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A Note on Skew Generalized Power Serieswise Reversible Property

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Abstract. The aim of this paper is to introduce and study (S,ω)-nil-reversible rings wherein we call a ring R is (S,ω)-nil-reversible if the left and right annihilators of every nilpotent element of R are equal. The researcher obtains various necessary or sufficient conditions for (S,ω)-nil-reversible rings are abelian, 2-primal, (S,ω)-nil-semicommutative and (S,ω)-nil-Armendariz. Also, he proved that, if R is completely (S,ω)-compatible (S,ω)-nil-reversible and J an ideal consisting of nilpotent elements of bounded index $\leq n$ in R, then R/J is $(S,\bar{\omega})$-nil-reversible. Moreover, other standard rings-theoretic properties are given.

1. **Introduction**

Throughout this paper, all rings are associated with identity unless otherwise stated. We write $P(R)$, $nil(R)$, $Mat_n(R)$, $T_n(R,)$ $S_n(R)$, $R[x]$, $\mathrm{End}(R)$ and $\mathrm{Aut}(R)$, respectively for the prime radical, the set of all nilpotent elements of R, full square matrices, upper square triangular matrices for a positive integer n with entries in R, the subring consisting of all upper square triangular matrices, the polynomial ring, the monoid of ring endomorphisms of R and the group of ring automorphisms of R.

The purpose of this article is to examine (S,ω)-nil-reversible rings, where $(S,\leq)$ is a strictly ordered monoid and $\omega : S \to End(R)$ is a monoid homomorphism. A ring R is considered (S,ω)-nil-reversible if the left and right annihilators of every nilpotent element in R are equal. The author provides various necessary or sufficient conditions for (S,ω)-nil-reversible rings to be abelian, 2-primal, (S,ω)-nil-semicommutative and (S,ω)-nil-Armendariz. I have an example to illustrate, a (S,ω)-nil-reversible ring may not necessarily be (S,ω)-semicommutative or (S,ω)-reversible. Additionally, it is shown that

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if R is completely (S, ω)-compatible and J is an ideal consisting of nilpotent elements with bounded index $\leq n$, then R/J is also $(S, \overline{\omega})$-nil-reversible. Furthermore, it is proven that a multiplicatively closed subset of a ring consisting of central non-zero divisors is (S, ω)-nil-reversible if and only if the entire ring itself is (S, ω)-nil-reversible. The article also covers other standard properties in ring theory.

A ring R is said to be reversible if $xy = 0$, then $yx = 0$, where $x, y \in R$ see Cohn [1]. The article [2] defines a semicommutative ring R as one where $xy = 0$ implies $xRy = 0$ for all $x, y \in R$. Rings with no nonzero nilpotent elements are called reduced rings and are symmetric, reversible, and semicommutative according to [3, P. 361] and [3, Proposition 1.3]. However, polynomial rings over reversible rings need not be reversible as shown in [4, Example 2.1]. In [5], strongly reversible rings are introduced as reversible rings over which polynomial rings are also reversible. A ring R is strongly reversible if $f(x)h(x) = 0$ implies $h(x)f(x) = 0$ for all polynomials $f(x), h(x) \in R[x]$. Reversible Armendariz rings satisfy this property, but reduced rings may not be strongly reversible in general. A ring is called a 2-primal ring if its nilradical coincides with its prime radical, and an NI-ring if its upper nilradical coincides with its set of nilpotent elements. A ring is an NI-ring if and only if its set of nilpotent elements forms an ideal, while 2-primal rings are NI-rings.

Armendariz ring defined by the reference [2]. If the products two polynomials $f(x)g(x) = 0$, then $a_i b_j = 0$, for all i, j. In our discussion, we use the following terminology: Given non-empty subsets A and D of a monoid S, an element $u_0 \in AD = \{st : s \in A, t \in D\}$ is considered a single product element (abbreviated as $s.p.$ element) in AD if it can be expressed singly in the form $u = st$. The following definition will be useful in the next section.

Definition 1.1. *The article [6] defines an ordered monoid $(S, \leq)$ as an artinian narrow unique product monoid (or a.n.u.p. monoid) if, for any two artinian and narrow subsets A and D of S, there exists a unique product element in the set AD that is upper principal. A minimal artinian narrow unique product monoid (or m.a.n.u.p. monoid) is defined as an ordered monoid $(S, \leq)$ where, for any two artinian and narrow subsets A and D of S, there exist minimum elements $a \in min(A)$ and $b \in min(D)$ such that their product ab is an upper principal element of the set AD. A monoid is said to be totally orderable if it can be ordered with a total order $\leq$, while a quasitotally ordered monoid is one where the order $\leq$ can be refined to a strictly total order $\preceq$.*

To start, we revisit the creation of the generalized power series ring, which was initially presented in [7]. Let $(S, \leq)$ be an ordered set. If every strictly decreasing sequence of elements in S is finite, then $(S, \leq)$ is said to be artinian.

Assume that S is a commutative monoid with the operation denoted additively and the neutral element denoted by 0. For a ring R, $(S, \leq)$ be a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ be a monoid homomorphism. Denote the image of s under ω as $\omega_s = \omega(s)$ for any $s \in S$.

