tau) + \gamma\max\{\sup_{\varpi\in[0,1]} \Upsilon h(\varpi), \sup_{\varpi\in[0,1]} \Upsilon\delta(\varpi)\}}\\
&\qquad\ge \frac{\alpha\tau}{\alpha\tau + \gamma\max\{\sup_{\varpi\in[0,1]} h(\varpi), \sup_{\varpi\in[0,1]} \delta(\varpi)\}}\\
&\qquad\Rightarrow \mathfrak{B}(\Upsilon h, \Upsilon\delta, \varrho\tau) \ge \mathfrak{B}(h, \delta, \tau),
\end{aligned}$$

and on the same technique, it is easy to deduce that

$$\mathfrak{O}(\Upsilon h, \Upsilon\delta, \varrho\tau) \leq \mathfrak{O}(h, \delta, \tau),$$
$$\mathfrak{S}(\Upsilon h, \Upsilon\delta, \varrho\tau) \leq \mathfrak{S}(h, \delta, \tau),$$

for some $\alpha, \gamma > 0$. Observe that the conditions of the Theorem 2.3 are fulfilled. Resultantly, Υ has a fixed-point fixed point; accordingly, the specified NFDE has a solution.

4. CONCLUSION

In this manuscript, we established numerous interesting conditions in the context of NMS. We provided numerous non-trivial examples and their graphical views via computational techniques. Also, we derived several coincident points and common fixed-point results for contraction mappings in the context of NMS, as well, we presented a graphical view of defined contractions. At the end, we provide a novel application to support the validity of our main result.

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