Int. J. Anal. Appl. (2023), 21:85

International Journal of Analysis and Applications

# On Prime $E$-ideals of Almost Distributive Lattices

**N. Rafi**[1], **Y. Monikarchana**[2], **Ravikumar Bandaru**[3], **Aiyared Iampan**[4,*]

[1] *Department of Mathematics, Bapatla Engineering College, Bapatla, Andhra Pradesh 522 101, India*

[2] *Department of Mathematics, Mohan Babu University, A. Rangampet, Tirupati, Andhra Pradesh 517 102, India*

[3] *Department of Mathematics, GITAM(Deemed to be University), Hyderabad Campus, Telangana 502 329, India*

[4] *Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[*] *Corresponding author:* aiyared.ia@up.ac.th

Abstract. In an almost distributive lattice (ADL), the idea of $E$-ideals is introduced, and their properties are discussed. In terms of a congruence, an equivalence is established between the minimal prime $E$-ideals of an ADL and its quotient ADL. Finally, topological investigations are performed on prime $E$-ideals and minimal prime $E$-ideals.

## 1. Introduction

In the article by Swamy and Rao [9], the concept of an Almost Distributive Lattice (ADL) was introduced as a generalization of Boolean algebras and distributive lattices. This allowed for the abstraction of various ring-theoretic generalizations. They also introduced the notion of an ideal in an ADL, noting that the set of principal ideals in an ADL forms a distributive lattice. This extension of lattice theory notions to ADLs was significant.

The concept of normal lattices was initially introduced by Cornish [2]. Later, Rao and Ravi Kumar presented the concept of a minimal prime ideal belonging to an ideal in an ADL [6]. In another paper by Rao and Ravi Kumar [7], the notion of a normal ADL was defined, providing equivalent

Received: Jun. 7, 2023.

2020 *Mathematics Subject Classification.* 06D99, 06D15.

*Key words and phrases.* almost distributive lattice(ADL); prime filter; $E$-ideal; $E$-normal ADL; congruence; compact; Hausdorff space; closure.

https://doi.org/10.28924/2291-8639-21-2023-85

ISSN: 2291-8639

conditions for an ADL to be considered normal in terms of its annulets. These papers contributed to the understanding of ADLs and their properties.

The study of $D$-filters in lattices and their properties was carried out by Kumar et al. [4]. They investigated the properties of $D$-filters in lattices, providing valuable insights.

In the same line of research, we investigated the notions of prime $E$-ideals and $E$-ideals in an ADL. The properties of these ideals are thoroughly examined, and it is established that every proper $E$-ideal must satisfy a set of equivalent conditions to become a prime $E$-ideal. It is also proven that every maximal $E$-ideal in an ADL is a prime $E$-ideal.

Furthermore, the paper introduces the concept of $\mathcal{O}^E(M)$ as the intersection of all minimal prime $E$-ideals contained in a prime $E$-ideal $M$ in an ADL $R$. An ADL is defined as $E$-normal, characterized in terms of relative dual annihilators with respect to an ideal $E$. An equivalence between the minimal prime $E$-ideals of an ADL and its quotient ADL is derived with respect to a congruence. The topological properties of the space of all prime $E$-ideals and the space of all minimal prime $E$-ideals in an ADL are also investigated.

## 2. Preliminaries

In this section, we recall certain definitions and important results from [5] and [9], those will be required in the text of the paper.

**Definition 2.1.** *[9] An algebra $R = (R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions:*

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$
(6) $0 \wedge a = 0$
(7) $a \vee 0 = a$, *for all* $a, b, c \in R$.

**Example 2.1.** *Every non-empty set $X$ can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on $X$ by*

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

*Then $(X, \vee, \wedge, x_0)$ is an ADL (where $x_0$ is the zero) and is called a discrete ADL.*

If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on $R$.

**Theorem 2.1.** *[9] If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in R$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$
(3) $\wedge$ *is associative in* $R$
(4) $a \wedge b \wedge c = b \wedge a \wedge c$
(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(7) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(8) $a \wedge a = a$ *and* $a \vee a = a$.

It can be observed that an ADL $R$ satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL $R$ a distributive lattice.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$. The set of all maximal elements of an ADL $R$ is denoted by $\mathcal{M}$.

As in distributive lattices [1,3], a non-empty subset $I$ of an ADL $R$ is called an ideal of $R$ if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$. Also, a non-empty subset $F$ of $R$ is said to be a filter of $R$ if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

The set $\mathfrak{I}(R)$ of all ideals of $R$ is a bounded distributive lattice with least element $\{0\}$ and greatest element $R$ under set inclusion in which, for any $I, J \in \mathfrak{I}(R)$, $I \cap J$ is the infimum of $I$ and $J$ while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal(filter) $P$ of $R$ is called a prime ideal(filter) if, for any $x, y \in R$, $x \wedge y \in P (x \vee y \in P) \Rightarrow x \in P$ or $y \in P$. A proper ideal(filter) $M$ of $R$ is said to be maximal if it is not properly contained in any proper ideal(filter) of $R$. It can be observed that every maximal ideal(filter) of $R$ is a prime ideal(filter). Every proper ideal(filter) of $R$ is contained in a maximal ideal(filter). For any subset $S$ of $R$ the smallest ideal containing $S$ is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$ and such an ideal is called the principal ideal of $R$. Similarly, for any $S \subseteq R$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$ and such a filter is called the principal filter of $R$.

For any $a, b \in R$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \wedge (b] = (a \wedge b]$. Hence the set $(\mathfrak{I}^{PI}(R), \vee, \cap)$ of all principal ideals of $R$ is a sublattice of the distributive lattice $(\mathfrak{I}(R), \vee, \cap)$ of all ideals of $R$. Also, we have that the set $(\mathfrak{F}(R), \vee, \cap)$ of all filters of $R$ is a bounded distributive lattice.

**Theorem 2.2.** *[6] Let $R$ be an ADL with maximal elements. Then $P$ is a prime ideal of $R$ if and only if $R \setminus P$ is a prime filter of $R$.*

**Definition 2.2.** *[5] An ADL $R$ is said to be an associate ADL, if the operation $\vee$ is associative on $R$.*

**Definition 2.3.** *[8] For any nonempty subset $A$ of an ADL $R$, define $A^+ = \{x \in R \mid a \vee x$ is maximal, for all $a \in A\}$. Here $A^+$ is called the dual annihilator of $A$ in $R$.*

For any $a \in R$, we have $\{a\}^+ = (a]^+$, where $(a]$ is the principal filter generated by $a$. An element $a$ of an ADL $R$ is called dual dense element if $(a]^+ = \mathcal{M}$ and the set $E$ of all dual dense elements in an ADL $R$ is an ideal if $E$ is non-empty.

## 3. $E$-ideals of ADLs

In this section, we present the concepts of prime $E$-ideals and $E$-ideals in an Abstract Distributive Lattice (ADL) and explore their properties. We observe that any proper $E$-ideal in an ADL can be transformed into a prime $E$-ideal based on a set of equivalent conditions. Additionally, we establish that the intersection of all minimal prime $E$-ideals contained in a prime $E$-ideal $M$ is denoted as $\mathcal{O}^E(M)$. Furthermore, we introduce the notion of $E$-normal ADLs, which are characterized in relation to the relative dual annihilators with respect to an ideal $E$. We establish an equivalence between the minimal prime $E$-ideals of an ADL and its quotient ADL with respect to a congruence.

**Definition 3.1.** *An ideal $G$ of $R$ is said to be an $E$-ideal of $R$ if $E \subseteq G$.*

Now we have the example of an $E$-ideal of an ADL.

**Example 3.1.** *Let $R = \{0, a, b, c, d, e, f, g\}$ and define $\vee$, $\wedge$ on $R$ as follows:*

| $\wedge$ | 0 | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c | d | e | f | g |
| b | 0 | a | b | c | d | e | f | g |
| c | 0 | c | c | c | 0 | 0 | c | 0 |
| d | 0 | d | e | 0 | d | e | g | g |
| e | 0 | d | e | 0 | d | e | g | g |
| f | 0 | f | f | c | g | g | f | g |
| g | 0 | g | g | 0 | g | g | g | g |

| $\vee$ | 0 | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | a | b | c | d | e | f | g |
| a | a | a | a | a | a | a | a | a |
| b | b | b | b | b | b | b | b | b |
| c | c | a | b | c | a | b | f | f |
| d | d | a | a | a | d | d | a | d |
| e | e | b | b | b | e | e | b | e |
| f | f | a | b | f | a | b | f | f |
| g | g | a | b | f | d | e | f | g |

*Then $(R, \vee, \wedge)$ is an ADL. Clearly, we have that $E = \{0, g\}$ and $G = \{0, c, f, g\}$ are ideals of $R$ satisfying $E \subseteq G$. Therefore $G$ is an $E$-ideal of $R$. Consider an ideal $H = \{0, c\}$ of $R$, but not an $E$-ideal.*

It is easy to verify the proof of the following result.

**Lemma 3.1.** *For any non-empty subset $A$ of an ADL $R$, $(A] \vee E$ is the smallest $E$-ideal of $R$ containing $A$.*

We denote $(A] \vee E$ by $A^E$, i.e., $A^E = (A] \vee E$. For, $A = \{a\}$, we denote simply $(a)^E$ for $\{a\}^E$. Clearly, we have that $(a)^E$ is the smallest $E$-ideal containing $a$, which is known as the principal $E$-ideal generated by $a$.

**Lemma 3.2.** *For any two elements $x, y$ of an ADL $R$ with maximal element $m$, we have the following:*

(1) $(0)^E = E$
(2) $(m)^E = R$
(3) $x \leq y$ *implies* $(x)^E \subseteq (y)^E$
(4) $(x \vee y)^E = (x)^E \vee (y)^E$
(5) $(x \wedge y)^E = (x)^E \cap (y)^E$
(6) $(x)^E = E$ *if and only if* $x \in E$.