Let F be the set of all functions $f : S \to R$ such that the support $\text{supp}(f) = \{s \in S : f(s) \neq 0\}$ is both artinian and narrow. For any $s \in S$ and $f, g \in F$, the set $X_s(f, g)$ consisting of all pairs $(u, v) \in \text{supp}(f) \times \text{supp}(g)$ such that $s = uv$ is finite. Therefore, we can define the product $fg : S \to R$ of f and g as follows: if $(u, v) \notin X_s(f, g)$ for all $(u, v) \in \text{supp}(f) \times \text{supp}(g)$, then $(fg)(s) = 0$, otherwise, $(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v))$ is conventionally considered to be 0. Using the previously defined pointwise addition and multiplication, the set F becomes a ring known as the ring of skew generalized power series with coefficients in R and exponents in S, denoted by $[[R^{S,\leq}, \omega]]$ (or simply $R[[S, \omega]]$ if the order $\leq$ is unambiguous), as described in [8]. A subset $P \subseteq R$ is considered to be S-invariant if it is ω_t-invariant for every $t \in S$, meaning that $\omega_t(P) \subseteq P$. For each element $d \in R$ and each element $t \in S$, we have the elements c_d and e_t in $[[R^{S,\leq}, \omega]]$ defined by

$$c_d(\lambda) = \begin{cases} d, & \lambda = 1, \\ \\ 0, & \lambda \in S\backslash\{1\}, \end{cases} \qquad e_t(\lambda) = \begin{cases} 1, & \lambda = t, \\ \\ 0, & \lambda \in S\backslash\{t\}. \end{cases}$$

The mapping $d \mapsto c_d$ is a ring embedding of R into the ring $[[R^{S,\leq}, \omega]]$, while the mapping $t \mapsto e_t$ is a monoid embedding of S into the multiplicative monoid of that same ring. Moreover, we have the relationship that $e_t c_d = c_{\omega_t(d)} e_t$.

2. (S, ω)-nil-reversible rings

In this section, we introduce the concept of (S, ω)-nil-reversible rings, which is a generalization of both (S, ω)-reversible rings and generalized power series reversible rings. We then utilize this concept to investigate the relationships between (S, ω)-nil-reversible rings and certain classes of rings.

Definition 2.1. *For a ring R, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to$End(R) a monoid homomorphism. R is to be (S, ω)-nil-reversible, if $fg \in [[nil(R)^{S,\leq}, \omega]]$, then $gf \in [[nil(R)^{S,\leq}, \omega]]$, for all $f, g \in [[R^{S,\leq}, \omega]]$.*

Remark 2.2. *By definition, it is clear that, skew generalized power series nil-reversible rings are closed under subrings.*

Definition 2.3. *In [6] A ring R is said to be S-compatible (or (S, ω)-compatible) if for every element d in the strictly ordered monoid S, the corresponding endomorphism ω_d of R is compatible. Similarly, a ring R is said to be S-rigid (or (S, ω)-rigid) if for every element d in S, the corresponding endomorphism ω_d of R is rigid. Here, $\omega : S \to End(R)$ is a monoid homomorphism that maps elements of the monoid S to endomorphisms of the ring R.*

Lemma 2.4. *[6] For a ring R, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. A ring R is reduced $\Leftrightarrow$ $[[R^{S,\leq}, \omega]]$ is reduced.*

Lemma 2.5. *[9] For $\omega : S \to End(R)$ a monoid homomorphism. Any elements $r, t \in R$ and $d \in S$. The following results are correct:*
(1) $rt \in nil(R) \Leftrightarrow r\omega_d(t) \in nil(R)$.
(2) $rt \in nil(R) \Leftrightarrow \omega_d(r)t \in nil(R)$.

We can provide an example of nil-reversible rings of skew generalized power series that do not fall under the categories of either skew generalized power series reversible or skew generalized power series semicommutative. It is important to note that skew generalized power series reversible rings are both skew generalized power series semicommutative and skew generalized power series nil-reversible by definition. This leads us to speculate that skew generalized power series nil-reversible rings may also be skew generalized power series semicommutative. However, the following examples disprove this possibility. To support this claim, we require the following propositions.

Proposition 2.6. *For a ring R, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Suppose R is an (S, ω)-compatible with $nil(R)$ an ideal, then R is (S, ω)-nil-reversible.*

Proof. Let $f, g \in [[R^{S,\leq}, \omega]]$, satisfying fg is nilpotent. There exists a positive integer ℓ such that $(fg)^\ell = 0$, so $(f(r)\omega_r(g(t)))^\ell = 0$ for each $r, t \in S$. Then by compatibility $f(r)g(t) \in nil(R)$. Hence $g(t)f(r)$ is nilpotent. Thus, gf is nilpotent. □

Proposition 2.7. *For a ring R, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Suppose R is an (S, ω)-compatible. A ring R is (S, ω)-nil-reversible ring if and only if for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is (S, ω)-nil-reversible.*

Proof. Assume that $f, g \in [[T_n(R)^{S,\leq}, \omega]]$, satisfying $fg \in [[nil(T_n(R))^{S,\leq}, \omega]]$. So by [10],

$$nil(T_n(R)) = \begin{pmatrix} nil(R) & R & R & \cdots & R \\ 0 & nil(R) & R & \cdots & R \\ 0 & 0 & nil(R) & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & nil(R) \end{pmatrix}.$$

If R is a ring with no nonzero nilpotent elements, then the nilradical of R is trivial, i.e., $nil(R) = 0$. Therefore, the nilradical of the n-th triangular matrix ring over R, denoted by $T_n(R)$, is also trivial. Hence, $nil(T_n(R))$ forms an ideal in $T_n(R)$. By Proposition 2.6, $T_n(R)$ is (S, ω)-nil-reversible. The if part follows Remark 2.2. □

Example 2.8. *For a ring R, (S, ω)-compatible, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Let R be (S, ω)-nil-reversible ring. Then*

$$T = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\}.$$

is (S,ω)-nil-reversible ring by Proposition 2.7. Note that $fg = 0$, where $f = c_{E_{23}} + c_{E_{13}}e_s$ and $g = c_{E_{12}} + c_{E_{22}}e_s$, but we have $gf \neq 0$. So T is not (S,ω)-reversible. In fact, T is not (S,ω)-semiccommutative by [11, Example 2.5] (with $n = 3$).

Also, let S be an (S,ω)-nil-reversible ring. Then the ring

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S; n \geq 3 \right\}.$$

is not (S,ω)-reversible by [11, Example 2.5]. But R_n is (S,ω)-nil-reversible by Proposition 2.7 since any subring of (S,ω)-nil-reversible ring is (S,ω)-nil-reversible. It is obvious that R_4 is not (S,ω)-semicommutative and it can be proved similarly that R_n is not (S,ω)-semicommutative for $n \geq 5$.

Proposition 2.9. *For a ring R, $(S,\leq)$ a strictly ordered monoid and $\omega : S \rightarrow End(R)$ a monoid homomorphism. Assume that R is (S,ω)-nil-reversible and (S,ω)-compatible. Suppose $g_1, g_2, \ldots, g_n \in [[R^{S,\leq},\omega]]$ satisfying $g_1g_2\cdots g_n \in [[nil(R)^{S,\leq},\omega]]$, then $g_1(v_1)g_2(v_2)$
$\cdots g_n(v_n) \in nil(R)$ for all $v_1, v_2, \ldots, v_n \in S$.*

Proof. It is clear by the definition. □

Corollary 2.10. *For a ring R, $(S,\leq)$ a strictly ordered monoid and $w : S \rightarrow End(R)$ a monoid homomorphism and R to be S-compatible. The following conditions are equal:*
*(1) If $g_1, g_2, \ldots, g_n \in [[R^{S,\leq},\omega]]$ satisfy $g_1g_2\cdots g_n \in [[nil(R)^{S,\leq},\omega]]$, then $g_1(v_1)g_2(v_2)$
$\cdots g_n(v_n) \in nil(R)$, for any $v_1, v_2, \ldots, v_n \in S$.*
(2) R is NI ring.

Proposition 2.11. *For a ring R, $(S,\leq)$ a strictly ordered monoid and $w : S \rightarrow End(R)$ a monoid homomorphism. Assume R is an (S,ω)-compatible. If R is (S,ω)-nil-reversible, then $nil(R[[S,\omega]]) \subseteq nil(R)[[S,\omega]]$.*

Proof. Let $g \in nil(R[[S,\omega]])$, suppose $g^{\ell} = 0$ where $\ell \in \mathbb{Z}^+$. Then by Proposition 2.9, $g(v) \in nil(R)$ for each $v \in S$. Thus $nil(R[[S,\omega]]) \subseteq nil(R)[[S,\omega]]$. □

Proposition 2.12. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $w : S \rightarrow End(R)$ a monoid homomorphism. Suppose that R to be S-compatible. If R is (S,ω)-nil-reversible, then*
(1) R is abelian.
() R is 2-primal..

Proof. Let R be a (S,ω)-nil-reversible ring.
(1) Let e be an idempotent element of R. For any $g(v) \in R, v \in S, eg(v) - eg(v)e \in nil(R)$. Note

that $(eg(v) - eg(v)e)e = 0$. By hypothesis, $e(eg(v) - eg(v)e) = 0$, so $eg(v) = eg(v)e$. Again, $g(v)e - eg(v)e \in nil(R)$ and $e(g(v)e - eg(v)e) = 0$. So by (S, ω)-nil-reversibility of R, we have $(g(v)e - eg(v)e)e = 0$, that is, $g(v)e = eg(v)e$. Hence, $eg(v) = g(v)e$.
(2) Note that $P(R) \subseteq nil(R)$. Suppose $g(v) \in nil(R)$. Then there is a positive integer $m \geq 2$ such that $(g(v))^m = 0$. Thus, $R(g(v))^{m-1}g(v) = 0$. This implies that $g(v)R(g(v))^{m-1} = 0$ as R is (S, ω)-nil-reversible. This yields $(Rg(v))^m = 0$, so $g(v) \in P(R)$. □

According to [9], a ring R is called (S, ω)-*nil*-Armendariz, if whenever $f, g \in R[[S, \omega]]$ satisfying $fg \in nil(R[[S, \omega]])$, then $f(r)\omega_r(g(t)) \in nil(R)$ for all $r, t \in S$.