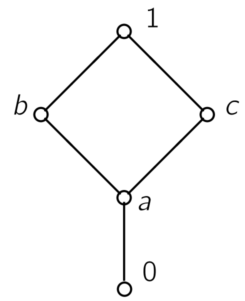
*Proof.* (1) Now $(0)^E = (0] \vee E = E$.
(2) Now $(m)^E = (m] \vee E = R \vee E = R$.
(3) Let $x \leq y$. Then $(x] \subseteq (y]$. Now $(x)^E = (x] \vee E \subseteq (y] \vee E = (y)^E$. Therefore $(x)^E \subseteq (y)^E$.
(4) Clearly, we have that $(x \vee y] = (x] \vee (y]$. Now, $(x \vee y)^E = (x \vee y] \vee E = (x] \vee (y] \vee E = ((x] \vee E) \vee ((y] \vee E)) = (x)^E \vee (y)^E$. Therefore $(x \vee y)^E = (x)^E \vee (y)^E$.
(5) Since $x \wedge y \leq y$ and $y \wedge x \leq x$ and hence $(x \wedge y] \subseteq (x]$ and $(y \wedge x] \subseteq (y]$. Since $(x \wedge y] = (y \wedge x]$, we get that $(x \wedge y] \subseteq (x] \cap (y]$. Let $t \in (x] \cap (y]$. Then $t \in (x]$ and $t \in (y]$. That implies $x \wedge t = t$ and $y \wedge t = t$. Therefore $x \wedge y \wedge t = t$ and hence $t \in (x \wedge y]$. Thus $(x] \cap (y] \subseteq (x \wedge y]$, which gives $(x \wedge y] = (x] \cap (y]$. Now $(x \wedge y)^E = (x \wedge y] \vee E = [(x] \cap (y]] \vee E = ((x] \vee E) \cap ((y] \vee E) = (x)^E \cap (y)^E$. Hence $(x \wedge y)^E = (x)^E \cap (y)^E$.
(6) Assume that $(x)^E = E$. Then $(x] \vee E = E$. That implies $(x] \subseteq E$ and hence $x \in E$. Conversely, assume that $x \in E$. Then $(x] \subseteq E$. This implies that $(x] \vee E \subseteq E$. Since $E \subseteq (x] \vee E$, we get that $E = (x] \vee E$. Therefore $(x)^E = E$. □

We denote $\mathfrak{I}(R), \mathfrak{I}^E(R)$and $\mathfrak{I}^{PEF}(R)$ as the set of all ideals, $E$-ideals and principal $E$-ideals of an ADL $R$ respectively.

**Theorem 3.1.** *$\mathfrak{I}^E(R)$ forms a distributive lattice contained in $\mathfrak{I}(R)$, and $\mathfrak{I}^{PEF}(R)$ forms a sublattice of $\mathfrak{I}^E(R)$.*

**Definition 3.2.** *An $E$-ideal $Q$ is said to be proper if $Q \subsetneq R$. A proper $E$-ideal $Q$ is said to be maximal if it is not properly contained in any proper $E$-ideal of $R$. A proper $E$-ideal $Q$ of an ADL $R$ is said to be a prime $E$-ideal if $Q$ is a prime filter of $R$.*

**Example 3.2.** *Consider a distributive lattice* $L = \{0, a, b, c, 1\}$ *and discrete ADL* $A = \{0', a'\}$.




*Clearly,*
$R = A \times L = \{(0', 0), (0', a), (0', b), (0', c), (0', 1), (a', 0), (a', a), (a', b), (a', c), (a', 1)\}$ *is an ADL with zero element* $(0, 0')$. *Clearly, the dense set* $E = \{(0', 0), (0', a)\}$. *Consider the E-ideals:*

$I_1 = \{(0', 0), (0', a), (0', b)\}$
$I_2 = \{(0', 0), (0', a), (0', c)\}$
$I_3 = \{(0', 0), (0', a), (a', 0), (a', a)\}$
$I_4 = \{(0', 0), (0', a), (0', c), (a', 0), (a', a)(a', c)\}$
$I_5 = \{(0', 0), (0', a), (0', b), (a', 0), (a', a), (a', b)\}$
$I_6 = \{(0', 0), (0', a), (0', b), (0', c), (0', 1)\}$

*Clearly,* $I_4, I_5$ *and* $I_6$ *are prime E-ideal. But* $I_1$ *is not a prime E-ideal, because* $(a', b) \wedge (0', c) = (0', a) \in I_1$, *but* $(a', b) \notin I_1$. *and* $(0', c) \notin I_1$. *And also,* $I_2$ *is not a prime E-ideal, because* $(0', b) \wedge (a', c) = (0', a) \in I_2$, *but* $(0', b) \notin I_2$ *and* $(a', c) \notin I_2$.

**Theorem 3.2.** *For any E-ideal Q of R, the following conditions are equivalent:*

(1) *Q is a prime E-ideal*
(2) *for any two E-ideals* $G, H$ *of* $R, G \cap H \subseteq Q \Rightarrow G \subseteq Q$ *or* $H \subseteq Q$
(3) *for any* $x, y \in R, (x)^E \cap (y)^E \subseteq Q \Rightarrow x \in Q$ *or* $y \in Q$.

*Proof.* (1) ⇒ (2) Assume (1). Let $G$ and $H$ be two $E$-ideals of $R$ such that $G \cap H \subseteq Q$. We prove that $G \subseteq Q$ or $H \subseteq Q$. Suppose $G \nsubseteq Q$ and $H \nsubseteq Q$. Choose $x, y \in R$ such that $x \in G \setminus Q$ and $y \in H \setminus Q$. By our assumption we have that $x \wedge y \notin Q$. Since $x \in G, y \in H$, which gives $x \wedge y \in G \cap H \subseteq Q$. Therefore $x \wedge y \in Q$, we get a contradiction. Thus $G \subseteq Q$ or $H \subseteq Q$.
(2) ⇒ (3) Assume (2). Let $x, y \in R$ with $(x)^E \cap (y)^E \subseteq Q$. Since $(x)^E$ and $(y)^E$ are $E$-ideals of $R$, and by our assumption, we get that $(x)^E \subseteq Q$ or $(y)^E \subseteq Q$. Hence $x \in Q$ or $y \in Q$.
(3) ⇒ (1) Assume (3). Let $x, y \in R$ with $x \wedge y \in Q$. Since $Q$ is an $E$-ideal, we have that $(x)^E \cap (y)^E = (x \wedge y)^E \subseteq Q$. By our assumption, we get that $x \in Q$ or $y \in Q$. Hence $Q$ is prime. □

**Theorem 3.3.** *Every maximal E-ideal of an ADL R is a prime E-ideal.*

*Proof.* Let $N$ be a maximal $E$-ideal of $R$. Let $a, b \in R$ with $a \notin N$ and $b \notin N$. Then $N \vee (a)^E = R$ and $N \vee (b)^E = R$. That implies $R = N \vee ((a)^E \cap (b)^E) = N \vee (a \wedge b)^E$. If $a \wedge b \in N$ then $N = R$, we get a contradiction. Therefore $a \wedge b \notin N$ and hence $N$ is prime. □

**Corollary 3.1.** *Let $N_1, N_2, N_3, \ldots, N_n$ and $N$ be maximal $E$-ideals of an ADL $R$ with $\bigcap_{i=1}^{n} N_i \subseteq N$, then $N_j \subseteq N$, for some $j \in \{1, 2, 3, \ldots, n\}$.*

**Theorem 3.4.** *A proper $E$-ideal $Q$ of an ADL $R$ is a prime $E$-ideal if and only if $R \setminus Q$ is a prime filter such that $(R \setminus Q) \cap E = \emptyset$.*

*Proof.* Assume that $Q$ is a prime $E$-ideal of $R$. Clearly, $R \setminus Q$ is a prime filter of $R$. We prove that $(R \setminus Q) \cap E = \emptyset$. If $(R \setminus Q) \cap E \neq \emptyset$, choose $x \in (R \setminus Q) \cap E$. That implies $x \in E \subseteq Q$, which gives a contradiction. Hence $(R \setminus Q) \cap E = \emptyset$. Conversely, assume that $R \setminus Q$ is a prime filter of $R$ such that $(R \setminus Q) \cap E = \emptyset$. Clearly, $Q$ is a prime ideal of $R$ and $E \subseteq R \setminus (R \setminus Q) = Q$. Therefore $Q$ is a prime $E$-ideal of $R$. □

**Theorem 3.5.** *Let $G$ be a $E$-ideal of an ADL $R$, and $K$ be any non-empty subset of $R$, which is closed under the operation $\wedge$ such that $G \cap K = \emptyset$. Then there exists a prime $E$-ideal $Q$ of $R$ containing $G$ such that $Q \cap K = \emptyset$.*

*Proof.* Let $K$ be a non-empty subset of $R$, which is closed under the operation $\wedge$ such that $G \cap K = \emptyset$. Consider $\mathfrak{F} = \{H \mid H \text{ is an } E-\text{ideal of } R, G \subseteq H \text{ and } H \cap K = \emptyset\}$. Clearly, it satisfies the hypothesis of the Zorn's lemma and hence $\mathfrak{F}$ has a maximal element say $Q$. That is, $Q$ is an $E$-ideal of $R$ such that $G \subseteq Q$ and $Q \cap K = \emptyset$. Let $x, y \in R$ be such that $x \wedge y \in Q$. We prove that $x \in Q$ or $y \in Q$. Suppose that $x \notin Q$ and $y \notin Q$. Then clearly $Q \vee (x)^E$ and $Q \vee (y)^E$ are $E$-ideals of $R$ such that $Q \subsetneq Q \vee (x)^E$ and $Q \subsetneq Q \vee (y)^E$. Since $Q$ is maximal in $\mathfrak{F}$, we get that $(Q \vee (x)^E) \cap K \neq \emptyset$ and $(Q \vee (y)^E) \cap K \neq \emptyset$. Choose $s \in (Q \vee (x)^E) \cap K$ and $t \in (Q \vee (y)^E) \cap K$. Then $s \in (Q \vee (x)^E), t \in (Q \vee (y)^E)$ and $s, t \in K$. Since $K$ is closed under $\wedge$, we get $s \wedge t \in K$. Now $s \wedge t = \{Q \vee (x)^E\} \cap \{Q \vee (y)^E\} = Q \vee \{(x)^E \cap (y)^E\} = Q \vee (x \wedge y)^E$. Since $x \wedge y \in Q$, we get that $s \wedge t \in Q$. Since $s \wedge t \in K$, we get that $s \wedge t \in Q \cap K$, which is a contradiction to $Q \cap K = \emptyset$. Therefore either $x \in Q$ or $y \in Q$. Thus $Q$ is a prime $E$-ideal of $R$. □