Proposition 2.13. *For a ring R, S-compatible, $(S, \leq)$ totally ordered monoid and $w : S \to End(R)$ a homomorphism of monoid. Then, every (S, ω)-nil-reversible rings are (S, ω)-nil-Armendariz.*

Proof. Suppose $0 \neq f, g \in R[[S, \omega]]$ satisfying $fg \in nil(R[[S, \omega]])$. Transfinite induction will be applied to the set that is strictly and totally ordered $(S, \leq)$ showing $f(r)g(t) \in nil(R)$ for any $r \in supp(f)$ and $t \in supp(g)$. In the $\leq'$ order, let s and d be the smallest elements in supp(f) and supp(g), respectively. If $r \in supp(f)$ and $t \in supp(g)$ satisfying $r + t = s + d$, then $s \leq' r$ and $d \leq' t$. If $s <' r$ then $s + d <' r + t = s + d$, a contradiction. Thus $r = s$. Similarly, $t = d$. Hence $0 = (fg)(s + d) = \sum_{(r,t)\in X_{s+d}(f,g)} f(r)\omega_r(g(t)) = f(s)\omega_s(g(d))$.

Now let $w \in S$ with $r + t <' w, f(r)g(t) = 0$. We need to show $f(r)\omega_r(g(t)) \in nil(R)$ for al $r \in supp(f)$ and $t \in supp(g)$ with $r + t = w$. Writing $X_w(f, g) = \{(r, t) \mid r + t = w$ as $\{(r_i, t_i) \mid i = 1, 2, \ldots, n\}$ such that $r_1 <' r_2 <' \cdots <' r_n$. Since S is cancellative, $r_1 = r_2$ and $r_1 + t_1 = r_2 + t_2 = w$ imply $t_1 = t_2$. Since $\leq'$ is a strict order, $r_1 <' r_2$ and $r_1 + t_1 = r_2 + t_2 = w$ imply $t_2 <' t_1$. Thus we have $t_n <' \cdots <' t_2 <' t_1$. Now,

$$0 = (fg)(w) = \sum_{(r,t)\in X_w(f,g)} f(r)\omega_r(g(t)) = \sum_{i=1}^{n} f(r_i)\omega_{r_i}(g(t_i)). \tag{2.1}$$

For each $i \geq 2$, $r_1 + t_i <' r_i + t_i = w$. Therefore, using the induction hypothesis, we can conclude that $f(r_1)g(t_i)$ belongs to the nilradical of R. Since R is a 2-primal ring (as shown in Proposition 2.12), this implies that $f(r_1)g(t_i)$ also belongs to the nilradical of R. Thus, by multiplying equation (2.1) on the right by $f(r_1)g(t_1)$, we get:

$$\Big(\sum_{i=1}^{n} f(r_i)g(t_i)\Big)f(r_1)g(t_1) = f(r_1)g(t_1)f(r_1)g(t_1) = 0.$$

Then $(f(r_1)g(t_1))^2 = 0$ and so $f(r_1)g(t_1) \in nil(R)$. Now (2.1) becomes

$$\sum_{i=2}^{n} f(r_i)g(t_i) = 0. \tag{2.2}$$

By performing a right-hand side multiplication of (2.2) with $f(r_2)g(t_2)$, we get $f(r_2)g(t_2) = 0$. By following the same method as described above, we can continue this process and establish proof

$f(r_i)g(t_i) = 0$ for all $i = 1, 2, \ldots, n$. Thus $f(r)g(t) \in nil(R)$ with $r + t = w$. Hence, utilizing transfinite induction, it follows that $f(r)\omega_r(g(t))$ belongs to the set of nilpotent elements in R for any $r \in supp(f)$ and $t \in supp(g)$. □

Lemma 2.14. *Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. Suppose that R is compatible with S. We now examine the conditions for R.*
(1) R is (S, ω)-nil-reversible.
(2) If AB is a nilpotent set, then so is BA for each subsets A, B in R.
(3) If KZ is nilpotent, then ZK is nilpotent for right ideals (or left) K, Z in R.
Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. The proof is analog with the proof of [11, Lemma 3.5] □

Lemma 2.15. *Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. Suppose that R is compatible with S. Then every (S, ω)-nil-reversible rings are (S, ω)-nil-semicommutative.*

Proof. Suppose $f, g \in R[[S, \omega]]$ satisfying $fg \in nil(R[[S, \omega]])$. Then $gf \in nil(R[[S, \omega]])$ and $g(t)\omega_t(h(w)\omega_w(f(r))) \in nil(R)$ for any $r, t, w \in S$ and $h(w) \in R$, so $f(t)h(w)g(r) \in nil(R)$ by compatibility. Thus, $fhg \in nil(R[[S, \omega]])$ by [4, Lemma 1.1]. Therefore, R is an (S, ω)-nil-semicommutative. □

Proposition 2.16. *Consider R is an NI ring and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. Suppose that R is compatible with S. If (S, ω)-nil-reversible with $nil(R)$ is an ideal of R, then*

$$nil(R)[[S, \omega]] = nil(R[[S, \omega]]).$$

Proof. Suppose $d \in nil(R)$, by Lemma 2.14, RdR is a nilpotent in R. Since R is compatible with S, for any $\lambda \in S$, $R\omega_\lambda(d)R$ is a nilpotent ideal of R and so $\omega_\lambda(d) \in nil(R)$. Thus $nil(R)$ is an invariant with S and so $nil(R)[[S, \omega]]$ is an ideal of $R[[S, \omega]]$. By Proposition 2.11, $nil(R[[S, \omega]]) \subseteq nil(R)[[S, \omega]]$. Therefore, it is enough to demonstrate that $nil(R)[[S, \omega]] \subseteq nil([[R^{S,\leq}, \omega]])$.

Suppose $f \in nil(R)[[S, \omega]]$ then for any $r \in S$, $f(r) \in nil(R)$. By Proposition 2.11, there is a positive integer ℓ such that $r \in S$, $(Rf(r)R)^\ell = 0$. Since R is compatible with S, then for any $g, h \in R[[S, \omega]], (gfh)^\ell = 0$. I have know, if $g \in nil(R)[[S, \omega]]$, then $g(t) \in nil(R)$. So $g \in nil(R[[S, \omega]])$. Thus, $nil(R)[[S, \omega]] \subseteq nil(R[[S, \omega]])$. Therefore, $nil(R)[[S, \omega]] = nil(R[[S, \omega]])$. □

Corollary 2.17. *Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. Suppose that R is compatible with S and (S, ω)-reversible. Then g is a nil element of $[[R^{S,\leq}, \omega]] \Leftrightarrow f(r) \in nil(R)$ for all $r \in S$.*

Proposition 2.18. *Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. Suppose that R is compatible with S. If a subdirect product of (S, ω)-nil-reversible rings is finite, then it is also an (S, ω)-nil-reversible ring.*

Proof. Suppose we have ideals J_k of R and R/J_k is $(S, \bar{\omega})$-nil-reversible for $k = 1, \ldots, l$ such that $\bigcap_{k=1}^{l} J_k = 0$. Assume that $f, g \in R[[S, \omega]]$ satisfying $fg \in nil(R[[S, \omega]])$. Then, we have $\overline{f}\overline{g} \in nil(R/J_k[[S, \overline{\omega}]])$. Since R/J_k is $(S, \bar{\omega})$-nil-reversible, we have $(f(r)g(t))^{d_{r,t,k}} \in J_k$ for all $r, t \in S$ and $k = 1, \ldots, l$, where $d_{r,t,k}$ is the maximum value of $d_{r,t}$ over all ideals. Thus, $(f(r)g(t))^{d_{r,t}} \in \bigcap_{k=1}^{l} J_k = 0$, which implies that $f(r)g(t) \in nil(R)$ for all $r, t \in S$. Therefore, we have $g(t)f(r) \in nil(R)$ as well, and so $gf \in nil(R[[S, \omega]])$ as desired. □

Proposition 2.19. *Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. Suppose that R is compatible with S and $e^2 = e \in R$. If R is (S, ω)-nil-reversible, then so is eRe.*

Proof. Suppose $c_e f c_e, c_e g c_e \in (eRe)[[S, \omega]]$ satisfying $(c_e f c_e)(c_e g c_e) \in nil(eRe)[[S, \omega]]$. Let e be an idempotent of R. c_e is clearly an idempotent element of $(eRe)[[S, \omega]]$, $c_e g = g c_e$ for each $g \in R[[S, \omega]]$. Then $(c_e f)(c_e g) \in nil(eR)[[S, \omega]]$. Since R is (S, ω)-nil-reversible, the elements $fg \in nil(R)[[S, \omega]]$, and so $gf \in nil(R)[[S, \omega]]$. Then there exists $\ell \in \mathbb{N}$ such that $((c_e f c_e)(c_e g c_e))^{\ell} = 0$. Therefore $(c_e g c_e)(c_e f c_e) \in nil(eRe)[[S, \omega]]$. □

Corollary 2.20. *Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. If and only if R is (S, ω)-nil-reversible, then both eR and $(1-e)R$ are also (S, ω)-nil-reversible for a central idempotent e of the ring R.*

Proof. Assume that eR and $(1-e)R$ are (S, ω)-nil-reversible. Since the nil skew generalized power series reversibility property finite direct products preserve the closure property of the set, $R \cong eR \times (1-e)R$ is (S, ω)-nil-reversible. The converse is true by Proposition 2.19. □

In [12], A homomorphic image of a nil-reversible ring may not be nil-reversible, so as (S, ω)-nil-reversible by the next example.

Example 2.21. *Let $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Assume that $R = D[[S, \leq]]$, where D is a division ring and $I =< xy >$, where $xy \neq yx$. As R is a domain, R is (S, ω)-nil-reversible. Clearly $\overline{yx} \in nil(R/I)[[S, \overline{\omega}]]$ and $\overline{x}(\overline{yx}) = \overline{xyx} = 0$. But, $(\overline{yx})\overline{x} = \overline{yx^2} \neq 0$. This implies R/I is not $(S, \overline{\omega})$-nil-reversible.*

Definition 2.22. *[9] Consider a ring R and a strictly ordered monoid $(S, \leq)$ with a monoid homomorphism $w : S \to End(R)$. To express the concept of a ring being completely compatible with a set S, we define it as follows: A ring R is said to be completely S-compatible if every ideal J of R yields an S-compatible quotient ring R/J. In order to refer to the homomorphism ω, we may alternatively refer to R as completely compatible with S..*

It is evident that any ring that is completely (S, ω)-compatible also qualifies as (S, ω)-compatible. Another way to express the complete (S, ω)-compatibility of a ring R is by stating that for any subset J of R and elements r and t in R, the condition $rt \in J$ is equivalent to $r\omega(t) \in J$. This description will be frequently referenced in our discussions.