**Corollary 3.2.** *For any $E$-ideal $G$ of an ADL $R$ with $x \notin G$, there exists a prime $E$-ideal $Q$ of $R$ such that $G \subseteq Q$ and $x \notin Q$.*

**Corollary 3.3.** *For any $E$-ideal $G$ of an ADL $R$, $G = \bigcap\{Q \mid Q \text{ is a prime } E-\text{ ideal of } R \text{ and } G \subseteq Q\}$.*

**Corollary 3.4.** *$E$ is the intersection of all prime $E$-ideals of $R$.*

*Proof.* Let $Q$ be any prime $E$-ideal of $R$. Clearly, we have that $E \subseteq \bigcap Q$. Let $Q$ be any prime $E$-ideal of an ADL $R$ and $x \in \bigcap Q$. Suppose $x \notin E$. Then there exists prime filter $N$ such that $x \in N$ and

$N \cap E = \emptyset$. That implies $x \notin R \setminus N$ and $E \subseteq R \setminus N$. Therefore $R \setminus N$ is a prime $E$-ideal of $R$ and $x \notin R \setminus N$, which is a contradiction. Therefore $x \in E$ and hence $\bigcap Q \subseteq E$. Thus $E = \bigcap Q$. □

**Theorem 3.6.** *In an ADL the following are equivalent:*

(1) *Every proper $E$-ideal is prime*
(2) $\mathfrak{I}^E(R)$ *is a chain*
(3) $\mathfrak{I}^{PEF}(R)$ *is a chain.*

*Proof.* (1) ⇒ (2) Assume (1). Clearly $(\mathfrak{I}^E(R), \subseteq)$ is a poset. Let $S$ and $T$ be two proper $E$-ideals of $R$. By (1), we have that $S \cap T$ is a prime $E$-ideal of $R$. Since $S \cap T \subseteq S \cap T$, we get $S \subseteq S \cap T \subseteq T$ or $T \subseteq S \cap T \subseteq S$. Hence $\mathfrak{I}^E(R)$ is a chain.
(2) ⇒ (3) It is obvious.
(3) ⇒ (1) Assume that (3). Let $G$ be a proper $E$-ideal of $R$. We prove that $G$ is prime. Let $x, y \in R$ such that $(x)^E \cap (y)^E \subseteq G$. By our assumption, we get that $(x)^E \subseteq (y)^E$ or $(y)^E \subseteq (x)^E$. That implies $x \in (x)^E = (x)^E \cap (y)^E \subseteq G$ or $y \in (y)^E = (x)^E \cap (y)^E \subseteq G$. Therefore $G$ is a prime $E$-ideal of $R$. □

Now we introduce the concept of a relative dual annihilator in the following definition.

**Definition 3.3.** *For any nonempty subset $S$ of $R$, define $(S, E) = \{a \in R \mid s \wedge a \in E, \ for\ all\ s \in S\}$. We call this set as relative dual annihilator of $S$ with respect to the ideal $E$.*

For $S = \{s\}$, we denote $(\{s\}, E)$ by $(s, E)$.

**Lemma 3.3.** *If $S, T$ are nonempty subsets of an ADL $R$, then we have the following:*

(1) $(R, E) = E = (\mathcal{M}, E)$
(2) $(E, E) = R$
(3) $E \subseteq (S, E)$
(4) $(S, E)$ *is a $E$-ideal of $R$*
(5) $S \subseteq E$ *if and only if* $(S, E) = R$
(6) *if* $S \subseteq T$, *then* $(T, E) \subseteq (S, E)$ *and* $((S, E), E) \subseteq ((T, E), E)$
(7) $S \subseteq ((S, E), E)$
(8) $(((S, E), E), E) = (S, E)$
(9) $(S, E) = ([S), E)$
(10) $\bigcap_{i \in \triangle} (S_i, E) = \left( \bigcup_{i \in \triangle} S_i, E \right)$
(11) $(S, E) \subseteq (S \cap T, (T, E))$
(12) *if* $S \subseteq T$, *then* $(S, (T, E)) = (S, E)$
(13) $(S \cup T, E) \subseteq (S, (T, E)) \subseteq (S \cap T, E)$
(14) $(S, (S, E)) = (S, E)$.

*Proof.* (1) Let $x \in (R, E)$. Then $a \wedge x \in E$, for all $a \in R$. That implies $x \wedge x \in E$. So that $x \in E$. Hence $(R, E) \subseteq E$. Let $x \in E$. Then $a \wedge x \in E$, for all $a \in R$. Thus $x \in (R, E)$. Therefore $E \subseteq (R, E)$ and hence $(R, E) = E$. Clearly, we have that $(\mathcal{M}, E) = E$.

(2) Let $x \in E$. Then $x \wedge a \in E$, for all $a \in R$. Since $x \wedge a \in E$, for all $x \in E$, we get that $a \in (E, E)$, for all $a \in R$. Therefore $R \subseteq (E, E)$ and hence $R = (E, E)$.

(3) Let $x \in E$. Then $y \wedge x \in E$, for all $y \in R$. Then $a \wedge x \in E$, for all $a \in S \subseteq R$. That implies $x \in (S, E)$. Therefore $E \subseteq (S, E)$.

(4) Let $a, b \in (S, E)$. Then $s \wedge a, s \wedge b \in E$, for all $s \in S$. This implies $(s \wedge a) \vee (s \wedge b) \in E$. Therefore $s \wedge (a \vee b) \in E$. Hence $a \vee b \in (S, E)$. Let $a \in (S, E)$ and $b \in R$ with $b \leq a$. Then $s \wedge a \in E$ and $s \wedge b \leq s \wedge a$, for all $s \in S$. Since $s \wedge a \in E$ and $E$ is an ideal, we get $s \wedge b \in E$. Hence $b \in (S, E)$, for all $s \in S$. Thus $(S, E)$ is an ideal of $R$. Since $E \subseteq (S, E)$, we get that $(S, E)$ is an $E$-ideal of $R$.

(5) Suppose $(S, E) = R$. Let $m \in \mathcal{M}$. Then $m \in (S, E)$. That implies $a = m \wedge a \in E$, for all $a \in S$. Hence $a \in E$, for all $a \in S$. Therefore $S \subseteq E$. Conversely, assume that $S \subseteq E$. Let $x \in R$. Since $E$ is an ideal, we get $a \wedge x \in E$, for all $a \in S \subseteq E$. Hence $x \in (S, E)$. Therefore $(S, E) = R$.

(6) Suppose $S \subseteq T$. Let $a \in (T, E)$. Then $t \wedge a \in E$, for all $t \in T$. Since $S \subseteq T$, we get that $s \wedge a \in E$, for all $s \in S$. That implies $a \in (S, E)$. Therefore $(T, E) \subseteq (S, E)$ and hence $((S, E), E) \subseteq ((T, E), E)$.

(7) Let $x \in (S, E)$. Then $s \wedge x \in E$, for all $s \in S$. That implies $x \wedge s \in E$, for all $x \in (S, E)$. That implies $s \in ((S, E), E)$, for all $s \in S$. Thus $S \subseteq ((S, E), E)$.

(8) By (7), we have that $(((S, E), E), E) \subseteq (S, E)$. Let $x \notin (((S, E), E), E)$. Then there exists an element $a \notin ((S, E), E)$ such that $a \wedge x \notin E$. Since $S \subseteq ((S, E), E)$, we have that $a \notin S$. So that $a \wedge x \notin E$ and $s \notin S$. Therefore $x \notin (S, E)$, it concludes that $(S, E) \subseteq (((S, E), E), E)$. Thus $(((S, E), E), E) = (S, E)$.

(9) Since $S \subseteq (S]$, we get that $((S], E) \subseteq (S, E)$. Let $x \in (S, E)$. Then $a \wedge x \in E$, for all $a \in S \subseteq (S]$. That implies $x \in ((S], E)$. Therefore $(S, E) \subseteq ((S], E)$. Therefore $(S, E) \subseteq ((S], E)$. Hence $(S, E) = ((S], E)$.

(10) Since $S_i \subseteq \bigcup\limits_{i \in \triangle} S_i$, for all $i \in \triangle$, we get that $\big(\bigcup\limits_{i \in \triangle} S_i, E\big) \subseteq (S_i, E)$, for all $i \in \triangle$. That implies $\big(\bigcup\limits_{i \in \triangle} S_i, E\big) \subseteq \bigcap\limits_{i \in \triangle} (S_i, E)$. Let $x \in \bigcap\limits_{i \in \triangle} (S_i, E)$. Then $x \in (S_i, E)$, for all $i \in \triangle$. That implies $a \wedge x \in E$, for all $a \in S_i \subseteq \bigcup S_i$. That implies $\bigcap\limits_{i \in \triangle} (S_i, E) \subseteq \big(\bigcup\limits_{i \in \triangle} S_i, E\big)$. Therefore $\bigcap\limits_{i \in \triangle} (S_i, E) = \big(\bigcup\limits_{i \in \triangle} S_i, E\big)$.

(11) Since $E$ is an ideal in $R$, we have that $E \subseteq (T, E)$ and hence we get that $(S, E) \subseteq (S, (T, E))$. Since $S \cap T \subseteq S$, we get that $(S, (T, E)) \subseteq (S \cap T, (T, E))$. Therefore $(S, E) \subseteq (S \cap T, (T, E))$.

(12) Let $S, T$ be two non empty subsets of $R$ such that $S \subseteq T$. Since $E \subseteq (T, E)$, we have that $(S, E) \subseteq (S, (T, E))$. Let $x \in (S, (T, E))$. Then $a \wedge x \in (T, E)$, for all $a \in S$. That implies $a \wedge x \in (S, E)$, for all $a \in S$. Since $a \wedge x \in (S, E)$, we get that $s \wedge (a \wedge x) \in E$, for all $s \in S$ and hence $a \wedge x \in E$, for all $a \in S$. Therefore $x \in (S, E)$ and hence $(S, (T, E)) \subseteq (S, E)$. Thus $(S, (T, E)) = (S, E)$.

(13) Clearly, we have that $(S \cup T, E) \subseteq (S, E)$ and $E \subseteq (T, E)$. So that $(S, E) \subseteq (S, (T, E))$. Also

$S \cap T \subseteq S$. It follows that $(S,(T,E)) \subseteq (S \cap T, E)$. Therefore $(S \cup T, E) \subseteq (S,(T,E)) \subseteq (S \cap T, E)$.
(14) It is clear by (12). □

**Proposition 3.1.** *Let $S$ and $T$ be any two ideals of and ADL $R$. Then we have the following:*

(1) $(S,E) \cap ((S,E),E) = E$
(2) $(S \vee T, E) = (S,E) \cap (T,E)$
(3) $((S \cap T, E), E) \subseteq ((S,E),E) \cap ((T,E),E)$.

*Proof.* (1) We have that $E \subseteq (S,E) \cap ((S,E),E)$. Let $x \in (S,E) \cap ((S,E),E)$. Then $x \in (S,E)$ and $x \in ((S,E),E)$. Since $x \in ((S,E),E))$, we have that $a \wedge x \in E$, for all $a \in (S,E)$. Since $x \in (S,E)$, we get that $x \in E$ and hence $(S,E) \cap ((S,E),E) \subseteq E$. Thus $(S,E) \cap ((S,E),E) = E$.
(2) Clearly, $S \subseteq S \vee T$ and $T \subseteq S \vee T$. Then $((S \vee T),E) \subseteq (S,E)$ and $((S \vee T),E) \subseteq (T,E)$. That implies $((S \vee T),E) \subseteq (S,E) \cap (T,E)$. Let $x \in (S,E) \cap (T,E)$. Then $x \in (S,E)$ and $x \in (T,E)$. That implies $s \wedge x \in E$, for all $s \in S$ and $t \wedge x \in E$, for all $t \in T$. That implies $(s \wedge x) \vee (t \wedge x) \in E$ and have $(s \vee t) \wedge x \in E$. Since $s \in S$ and $t \in T$, we get $s \vee t \in S \vee T$. Therefore $(s \vee t) \wedge x \in E$, for all $s \vee t \in S \vee T$. That implies $x \in (S \vee T, E)$. Therefore $(S,E) \cap (T,E) \subseteq (S \vee T, E)$. Hence $(S,E) \cap (T,E) = (S \vee T, E)$.
(3) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, we get that $(S,E) \subseteq (S \cap T, E)$ and $(T,E) \subseteq (S \cap T, E)$. That implies $((S \cap T, E), E) \subseteq ((S,E),E)$ and $((S \cap T, E), E) \subseteq ((T,E),E)$. Hence $((S \cap T, E), E) \subseteq ((S,E),E) \cap ((T,E),E)$. □

**Theorem 3.7.** *For any non-empty subset $S$ of an ADL $R$, $(S,E) = \bigcap_{s \in S} ((s],E)$.*

*Proof.* Let $x \in \bigcap_{s \in S} ((s],E)$. Then $x \in ((s],E)$, for all $s \in S$. That implies $t \wedge x \in E$, for all $t \in (s]$ and for all $s \in S$. It follows that $s \wedge x \in E$ for all $s \in S$. Therefore $x \in (S,E)$. Hence $x \in \bigcap_{s \in S} ((s],E) \subseteq (S,E)$. Let $s$ be any element of $S$. Take $t \in (s]$. Then $s \wedge t = t$. Now, $x \in (S,E)$. That implies $s \wedge x \in E$, for all $s \in S$. So that $t \wedge x = t \wedge s \wedge x \in E$, for all $t \in (s]$ and for all $s \in S$. That implies $x \in ((s],E)$, for all $s \in S$. Therefore $x \in \bigcap_{s \in S} ((s],E)$ and hence $(S,E) \subseteq \bigcap_{s \in S} ((s],E)$. Thus $(S,E) = \bigcap_{s \in S} ((s],E)$. □

**Corollary 3.5.** *Let $x \in R$ and $S$ be arbitrary subset of $R$. Then $(S,(x]) = \bigcap_{a \in S} (a,(x])$.*

**Corollary 3.6.** *For any $x, y \in R$ we have the following:*

(1) $((x],E) = (x,E)$
(2) $x \leq y \Rightarrow (y,E) \subseteq (x,E)$
(3) $(x \vee y, E) = (x,E) \cap (y,E)$
(4) $((x \wedge y, E), E) = ((x,E),E) \cap ((y,E),E)$
(5) $(x,E) = R \Leftrightarrow x \in E$.

**Theorem 3.8.** *Let $G$ be an $E$-ideal of an ADL $R$. Then*

(1) $G \cap (G, E) = E$
(2) $((G \vee (G, E)), E) = E$.

*Proof.* (1) It is clear.
(2) Clearly, $((G \vee (G, E)), E) \subseteq (G, E) \cap ((G, E), E)$. Let $a \in (G, E) \cap ((G, E), E)$. Let $b \in G \vee (G, E)$. Then $b = c \vee d$, for some $c \in G$ and $d \in (G, E)$. That implies $a \wedge c \in E$ and $a \wedge d \in E$. Now $a \wedge b = a \wedge (c \vee d) = (a \wedge c) \vee (a \wedge d) \in E$, for all $b \in G \vee (G, E)$. Therefore $a \in ((G \vee (G, E)), E)$ and hence $(G, E) \cap ((G, E), E) \subseteq ((G \vee (G, E)), E)$. Thus $E = (G, E) \cap ((G, E), E) = ((G \vee (G, E)), E)$. □

Consider two ADLs $R_1$ and $R_2$ with zero elements $0$ and $0'$ respectively. Let $\mathcal{M}$ and $\mathcal{M}'$ be denotes the set of all maximal elements of ADLs $R_1$ and $R_2$ respectively.

**Lemma 3.4.** *Let $R_1$ and $R_2$ be two ADLs with $m \in \mathcal{M}$ and $m' \in \mathcal{M}'$. Then for any $(x, y) \in R_1 \times R_2$, we have the following:*

(1) $(x, y)^+ = (a)^+ \times (y)^+$
(2) $(x, y)^+ = (m, m')$ *if and only if* $(x)^+ = \mathcal{M}$ *and* $(y)^+ = \mathcal{M}'$
(3) $((x, y), E) = (a, E) \times (y, E)$.

Let $E_1$ and $E_2$ be dual dense sets of $R_1$ and $R_2$ respectively. From the above result, it can be concluded that $E = E_1 \times E_2$ is a dual dense set of $R_1 \times R_2$. Further, every dual dense set of $R_1 \times R_2$ is form the form $E_1 \times E_2$.

**Theorem 3.9.** *Let $M_i$ be a prime $E_i-$ideals of ADLs $R_i$, for $i = 1, 2$. Then $M_1 \times R_2$ and $R_1 \times M_2$ are prime $E$-ideals of $R_1 \times R_2$.*

*Proof.* Since $E_1 \subseteq M_1$ and $E_2 \subseteq M_2$, we get $E_1 \times E_2 \subseteq M_1 \times R_2$ and $E_1 \times E_2 \subseteq R_1 \times M_2$. That implies $M_1 \times R_2$ and $R_1 \times M_2$ are $E$-ideals of $R_1 \times R_2$. Let $(a, b), (c, d) \in R_1 \times R_2$ with $(a, b) \wedge (c, d) \in M_1 \times R_2$. Then $a \wedge c \in M_1$. Since $M_1$ is a prime $E_1-$ideal of $R_1$, we get $a \in M_1$ or $c \in M_1$. Thus $(a, b) \in M_1 \times R_2$ or $(c, d) \in M_1 \times R_2$. Therefore $M_1 \times R_2$ is a prime $E$-ideal of $R_1 \times R_2$. Similarly, we can prove that $R_1 \times M_2$ is also a prime $E$-ideal of $R_1 \times R_2$. □

**Theorem 3.10.** *Let $R_1$ and $R_2$ be two ADLs with zero elements $0$ and $0'$ respectively. For any prime $E$-ideal $P$ of $R_1 \times R_2$, $P$ is of the form $P_1 \times R_2$ or $R_1 \times P_2$, where $P_i$ is a prime $E_i-$ideal of $R_i$, for $i = 1, 2$.*

*Proof.* Let $P$ be a prime $E$-ideal of $R_1 \times R_2$. Consider $P_1 = \pi_1(P) = \{x_1 \in R_1 \mid (x_1, x_2) \in P, \text{ for some } x_2 \in R_2\}$ and $P_2 = \pi_2(P) = \{x_2 \in R_2 \mid (x_1, x_2) \in P, \text{ for some } x_1 \in R_1\}$. It is easy to verify that $P_i$ is $E_i-$ideals of $R_i$, for $i = 1, 2$. We first show that $P_i$ is prime $E_i-$ideals of $R_i$, for $i = 1, 2$. Suppose $P_1 = R_1$ and $P_2 = R_2$. Let $(a, b) \in R_1 \times R_2$. Then there exist $x \in R_1$ and $y \in R_2$ such that $(a, y) \in P$ and $(x, b) \in P$. Since $(a, 0') \wedge (a, y) \in P$ and $(0, b) \wedge (x, b) \in P$, we get $(a, 0') \in P$

and $(0, b) \in P$. Therefore $(a, b) = (a, 0') \vee (0, b) \in P$. Hence $P = R_1 \times R_2$, which is a contradiction to that $P$ is proper. Next suppose that $P_1 \neq R_1$ and $P_2 \neq R_2$. Choose $a \in R_1 \setminus P_1$ and $b \in R_2 \setminus P_2$. Then $(a, y) \notin P$ for all $y \in R_2$ and $(x, b) \notin P_1$ for all $x \in R_1$. In particular, $(a, 0') \notin P$ and $(0, b) \notin P$. Since $P$ is prime, we get $(0, 0') \notin P$, which is a contradiction. From the above observations, we get that either $P_1 = R_1$ and $P_2 \neq R_2$ or $P_1 \neq R_1$ and $P_2 = R_2$.

Case (i): Suppose $P_1 = R_1$ and $P_2 \neq R_2$. Let $x_2, y_2 \in R_2$ be such that $x_2 \wedge y_2 \in P_2$. Then there exists $a \in R_1 = P_1$ such that $(a, x_2 \wedge y_2) \in P$. Therefore $(a, x_2) \wedge (a, y_2) = (a \wedge a, (x_2 \wedge y_2)) = (a, x_2 \wedge y_2) \in P$. Since P is prime, we get $(a, x_2) \in P$ or $(a, y_2) \in P$. Hence $x_2 \in P_2$ or $y_2 \in P_2$. Therefore $P_2$ is a prime $E_2-$ideal of $R_2$. We now show that $P = R_1 \times P_2$. Clearly $P \subseteq R_1 \times P_2$. On the other hand, suppose $(a, y) \in R_1 \times P_2$. Since $P_1 = R_1$, there exists $b \in R_2$ such that $(a, b) \in P$ and there exists $x \in R_1$ such that $(x, y) \in P$. Since $(a, 0') \wedge (a, b) = (a, 0')$ and $(0, y) \wedge (x, y) = (0, y)$, we get $(a, 0') \in P$ and $(0, y) \in P$. Since $P$ is an ideal, it gives $(a, y) = (a, 0') \vee (0, y) \in P$. Hence $R_1 \times P_2 \subseteq P$. Therefore $P = R_1 \times P_2$.