Theorem 2.23. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is completely S-compatible (S, ω)-nil-reversible and J an ideal consisting of nilpotent elements of bounded index $\leq n$ in R, then R/J is $(S, \overline{\omega})$-nil-reversible.*

Proof. Suppose $\bar{f}, \bar{g} \in (R/J)[[S, \overline{\omega}]]$ satisfying $\bar{f}\bar{g} \in nil(R/J)[[S, \overline{\omega}]]$. Assuming that the order $(S, \leq)$ can be improved to a strict total order $\leq$ on S, we will utilize transfinite induction on the strictly totally ordered set $(S, \leq)$ to demonstrate that $\bar{g}\bar{f} \in nil(R/J)[[S, \overline{\omega}]]$. To begin with, demonstrate through transfinite induction that $g(t)f(s) \in nil(R)$ for every $s \in supp(f)$ and $t \in supp(g)$. Given that $supp(f)$ and $supp(g)$ are non-empty subsets of S, there exist finite and non-empty sets of minimal elements in $supp(f)$ and $supp(g)$, respectively. Let s_0 and t_0 be the minimum elements in the $\leq$ order of these sets. By the same of the proof of [9, Theorem 2.25], we need to show $f(s_0)\omega_{s_0}(g(t_0)) = 0$. Therefore, by transfinite induction, we can proof that $f(s)g(t) = 0$. Since $\bar{f}\bar{g} \in nil(R/J)[[S, \overline{\omega}]]$. Then, there is a positive integer $\ell \in \mathbb{N}$ such that $(\bar{f}\bar{g})^{\ell} = \bar{0}$. So $(f(s)g(t))^{\ell} \in J$, for any $s, t \in S$. Since $J \subseteq nil(R), (f(s)g(t))^{\ell} = 0$. Hence $f(s)g(t) \in nil(R)$ by compatibility, so $g(t)f(s) \in nil(R)$, by R is (S, ω)-nil-reversible, $gf \in nil(R)[[S, \omega]]$. Thus $\bar{g}\bar{f} \in nil(R/J)[[S, \overline{\omega}]]$. Therefore R/J is $(S, \overline{\omega})$-nil-reversible. □

In the next, we utilize the prime radical of a ring to provide descriptions of skew generalized power series that exhibit nil-reversibility.

Corollary 2.24. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is completely (S, ω)-compatible (S, ω)-nil-reversible, then $R/P(R)$ is $(S, \overline{\omega})$-nil-reversible.*

Proof. The proof can be derived from Theorem 2.23 due to the fact that all elements in $P(R)$ are nilpotent. □

Proposition 2.25. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. Let J be a reduced ideal of a ring R such that R/J is $(S, \overline{\omega})$-nil-reversible. Then R is (S, ω)-nil-reversible.*

Proof. Suppose $f, g \in [[R^{S,\leq}, \omega]]$ satisfying $fg \in nil(R)[[S, \omega]]$. Then $\bar{f}\bar{g} \in nil(R/J)[[S, \overline{\omega}]]$ and so $\bar{g}\bar{f} \in nil(R/J)[[S, \overline{\omega}]]$ since R/J is $(S, \overline{\omega})$-nil-reversible. There is a positive integer $\ell \in \mathbb{N}$ and $(\bar{f}\bar{g})^{\ell} = \bar{0}$. Therefore $(f(s)g(t))^{\ell} \in J$ for any $s, t \in S$. Since J is reduced, we have $f(s)g(t) = 0$ yields $g(t)f(s) = 0$. Thus, $gf \in nil(R)[[S, \omega]]$. Therefore, R is (S, ω)-nil-reversible. □

3. Weak annihilator of reversible property of skew generalized power series rings

The concept of weak annihilators and its properties were introduced by Ouyang in [13], with a focus on subsets X of a ring R put $Nr_R(X) = \{a \in R | Xa \in nil(R)\}$ and $Nl_R(X) = \{b \in R | bX \in nil(R)\}$. It can be easily calculated that $Nr_R(X) = Nl_R(X)$. The set $Nr_R(X)$ is called the weak annihilator of X. If R is a NI-ring, it is evident that $Nr_R(X)$ forms an ideal of R. Additionally, if R is reduced, then we have $r_R(X) = Nr_R(X) = l_R(X) = Nl_R(X)$, and more information and findings on weak annihilators can be found in [14].

Our investigation now focuses on the correlation between weak annihilators in a ring R and those in a skew generalized power series ring $[[R^{S,\leq}, \omega]]$. Let R be a ring and $\gamma = C(f)$ be the content of f, defined as $C(f) = \{f(u) | u \in supp(f)\} \subseteq R$. As $R \simeq c_R$, we can equate the content of f with

$$c_{C(f)} = \{c_{f(u_i)} | u_i \in supp(f)\} \subseteq [[R^{S,\leq}, \omega]].$$

Then we have two maps $\phi : NrAnn_R(id(R)) \to NrAnn_{[[R^{S,\leq},\omega]]}(id([[R^{S,\leq}, \omega]]))$ and $\psi : NlAnn_R(id(R)) \to NlAnn_{[[R^{S,\leq},\omega]]}(id([[R^{S,\leq}, \omega]]))$ defined by $\phi(I) = I[[R^{S,\leq}, \omega]]$ and $\psi(J) = [[R^{S,\leq}, \omega]]J$ for each $I \in NrAnn_R(id(R)) = \{Nr_R(U) | U$ is an ideal of $R\}$ and $J \in NlAnn_R(id(R)) = \{Nl_R(U) | U$ is an ideal of $R\}$, respectively. It is evident that ϕ is a one-to-one function. The subsequent theorem demonstrates that ϕ and ψ are both bijective mappings if and only if R is (S, ω)-nil-reversible.

Theorem 3.1. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \to End(R)$ a monoid homomorphism. If R is reduced and $nil(R)$ is a nilpotent ideal of R, then the following are equivalent:*
(1) R is (S, ω)-nil-reversible ring.
(2) The function $\phi : NrAnn_R(id(R)) \to NrAnn_{[[R^{S,\leq},\omega]]}(id([[R^{S,\leq}, \omega]]))$ is bijective, where $\phi(I) = I[[R^{S,\leq}, \omega]]$ for each $I \in NrAnn_R(id(R))$.
(3) The function $\psi : NlAnn_R(id(R)) \to NlAnn_{[[R^{S,\leq},\omega]]}(id([[R^{S,\leq}, \omega]]))$ is bijective, where $\psi(J) = [[R^{S,\leq}, \omega]]J$ for every $J \in NlAnn_R(id(R))$.

Proof. (1)⇒(2) Suppose $Y \subseteq [[R^{S,\leq}, \omega]]$ and $\gamma = \cup_{f \in Y} C(f)$. By Proposition 2.9 it is enough to prove $Nr_{[[R^{S,\leq},\omega]]}(f) = Nr_R C(f) R[[S, \omega]]$ for every $f \in Y$. We know that, if $g \in Nr_{R[[S,\omega]]}(f)$. Then $fg \in nil(R)[[S, \omega]]$. According to the premise $f(d_i)\omega_{d_i}(g(t_j)) \in nil(R)$ for each $d_i \in supp(f)$ for all $t_j \in supp(g)$. For element $d_i \in supp(f)$ for every $t_j \in supp(g)$, $0 = f(d_i)\omega_{d_i}(g(t_j)) = (c_{f(d_i)} g)(t_j)$ and it follows that $g \in Nr_R \cup_{d_i \in supp(f)} c_{f(d_i)} R[[S, \omega]] = Nr_R C(f) R[[S, \omega]]$. So $Nr_{R[[S,\omega]]}(f) \subseteq Nr_R C(f) R[[S, \omega]]$.

Conversely, suppose $g \in Nr_R C(f) R[[S, \omega]]$, so $c_{f(d_i)} g \in nil(R)[[S, \omega]]$ for all $d_i \in supp(f)$. Thus, $(c_{f(d_i)} g)(t_j) = f(d_i)\omega_{d_i}(g(t_j)) \in nil(R)$ for all $d_i \in supp(f)$ and $t_j \in supp(g)$. Therefore,

$$(fg)(s) = \sum_{(d_i,t_j) \in X_s(f,g)} f(d_i)\omega_{d_i}(g(t_j)) = 0$$

it is evident that $g \in Nr_{R[[S,\omega]]}(f)$. Hence $Nr_R C(f)R[[S,\omega]] \subseteq Nr_{R[[S,\omega]]}(f)$ therefore $Nr_R C(f)R[[S,\omega]] = Nr_{R[[S,\omega]]}(f)$. So

$$Nr_{R[[S,\omega]]}(Y) = \cap_{f\in Y} Nr_{R[[S,\omega]]}(f) = \cap_{f\in Y} C(f)R[[S,\omega]] = Nr_R(\gamma)R[[S,\omega]].$$

(2)⇒(1) Assume the elements $f, g \in R[[S,\omega]]$ satisfying $fg \in nil(R)[[S,\omega]]$. Then $g \in Nr_{R[[S,\omega]]}(f)$ according to the premise $Nr_{R[[S,\omega]]}(f) = \gamma R[[S,\omega]]$ for any right ideal γ of R. Inversely, $0 = fc_{g(t_j)}$ and for every $d_i \in supp(f)$, $(fc_{g(t_j)})(d_i) = f(d_i)g(t_j) \in nil(R)$ for every $d_i \in supp(f)$ and $t_j \in supp(g)$. Thus by reduced ring, $g(t_j)f(d_i) \in nil(R)$, then $gf \in nil(R)[[S,\omega]]$. Thus, R is (S,ω)-nil-reversible. The demonstration of the equivalence between (1)⇔(3) follows a similar approach to that used for proving the equivalence between (1)⇔(2). □

According to [6], a ring R is defined (S,ω)-Armendariz if for tow polynomial $f, g \in R[[S,\omega]]$ satisfying $fg = 0$, then $f(r)\omega_r(g(t)) = 0$ for all $r, t \in S$. Now we given a strong condition under which $R[[S,\omega]]$ is nil-reversible.