Case (ii): Suppose $P_1 \neq R_1$ and $P_2 = R_2$. Similarly, we can prove that $P_1$ is prime $E_1-$ideal of $R_1$ and $P = P_1 \times R_2$. □

**Theorem 3.11.** *Let $S$ be a sub ADL of an ADL $R$ and $P$ is a prime $E$-ideal of $S$. Then there exists a prime $E$-ideal $Q$ of $R$ such that $Q \cap S = P$.*

*Proof.* Let $P$ be a prime $E$-ideal of $S$. Then $S \setminus P$ is a prime filter of $S$. Consider $I = (P]$. Then $P \subseteq I \cap S$. Suppose $I \cap (S \setminus P) \neq \emptyset$. Choose $x \in I \cap (S \setminus P)$. Then $x \in I$ and $x \in (S \setminus P)$. Since $x \in I = (P]$, there exists $a_1 \vee a_2 \vee \ldots \vee a_n \in P$ such that $x = y \wedge (a_1 \vee a_2 \vee \ldots \vee a_n)$. Since $P$ is an ideal of $S$, we get $a_1 \vee a_2 \vee \ldots \vee a_n \in P$ and hence $x \in P$. Since $x \in (S \setminus P)$, we get a contradiction. Hence $I \cap (S \setminus P) = \emptyset$. Then there exists a prime $E$-ideal $Q$ of $R$ such that $I \subseteq Q$ and $Q \cap (S \setminus P) = \emptyset$. Since $I \subseteq Q$, we get $I \cap S \subseteq Q \cap S$. Since $Q \cap (S \setminus P) = \emptyset$, we get $Q \subseteq P$. Hence, both observations lead to $P \subseteq I \cap S \subseteq Q \cap S \subseteq P \cap S \subseteq P$. Therefore $P = Q \cap S$. □

Now, we have the following definition.

**Definition 3.4.** *A prime $E$-ideal $M$ of an ADL $R$ containing an $E$-ideal $G$ is said to be a minimal prime $E$-ideal belonging to $G$ if there exists no prime $E$-ideal $N$ such that $G \subseteq N \subseteq M$.*

Note that if we take $E = G$ in the above definition then we say that $M$ is a minimal prime $E$-ideal.

**Example 3.3.** *From the Example 3.2, we have that $I_6$ is a prime $E$-ideal and $I_1$ is a $E$-ideal of $R$. Clearly $I_1 \subseteq I_6$. Clearly there is no $E$-ideal $N$ of $R$ such that $I_1 \subseteq N \subseteq I_6$. Hence $I_6$ is a minimal prime $E$-ideal belonging to $I_1$.*

**Proposition 3.2.** *Let $G$ be an $E$-ideal and $M$, a prime $E$-ideal of $R$ with $G \subseteq M$. Then $M$ is a minimal prime $E$-ideal belonging to $G$ if and only if $R \setminus M$ is a maximal filter with $(R \setminus M) \cap G = \emptyset$.*

*Proof.* Clearly, $R \setminus M$ is a proper filter and we have $(R \setminus M) \cap G = \emptyset$. We prove that $R \setminus M$ is maximal. Let $N$ be any proper filter of $R$ such that $N \cap G = \emptyset$ and $R \setminus M \subseteq N$. Then $G \subseteq R \setminus N \subseteq M$. By the minimality of $M$, we get $R \setminus N = M$. Therefore $R \setminus M$ is a maximal filter with respect to the property $(R \setminus M) \cap G = \emptyset$. Conversely, assume that $R \setminus M$ be a maximal filter with respect to the property $(R \setminus M) \cap G = \emptyset$. We prove that $M$ is minimal. If $N$ is any prime $E$-ideal of $R$ such that $E \subseteq G \subseteq N \subseteq M$. Clearly, $R \setminus N$ is a filter such that $R \setminus M \subseteq R \setminus N$ and $(R \setminus N) \cap G = \emptyset$, which is a contradiction. Therefore $M$ is a minimal prime $E$-ideal belonging to $G$. □

**Theorem 3.12.** *Let $G$ be an $E$-ideal and $M$, a prime $E$-ideal of $R$ with $G \subseteq M$. Then $M$ is a minimal prime $E$-ideal belonging to $G$ if and only if for any $a \in M$, there exists $b \notin M$ such that $a \wedge b \in G$.*

*Proof.* Assume that $M$ is a minimal prime $E$-ideal belonging to $G$. Then $R \setminus M$ is a maximal filter with respect to the property that $(R \setminus M) \cap G = \emptyset$. Let $a \in M$. Then $a \notin R \setminus M$. That implies $R \setminus M \subset (R \setminus M) \vee [a)$. By the maximality of $R \setminus M$, we get that $((R \setminus M) \vee [a)) \cap G \neq \emptyset$. Choose $s \in ((R \setminus M) \vee [a)) \cap G$. Then there exists $b \in R \setminus M$ such that $s = b \wedge a$ and $s \in G$. Therefore $b \wedge a \in G$. Conversely, assume that for any $a \in M$, there exists $b \notin M$ such that $a \wedge b \in G$. Suppose $M$ is not a minimal prime $E$-ideal belonging to $G$. Then there exists a prime $E$-ideal $N$ of $R$ such that $E \subseteq G \subseteq N \subseteq M$. Choose $a \in M \setminus N$. Then, by the our assumption, there exists $b \notin M$ such that $a \wedge b \in G \subseteq N$. Since $a \notin N$, we get that $b \in N \subseteq M$, which is a contradiction. Therefore $M$ is a minimal prime $E$-ideal belonging to $G$. □

**Corollary 3.7.** *A prime $E$-ideal $M$ of an ADL $R$ is minimal if and only if for any $a \in M$ there exists $b \notin M$ such that $a \wedge b \in E$.*

**Definition 3.5.** *For any prime $E$-ideal $M$ of $R$, define the set $\mathcal{O}^E(M)$ as follows:*

$$\mathcal{O}^E(M) = \{x \in R \mid x \in (y, E),\ \text{for some } y \notin M\}.$$

*Clearly, observe that* $\mathcal{O}^E(M) = \bigcup_{y \notin M} (y, E)$.

**Lemma 3.5.** *Let $M$ be prime $E$-ideal of an ADL $R$. Then $\mathcal{O}^E(M)$ is an $E$-ideal such that $\mathcal{O}^E(M)$ is contained in $M$.*

*Proof.* Let $a, b \in \mathcal{O}^E(M)$. There exist elements $s \notin M$ and $t \notin M$ such that $a \in (s, E)$ and $b \in (t, E)$. That implies $((s, E), E) \subseteq (a, E)$ and $((t, E), E) \subseteq (b, E)$. So that $((s \wedge t, E), E) = ((s, E), E) \cap ((t, E), E) \subseteq (a, E) \cap (b, E) = (a \vee b, E)$. Hence $a \vee b \in ((a \vee b, E), E) \subseteq (((s \wedge t, E), E), E) = (s \wedge t, E)$. Since $s \wedge t \notin M$, we get that $a \vee b \in \mathcal{O}^E(M)$. Let $a \in \mathcal{O}^E(M)$ and $b \leq a$. There exists $s \notin M$ such that $a \in (s, E)$. Since $(s, E)$ is an ideal, we get that $b \in (s, E)$. Therefore $b \in \mathcal{O}^E(M)$ and hence $\mathcal{O}^E(M)$ is an ideal of $R$. Clearly, we have that $E \subseteq \mathcal{O}^E(M)$. Thus $\mathcal{O}^E(M)$ is an $E$-ideal of $R$. Let $a \in \mathcal{O}^E(M)$. Then there exists $s \notin M$ such that $a \in (s, E)$. That implies $a \wedge s \in E \subseteq M$. Since $M$ is prime, we get that $a \in M$. Hence $\mathcal{O}^E(M) \subseteq M$. □

**Corollary 3.8.** *For any prime $E$-ideal $M$ of $R$, $M$ is minimal if and only if $\mathcal{O}^E(M) = M$.*

**Theorem 3.13.** *Every minimal prime $E$-ideal of $R$ belonging to $\mathcal{O}^E(M)$ is contained in $M$.*

*Proof.* Let $N$ be any minimal prime $E$-ideal belonging to $\mathcal{O}^E(M)$. We prove that $N \subseteq M$. Suppose $N \nsubseteq M$. Choose $a \in N \setminus M$. Then there exists $b \notin N$ such that $a \wedge b \in \mathcal{O}^E(M)$. Hence $a \wedge b \in (s, E)$, for some $s \notin M$. That implies $b \wedge (a \wedge s) \in E \subseteq M$. Since $a \notin M, s \notin M$, and $M$ is prime, we get $a \wedge s \notin M$. Therefore $b \in \mathcal{O}^E(M) \subseteq N$, which is a contradiction. Hence $N \subseteq M$. □

**Theorem 3.14.** *For any prime $E$-ideal $M$ of an ADL $R$, $\mathcal{O}^E(M)$ is the intersection of all minimal prime $E$-ideals contained in $M$.*

*Proof.* Let $M$ be a prime $E$-ideal of $R$. By Zorn's lemma, $M$ contains a minimal prime $E$-ideal. Let $\{S_\alpha\}_{\alpha \in \Delta}$ be the set of all minimal prime $E$-ideals contained in $M$. Let $x \in \mathcal{O}^E(M)$. Then $x \in (a, E)$, for some $a \notin M$. Since each $S_\alpha \subseteq M$, we have that $a \notin S_\alpha$, for all $\alpha \in \Delta$ . Since $x \wedge a \in E \subseteq S_\alpha$ and $a \notin S_\alpha$, for all $\alpha \in \Delta$, we get $x \in S_\alpha$ for all $\alpha \in \Delta$. Hence $x \in \bigcap\limits_{\alpha \in \Delta} S_\alpha$. Therefore $\mathcal{O}^E(M) \subseteq \bigcap\limits_{\alpha \in \Delta} S_\alpha$. Let $x \notin \mathcal{O}^E(M)$. Consider $S = (R \setminus M) \vee [x)$. Suppose $E \cap S \neq \emptyset$. Choose $a \in E \cap S$. Since $a \in S$, we get $a = t \wedge x$, for some $t \in R \setminus M$. Since $a \in E$, we get that $t \wedge x \in E$. Hence $x \in (t, E)$, where $t \notin M$. Thus $x \in \mathcal{O}^E(M)$, which is a contradiction. Therefore $S \cap E = \emptyset$. Let $M$ be a maximal filter such that $S \subseteq M$ and $M \cap E = \emptyset$. Then $R \setminus M$ is a minimal prime $E$-ideal such that $R \setminus M \subseteq M$ and $x \notin R \setminus M$, since $x \in S \subseteq M$. Hence $x \notin \bigcap\limits_{\alpha \in \Delta} S_\alpha$. Therefore $\bigcap\limits_{\alpha \in \Delta} S_\alpha \subseteq \mathcal{O}^E(M)$. □

**Proposition 3.3.** *Let $M_1$ and $M_2$ be two prime $E$-ideals in an ADL $R$ with $M_1 \subseteq M_2$. Then $\mathcal{O}^E(M_2) \subseteq \mathcal{O}^E(M_1)$.*

*Proof.* Let $x \in \mathcal{O}^E(M_2)$. Then there exists an element $a \notin M_2$ such that $x \in (a, E)$. That implies $x \in (a, E)$ and $a \notin M_1$. So that $x \in \mathcal{O}^E(M_1)$. Therefore $\mathcal{O}^E(M_2) \subseteq \mathcal{O}^E(M_1)$. □

**Proposition 3.4.** *For any non zero element $a \in R$ with $a \notin E$, there is a minimal prime $E$-ideal not containing $a$.*

*Proof.* Let $a$ be any non zero element of $R$ with $a \notin E$. By Corollary 3.2, there exists a prime $E$-ideal $P$ of $R$ such that $a \notin P$. Consider $\mathfrak{F} = \{Q \mid Q \text{ is a prime } E-\text{ideal of } R, a \notin Q \text{ and } Q \subseteq P\}$. It satisfies the hypothesis of Zorn's Lemma. So that $\mathfrak{F}$ has a minimal element say $M$. i.e. $M$ is minimal and $a \notin M$. □

**Theorem 3.15.** *For any prime $E$-ideal $M$ of an ADL $R$, the following are equivalent:*

(1) *$M$ is minimal prime $E$-ideal*
(2) *$M = \mathcal{O}^E(M)$*
(3) *for any $x \in R, M$ contains precisely one of $x$ or $(x, E)$.*

*Proof.* (1) ⇒ (2) Assume (1). Let $x \in M$. Then there exists $y \notin M$ such that $x \wedge y \in E$. This implies that $x \in \mathcal{O}^E(M)$. So that $M \subseteq \mathcal{O}^E(M)$. Since $\mathcal{O}^E(M) \subseteq M$, we get that $M = \mathcal{O}^E(M)$.
(2) ⇒ (3) Assume (2). Let $x \in R$. Suppose $x \notin M$. Let $a \in (x, E)$. Then $a \wedge x \in E$. That implies $a \wedge x \in M$. So that $a \in M$. Since $x \notin M$. Therefore $(x, E) \subseteq M$.
(3) ⇒ (1) Let $Q$ be any prime $E$-ideal of $R$ with $Q \subsetneq M$. Then choose $x \in M$ such that $x \notin Q$. That implies $(x, E) \subseteq Q \subsetneq M$. So that $(x, E) \subsetneq M$ which is a contradiction. □

**Corollary 3.9.** *Let $P$ be a minimal prime $E$-ideal of an ADL $R$ and $a \in R$. Then $a \in P$ if and only if $((a, E), E) \subseteq P$.*

*Proof.* Assume that $a \in P$. Then $(a, E) \nsubseteq P$. Let $t \in ((a, E), E)$. Then $(a, E) \subseteq (t, E)$. Suppose $t \notin P$. Then $(a, E) \subseteq (t, E) \subseteq P$, which is a contradiction. That implies $t \in P$, which gives $((a, E), E) \subseteq P$. The converse follows from the fact that $a \in ((a, E), E)$. □

**Definition 3.6.** *An ADL $R$ with maximal elements is called an $E$- semi complemented if for each non maximal element $x \in R$, there exists a non zero element $y \notin E$ such that $x \wedge y \in E$.*

**Example 3.4.** *From the Example 3.2, clearly we have that $R$ is an $E$-semi complemented ADL.*

**Theorem 3.16.** *Let $R$ be an ADL with maximal elements. Then $R$ is $E$-semi complemented if and only if the intersection of all maximal filters disjoint with $E$ is $\mathcal{M}$.*

*Proof.* Assume that $R$ is $E$-semi complemented. Consider

$$K = \bigcap \{M \mid M \text{ is a maximal filter of } R \text{ and } M \cap E = \emptyset\}.$$

We have to prove that $K = \mathcal{M}$. Let $x \in K$ with $x$ is not a maximal element. Then $x \in M$, for all maximal filter $M$ disjoint with $E$. Then $x \notin E$. Since $x$ is non maximal and $R$ is $E$- semi complemented, there exists a non zero element $y \notin E$ such that $x \wedge y \in E$. Then $x \wedge y \notin M$. That implies $M \vee [x \wedge y) = R$. Since $y \notin E$, there exists a minimal prime $E$-ideal $N$ of $R$ such that $y \notin N$. That implies $y \in R \setminus N$ and $(R \setminus N) \cap E = \emptyset$, where $R \setminus N$ is maximal filter of $R$. So that $x, y \in R \setminus N$. We have $x \wedge y \in R \setminus N$. Therefore $(R \setminus N) \cap E \neq \emptyset$, which is a contradiction. Therefore $x$ is a maximal element. Hence $K = \mathcal{M}$. Conversely, assume that $\bigcap \{M \mid M \text{ is a maximal filter of } R \text{ and } M \cap E = \emptyset\} = \mathcal{M}$. Let $x$ be any non maximal element of $R$. Then there exists a maximal filter $M$ such that $x \notin M$ and $M \cap E = \emptyset$. That implies $M \vee [x) = R$. So that $a \wedge x = 0$, for some $a \in M$. Since $a \in M$ and $M \cap E = \emptyset$, we get $a \notin E$. Clearly, $a \wedge x \in E$. That is, for any non maximal element $x$ of $R$, there exists a non zero element $a \notin E$ such that $a \wedge x \in E$. Hence $R$ is $E$-semi complemented. □

**Definition 3.7.** *An ADL $R$ is said to be $E$-normal if for any $a, b \in R$ such that $a \wedge b \in E$, there exists $x \in (a, E)$ and $y \in (b, E)$ such that $x \vee y$ is maximal.*

From the Example 3.2, clearly we have that $R$ is a $D-$normal ADL. The following result is a direct consequence of the above definition.

**Theorem 3.17.** *$R$ is $E$-normal if and only if $(a, E) \vee (b, E) = R$, for any $a, b \in R$, with $a \wedge b \in E$.*

**Definition 3.8.** *Two $E$-ideals $G_1$ and $G_2$ of $R$ are said to be co-maximal if $G_1 \vee G_2 = R$.*

**Example 3.5.** *From the Example 3.2, we have that $I_2, I_3, I_4, I_5$ are $E$-ideals of $R$. Clearly, $I_4 \vee I_5 = R$. Therefore $I_4$ and $I_5$ are co-maximal. Also, we have $I_2 \vee I_3 \neq R$. Therefore $I_2, I_3$ are not co-maximal.*

**Theorem 3.18.** *In an ADL $R$, the following are equivalent:*

(1) *for any $a, b \in R$ with $a \wedge b \in E, (a, E) \vee (b, E) = R$*
(2) *for any $a, b \in R, (a, E) \vee (b, E) = (a \wedge b, E)$*
(3) *any two distinct minimal prime $E$-ideals are co-maximal*
(4) *every prime $E$-ideal contains a unique minimal prime $E$-ideal*
(5) *for any prime $E$-ideal $P$, $\mathcal{O}^E(P)$ is prime.*

*Proof.* (1) $\Rightarrow$ (2) Assume (1). Let $a, b \in R$ with $x \in (a \wedge b, E)$. Then $x \wedge (a \wedge b) \in E$ and hence $(x \wedge a) \wedge (x \wedge b) \in E$. By (1), we have that $(x \wedge a, E) \vee (x \wedge b, E) = R$. That implies $x \in (x \wedge a, E) \vee (x \wedge b, E)$. Then there exists $r \in (x \wedge a, E)$ and $s \in (x \wedge b, E)$ such that $x = r \vee s$. Since $r \in (x \wedge a, E)$, $s \in (x \wedge b, E)$ we get that $r \wedge x \in (a, E)$ and $s \wedge x \in (b, E)$. That implies $(x \wedge r) \vee (x \wedge s) \in (a, E) \vee (b, E)$ and hence $x \wedge (r \vee s) \in (a, E) \vee (b, E)$. Since $x = r \vee s$, we get that $x \in (a, E) \vee (b, E)$. Therefore $(a \wedge b, E) \subseteq (a, E) \vee (b, E)$. Since $(a, E) \vee (b, E) \subseteq (a \wedge b, E)$, we get that $(a, E) \vee (b, E) = (a \wedge b, E)$, for all $a, b \in R$.
(2) $\Rightarrow$ (3) Assume (2). Let $M$ and $N$ be two distinct minimal prime $E$-ideals of $R$. Choose elements $x, y \in R$ such that $x \in M \setminus N$ and $y \in N \setminus M$. Since $M$ and $N$ are minimal, $x \wedge a \in E$, $y \wedge b \in E$, for some $a \notin M$, $b \notin N$. That implies $x \wedge a \wedge y \wedge b \in E$ and hence $R = (x \wedge a \wedge y \wedge b, E)$. By (2), we get that $(x \wedge b, E) \vee (a \wedge y, E) = R$. Since $a \notin M$ and $y \notin M$, we get that $a \wedge y \notin M$. That implies $(a \wedge y, E) \subseteq M$. Similarly, we have that $(x \wedge b, E) \subseteq N$. That implies $((x \wedge b) \wedge (a \wedge y), E) \subseteq M \vee N$ and hence $R = M \vee N$. Therefore $M$ and $N$ are co-maximal.
(3) $\Rightarrow$ (4) Assume (3). Let $M$ be a prime $E$-ideal of $R$. Suppose $M$ contains two distinct minimal prime $E$-ideals, say $N_1$ and $N_2$. By (3), we get that $R = N_1 \vee N_2 \subseteq M$, we get a contradiction. Therefore every prime $E$-ideal contains a unique minimal prime $E$-filter.
(4) $\Rightarrow$ (5) Assume that every prime $E$-ideal of $R$ contains a unique minimal prime $E$-ideal. Then by Corollary 3.8, we get that $\mathcal{O}^E(P)$ is a prime $E$-ideal.
(5) $\Rightarrow$ (1) Assume (5). Let $a, b \in R$ be such that $a \wedge b \in E$. Suppose $(a, E) \vee (b, E) \neq R$. Then there exists a maximal $E$-ideal $M$ such that $(a, E) \vee (b, E) \subseteq M$. That implies $(a, E) \subseteq M$ and $(b, E) \subseteq M$. That implies $a \notin \mathcal{O}^E(M)$ and $b \notin \mathcal{O}^E(M)$. Since $\mathcal{O}^E(M)$ is prime, we get $a \wedge b \notin \mathcal{O}^E(M)$. So that $E \nsubseteq \mathcal{O}^E(M)$, which is a contradiction. Therefore $(a, E) \vee (b, E) = R$. □