Theorem 3.2. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \rightarrow End(R)$ a monoid homomorphism. If R is (S,ω)-compatible. Assume that R is (S,ω)-Armendariz ring, then R is (S,ω)-nil-reversible if and only if $R[[S,\omega]]$ is nil-reversible.*

Proof. Assume that R is (S,ω)-nil-reversible. Let $f, g \in R[[S,\omega]]$ be such that $fg \in nil(R)[[S,\omega]]$. By Proposition 2.16, $nil(R)[[S,\omega]] = nil(R[[S,\omega]])$. So $f(r_i)g(t_j) \in nil(R)$ for every $r, t \in S, \forall\, i, j$. By condition that R is (S,ω)-Armendariz, $f(r_i)\omega_{r_i}(g(t_j)) = 0$, for all i, j. By compatibility nil-reversibility, $g(t_j)f(r_i) \in nil(R)$ for all i, j. So, $gf \in nil(R)[[S,\omega]]$. Thus, $R[[S,\omega]]$ is nil-reversible. The converse is clear. □

Theorem 3.3. *Consider a ring R and a strictly ordered monoid $(S,\leq)$ with a monoid homomorphism $w : S \rightarrow End(R)$. Suppose that R is compatible with S. Let Δ denotes a multiplicatively closed subset of R consisting of central non-zero divisors. Then R is (S,ω)-nil-reversible if and only if $\Delta^{-1}R$ is (S,ω)-nil-reversible.*

Proof. Suppose R is (S,ω)-nil-reversible and $p_i, d_j, u, v \in R$. Let $u^{-1}C_{p_i}, v^{-1}C_{d_j} \in \Delta^{-1}R[[S,\omega]]$ for all i, j satisfying that $u^{-1}C_{p_i}v^{-1}C_{d_j} \in nil(\Delta^{-1}R[[S,\omega]])$. Then $(u^{-1}C_{p_i}v^{-1}C_{d_j})^{\ell} = 0$ for some positive integer ℓ. This implies $(C_{p_i}C_{d_j})^{\ell} = 0$, so $p_i d_j \in nil(R)$ by using Lemma 2.5 freely. For any $u^{-1}C_{p_i}, v^{-1}C_{d_j} \in \Delta^{-1}R[[S,\omega]]$ having the property that $(u^{-1}C_{p_i})(v^{-1}C_{d_j}) = 0$, we have $(uv)^{-1}C_{p_i}C_{d_j} = 0, C_{p_i}C_{d_j} = 0$ for every i, j. By condition that, R is (S,ω)-nil-reversible, $d_j p_i \in nil(R)$, so $(v^{-1}u^{-1})C_{d_j}C_{p_i} = 0$ which further yields $(v^{-1}C_{d_j})(u^{-1}C_{p_i}) \in nil(\Delta^{-1}R[[S,\leq]])$. Hence $\Delta^{-1}R$ is (S,ω)-nil-reversible. The converse part is clear. □

A McCoy ring is a generalization of a reversible ring, defined as a ring where the equation $f(x)g(x) = 0$ implies the existence of a non-zero element d such that $f(x)d = 0$. Left McCoy rings are defined similarly. McCoy rings are both left and right McCoy rings. It is known that every reversible ring is McCoy. However, it cannot be assumed that if a ring R is (S,ω)-nil-reversible, then it is also (S,ω)-McCoy. An example exists that disproves this assumption.

Example 3.4. *Assume that R is a reduced ring, a strictly ordered monoid $(S,\leq)$ with a monoid homomorphism $w : S \to End(R)$. Let*

$$T_n(R)) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in R \right\}.$$

Then $T_n(R)$ is not (S,ω)-McCoy by the similar as argument of [16, Example 2.6], but $T_n(R)$ to be (S,ω)-nil-reversible by Proposition 2.7.

As per Lambek [17], a ring R is considered symmetric if for any $x, y, z \in R$, the condition $xyz = 0$ implies $xzy = 0$. It can be easily observed that commutative rings are symmetric and symmetric rings are reversible rings.

Theorem 3.5. *Let R be a ring, (S,ω)-compatible and reversible right Noetherian ring, $(S,\leq)$ a strictly ordered monoid with $nil(R)$ is a nilpotent ideal of R and $\omega : S \to End(R)$ a monoid homomorphism. The ring R to be (S,ω)-nil-symmetric if and only if so is $[[R^{S,\leq},\omega]]$.*

Proof. Assume a ring R is (S,ω)-nil-symmetric such that $f, g, h \in [[R^{S,\leq},\omega]]$ satisfying $fgh \in nil([[R^{S,\leq},\omega]])$. Hence by Proposition 2.9, $f(r)g(d)h(t) \in nil(R)$ for any $r, d, t \in S$. By assumption R is nil-symmetric, then $f(r)h(t)g(d) \in nil(R)$. For all $s \in S$. Thus

$$(fhg)(s) = \sum_{(r,t,d)\in X_s(f,h,g)} f(r)\omega_r(h(t)\omega_t(g(d))).$$

So, the reversibility of R, $fhg \in nil([[R^{S,\leq},\omega]])$, it follows that $[[R^{S,\leq},\omega]]$ is nil-symmetric. On the other hand, if $[[R^{S,\leq},\omega]]$ is nil-symmetric, then R is (S,ω)-nil-symmetric because subrings of (S,ω)-nil-symmetric rings is to be (S,ω)-nil-symmetric. □

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