**Theorem 3.19.** *In an ADL $R$ with maximal elements, the following conditions are equivalent:*

(1) *$R$ is $E$-normal*

(2) *for any two distinct maximal filters* $G_1$ *and* $G_2$ *of* $R$ *with* $G_1 \cap E = \emptyset$, $G_2 \cap E = \emptyset$ *there exist* $a \notin G_1$ *and* $b \notin G_2$ *such that* $a \vee b$ *is maximal*

(3) *for any maximal filter* $G$ *with* $G \cap E = \emptyset$, $G$ *is the unique maximal filter containing* $R \setminus \mathcal{O}^E(P)$.

*Proof.* (1) $\Rightarrow$ (2) Assume that $R$ is $E$-normal.
Let $G_1$ and $G_2$ be two distinct maximal filters of $R$ with $G_1 \cap E = \emptyset, G_2 \cap E = \emptyset$. Then $R \setminus G_1$ and $R \setminus G_2$ are distinct minimal prime $E$-ideals of $R$. By our assumption, we get $R \setminus G_1$ and $R \setminus G_2$ are co-maximal. That is, $(R \setminus G_1) \vee (R \setminus G_2) = R$. Then, there exist $a \in R \setminus G_1$ and $b \in R \setminus G_2$ such that $a \vee b$ is maximal.
(2) $\Rightarrow$ (3) Assume (2). Let $G$ be any maximal filter of $R$ with $G \cap E = \emptyset$ and $R \setminus \mathcal{O}^E(P) \subseteq G$. Let $G_1$ be any maximal filter of $R$ with $G_1 \cap E = \emptyset$ and $R \setminus \mathcal{O}^E(P) \subseteq G_1$. We prove that $G = G_1$. Suppose $G \neq G_1$. By our assumption, there exists $a \notin G$ and $b \notin G_1$ such that $a \vee b$ is maximal. That implies $a, b \notin R \setminus \mathcal{O}^E(P)$. So that $a, b \in \mathcal{O}^E(P)$. This implies that $a \vee b \in \mathcal{O}^E(P)$. Therefore $\mathcal{O}^E(P) = R$, which is a contradiction. We conclude that $G = G_1$.
(3) $\Rightarrow$ (1) For any maximal filter $G$ with $G \cap E = \emptyset$, $G$ is the unique maximal filter containing $R \setminus \mathcal{O}^E(P)$. Let $P$ be a prime $E$-ideal of $R$. Suppose $P$ contains two minimal prime $E$-ideals say $Q_1$ and $Q_2$. That is, $Q_1 \subseteq P$ and $Q_2 \subseteq P$. That implies $\mathcal{O}^E(P) \subseteq \mathcal{O}^E(Q_1)$ and $\mathcal{O}^E(P) \subseteq \mathcal{O}^E(Q_2)$. We get $P \subseteq \mathcal{O}^E(Q_1)$ and $P \subseteq \mathcal{O}^E(Q_2)$. So that $Q_2 \subseteq Q_1$ and $Q_1 \subseteq Q_2$. This concludes that $Q_1 = Q_2$. ☐

Let $F$ be a filter of $R$. For any $x, y \in R$, define a binary relation $\phi_F$ on $R$ as $\phi_F = \{(x, y) \in R \times R \mid x \wedge a = y \wedge a, \text{ for some } a \in F\}$.

**Proposition 3.5.** *For any filter* $F$ *of an associative ADL* $R$, $\phi_F$ *is a congruence relation on* $R$.

For any ADL $R$, it can be easily verified that the quotient $R/\phi_F$ is also an ADL with respect to the following operations: $[a]_{\phi_F} \wedge [b]_{\phi_F} = [a \wedge b]_{\phi_F}$ and $[a]_{\phi_F} \vee [b]_{\phi_F} = [a \vee b]_{\phi_F}$ where $[a]_{\phi_F}$ is the congruence class of $a$ modulo $\phi_F$ . It can be routinely verified that the mapping $\Phi : R \to R/\phi_F$ defined by $\Phi(a) = [a]_{\phi_F}$ is a homomorphism.

**Theorem 3.20.** *In an ADL* $R$, *we have the following:*

(1) *if* $x$ *is a dual dense element of* $R$, *then* $[x]_{\phi_F}$ *is a dual dense element of* $R/\phi_F$

(2) *if* $G$ *is a* $E$*-ideal of* $R/\phi_F$, *then* $\Phi^{-1}(G)$ *is a* $E$*-ideal of* $R$

(3) *if* $G$ *is a prime* $E$*-ideal of* $R/\phi_F$, *then* $\Phi^{-1}(G)$ *is a prime* $E$*-ideal of* $R$.

**Definition 3.9.** *Let* $F$ *be a filter of an ADL* $R$. *For any ideal* $G$ *of* $R$, *define* $\widetilde{G} = \{[a]_{\phi_F} \mid a \in G\}$.

The following result can be proved easily.

**Lemma 3.6.** *Let* $G$ *be an* $E$*-ideal of* $R$. *Then* $\widetilde{G}$ *is an* $E$*-ideal of* $R/\phi_F$.

**Proposition 3.6.** *Let* $G$ *be a prime* $E$*-ideal and* $F$ *a filter of an ADL* $R$ *such that* $G \cap F = \emptyset$. *We have the following:*

(1) *$x \in G$ if and only if $[x]_{\phi_F} \in \widetilde{G}$*
(2) $\widetilde{G} \cap \widetilde{F} = \emptyset$
(3) *if $G$ is a prime $E$-ideal of $R$, then $\widetilde{G}$ is a prime $E$-ideal of $R/_{\phi_F}$.*

*Proof.* (1) Assume that $x \in G$. Then we have $[x]_{\phi_F} \in \widetilde{G}$. Conversely assume that $[x]_{\phi_F} \in \widetilde{G}$. Then there exists $y \in G$ such that $[x]_{\phi_F} = [y]_{\phi_F}$. That implies $(x, y) \in \phi_F$. So there exists $a \in F$ such that $x \wedge a = y \wedge a \in G$. Since $G \cap F = \emptyset$, we get $a \notin G$. Since $x \wedge a \in G$ and $a \notin G$, we get that $x \in G$.
(2) Suppose $\widetilde{G} \cap \widetilde{F} \neq \emptyset$. Then choose an element $x \in R$ such that $[x]_{\phi_F} \in \widetilde{G} \cap \widetilde{F}$. Then $[x]_{\phi_F} \in \widetilde{G}$ and $[x]_{\phi_F} \in \widetilde{F}$. Since $[x]_{\phi_F} \in \widetilde{G}$ and by (1), we get $x \in G$. Since $[x]_{\phi_F} \in \widetilde{F}$, there exists $y \in F$ such that $[x]_{\phi_F} = [y]_{\phi_F}$. Then $(x, y) \in \phi_F$. So there exist $a \in F$ such that $x \wedge a = y \wedge a$. Since $y \wedge a \in F$, we get that $x \wedge a \in F$. Since $x \in G$, we have that $x \wedge a \in G \cap F$. That implies $G \cap F \neq \emptyset$, we get a contradiction. Hence $\widetilde{G} \cap \widetilde{F} = \emptyset$.
(3) Clearly, we have that $\widetilde{G}$ is a proper ideal of $R/_{\phi_F}$. Let $[x]_{\phi_F} \in \widetilde{E}$. Then $x \in E \subseteq G$. That implies $[x]_{\phi_F} \in G$ and hence $\widetilde{G}$ is an $E$-ideal of $R/_{\phi_F}$. Let $[x]_{\phi_F}, [y]_{\phi_F} \in R/_{\phi_F}$ such that $[x]_{\phi_F} \wedge [y]_{\phi_F} \in \widetilde{G}$. Then $[x \wedge y]_{\phi_F} \in \widetilde{G}$. By (1) we have that $x \wedge y \in G$. Since $G$ is prime, we get that $x \in G$ *or* $y \in G$ Again by(1) we get that $[x]_{\phi_F} \in \widetilde{G}$ or $[y]_{\phi_F} \in \widetilde{G}$. Hence $\widetilde{G}$ is a prime $E$-ideal in $R/_{\phi_F}$. □

**Proposition 3.7.** *Let $F$ be a filter of an ADL $R$. Then there is an order isomorphism of the set of all prime $E$-ideals of $R$ disjoint from $F$ onto the set of all prime $E$-ideals of $R/_{\phi_F}$ .*

*Proof.* Let $G$ and $H$ be two prime $E$-ideals of $R$ such that $G \cap F = \emptyset$ and $H \cap F = \emptyset$. Then by Proposition 3.6(1), we get that $G \subseteq H$ if and only if $\widetilde{G} \subseteq \widetilde{H}$. Let $G$ be a prime $E$-ideal of $R$ with $G \cap F = \emptyset$. Then by Proposition 3.6(3), we get that $\widetilde{G}$ is a prime $E$-ideal of $R/_{\phi_F}$. Let $Q$ be a prime $E$-ideal of $R/_{\phi_F}$. Consider $G = \{a \in R | [a]_{\phi_F} \in Q\}$. Since $Q$ is a $E$-ideal of $R/_{\phi_F}$, we get that $G$ is a $E$-ideal of $R$. Let $a, b \in R$ with $a \wedge b \in G$. Then $[a]_{\phi_F} \wedge [b]_{\phi_F} = [a \wedge b]_{\phi_F} \in Q$. Since $Q$ is prime, we get $[a]_{\phi_F} \in Q$ or $[b]_{\phi_F} \in Q$. Therefore $a \in G$ or $b \in G$. Hence $G$ is a prime $E$-ideal of $R$. Clearly $\widetilde{G} = Q$. Suppose $G \cap F \neq \emptyset$. Then choose an element $s \in G \cap F$. That implies $[s]_{\phi_F} \in Q$ and $s \in F$. Let $[b]_{\phi_F} \in R/_{\phi_F}$. Since $s \in F$ and $b \wedge s = b \wedge s \wedge s$, we get that $(b, b \wedge s) \in F$. That implies $[b]_{\phi_F} = [b \wedge s]_{\phi_F} = [b]_{\phi_F} \wedge [s]_{\phi_F} \in Q$. Therefore $[b]_{\phi_F} \in Q$. and hence $R/_{\phi_F} = Q$, which is a contradiction. Thus $G \cap F = \emptyset$. □

**Corollary 3.10.** *Let $R$ be an ADL. Then the above map induces a one-to-one correspondence between the set of all minimal prime $E$-ideals of $R$ which are disjoint from $F$ and the set of all minimal prime $E$-ideals of $R/_{\phi_F}$.*

**Theorem 3.21.** *For any filter $F$ of an ADL $R$, the following are equivalent:*

(1) *any two distinct minimal prime $E$-ideals of $R$ are co-maximal*
(2) *any two distinct minimal prime $E$-ideals of $R/_{\phi_F}$ are co-maximal.*

*Proof.* (1) ⇒ (2) Assume (1). Let $G_1, G_2$ be two distinct minimal prime $E$-ideals of $R/_{\phi_F}$. Then by the corollary 3.10, there exist two minimal prime $E$-ideals $H_1$ and $H_2$ of $R$ such that $H_1 \cap F = \emptyset$ and $H_2 \cap F = \emptyset$. Also $\widetilde{H_1} = G_1$ and $\widetilde{H_2} = G_2$. Since $G_1$ and $G_2$ are distinct, we get that $H_1$ and $H_2$ are distinct. By the assumption, we have $H_1 \vee H_2 = R$. Let $a \in R$. There exist $a_1 \in H_1$ and $a_2 \in H_2$ such that $a = a_1 \vee a_2$. Since $a_1 \in H_1$ and $a_2 \in H_2$ we get $[a_1]_{\phi_F} \in \widetilde{H}_1 = G_1$ and $[a_2]_{\phi_F} \in \widetilde{H}_2 = G_2$. Now, $[a]_{\phi_F} = [a_1 \vee a_2]_{\phi_F} = [a_1]_{\phi_F} \vee [a_2]_{\phi_F} \in G_1 \vee G_2$. That implies $[a]_{\phi_F} \in G_1 \vee G_2$, for all $a \in R$.Therefore $G_1 \vee G_2 = R/_{\phi_F}$.

(2) ⇒ (1) Assume (2). Let $P$ be a prime $E$-ideal of $R$. Suppose $P$ contains two distinct minimal prime $E$-ideals, say $G_1$ and $G_2$. Consider $K = R \setminus P$. Clearly $K$ is a filter of $R$ and $G_1 \cap K = \emptyset = G_2 \cap K$. By Corollary 3.10, we get that $\widetilde{G_1}$ and $\widetilde{G_2}$ are distinct minimal prime $E$-ideals of $R/_{\phi_F}$ such that $\widetilde{G_1}, \widetilde{G_2} \subseteq \widetilde{P}$. That implies $\widetilde{P}$ is containing two distinct minimal prime $E$-ideals of $R/_{\phi_F}$ , which is a contradiction. Hence $P$ contains a unique minimal prime $E$-ideal. By Theorem 3.18, any two distinct minimal prime $E$-ideals of $R$ are co-maximal. □

## 4. On the space prime $E$-ideals

In this section, some topological properties of the space of all prime $E$-ideals and the space of all minimal prime $E$-ideals of an ADL are studied.

Let us denote the set of all prime $E$-ideals of an ADL $R$ by $Spec_I^E(R)$. For any $A \subseteq R$, define $\alpha(A) = \{P \in Spec_I^E(R) | A \nsubseteq P\}$ and for any $a \in R$, $\alpha(a) = \{P \in Spec_I^E(R) | a \notin P\}$. Then we have the following result whose proof is straightforward.

**Lemma 4.1.** *Let $R$ be an ADL and $a, b \in R$. Then the following conditions hold:*

(1) $\bigcup_{a \in R} \alpha(a) = Spec_I^E(R)$
(2) $\alpha(a) \cap \alpha(b) = \alpha(a \wedge b)$
(3) $\alpha(a) \cup \alpha(b) = \alpha(a \vee b)$
(4) $\alpha(a) = \emptyset$ *if and only if* $a \in E$
(5) $\alpha(a) = Spec_I^E(R)$ *if and only if* $a \in \mathcal{M}$.

From the above result, it can be easily observed that the collection $\{\alpha(a) | a \in R\}$ forms a base for a topology on $Spec_I^E(R)$. The topology generated by this base is precisely $\{\alpha(A \mid A \subseteq R\}$ and is called the hull-kernel topology on $Spec_I^E(R)$. Under this topology, we have the following result.

**Theorem 4.1.** *In an ADL $R$, we have the following:*

(1) *for any* $a \in R, \alpha(a)$ *is compact in* $Spec_I^E(R)$
(2) *if $C$ is a compact open subset of* $Spec_I^E(R)$, *then* $C = \alpha(a)$ *for some* $a \in R$
(3) $Spec_I^E(R)$ *is a* $T_0$*-space*
(4) *the map* $a \mapsto \alpha(a)$ *is an epimorphism from $R$ onto the lattice of all compact open subsets of* $Spec_I^E(R)$.

*Proof.* (1) Let $a \in R$. Let $X \subseteq R$ be such that $\alpha(a) \subseteq \bigcup_{x \in X} \alpha(x)$. Let $J$ be a $E$-ideal generated by the set $X$. Suppose $a \notin J$. Then there exists a prime $E$-ideal $P$ such that $J \subseteq P$ and $a \notin P$. Since $X \subseteq J \subseteq P$, we get $P \notin \alpha(x)$ for all $x \in X$. Since $a \notin P$, we get $P \in \alpha(a)$, which is a contradiction. Hence $a \in J$. So we can write $a = (\bigvee_{i=1}^{n} x_i) \wedge a$ for some $x_1, x_2, \ldots, x_n \in X$ and $n \in N$. Then, we get $\alpha(a) = \alpha((\bigvee_{i=1}^{n} x_i) \wedge a) \subseteq \alpha(\bigvee_{i=1}^{n} x_i) = \bigcup_{i=1}^{n} \alpha(x_i)$ which is finite subcover for $\alpha(a)$. Therefore $\alpha(a)$ is compact.

(2) Let $C$ be a compact open subset of $Spec_I^E(R)$. Since $C$ is open, we get $C = \bigcup_{x \in X} \alpha(x)$ for some $X \subseteq R$. Since $C$ is compact, there exist $x_1, x_2, \ldots, x_n \in X$ such that $C = \bigcup_{i=1}^{n} \alpha(x_i) = \alpha(\bigvee_{i=1}^{n})$ Therefore $C = \alpha(x)$ for some $x \in R$.

(3) Let $P$ and $Q$ be two distinct prime $E$-ideals of $R$. Without loss of generality, assume that $P \nsubseteq Q$. Choose $x \in R$ such that $x \in P$ and $x \notin Q$. Hence $P \notin \alpha(x)$ and $Q \in \alpha(x)$. Therefore $Spec_I^E(R)$ is a $T_0$-space.

(4) It can be obtained from (1), (2) and by the above lemma. □

**Proposition 4.1.** *In an ADL $R$, the following are equivalent:*

1. *$Spec_I^E(R)$ is a Hausdorff space*
2. *for each $P \in Spec_I^E(R)$, $P$ is the unique member of $Spec_I^E(R)$ such that $\mathcal{O}^E(P) \subseteq P$*
3. *every prime $E$-ideal is minimal*
4. *every prime $E$-ideal is maximal.*

*Proof.* (1) $\Rightarrow$ (2) Assume (1). Let $P \in Spec_I^E(R)$. Clearly $\mathcal{O}^E(P) \subseteq P$. Suppose $Q \in Spec_I^E(R)$ such that $Q \neq P$ and $\mathcal{O}^E(P) \subseteq Q$. Since $Spec_E^I(R)$ is Hausdorff, there exists $a, b \in R$ such that $P \in \alpha(a), Q \in \alpha(b)$ and $\alpha(a \wedge b) = \alpha(a) \cap \alpha(b) = \emptyset$. Hence $a \notin P, b \notin Q$ and $a \wedge b \in E$. Therefore $b \in \mathcal{O}^E(P) \subseteq Q$, which is a contradiction to that $b \notin Q$. Hence $P = Q$. Therefore $P$ is the unique member of $Spec_I^E(R)$ such that $\mathcal{O}^E(P) \subseteq P$.

(2) $\Rightarrow$ (3) Assume (2). Let $P$ be a prime $E$-ideal of $R$. Let $Q$ be a prime $E$-ideal in $R$ such that $Q \subseteq P$. Hence $\mathcal{O}^E(Q) \subseteq Q \subseteq P$. Therefore $P$ is a minimal prime $E$-ideal of $R$.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) Assume (4). Let $P$ and $Q$ be two distinct elements of $Spec_I^E(R)$. Hence $\mathcal{O}^E(Q) \nsubseteq P$. Choose $a \in \mathcal{O}^E(Q)$ such that $a \notin P$. Since $a \in \mathcal{O}^E(Q)$, there exists $b \notin Q$ such that $a \in (b, E)$. Hence $a \wedge b \in E$. Thus it yields, $P \in \alpha(a), Q \in \alpha(b)$. Since $a \wedge b \in E$, we get that $\alpha(a) \cap \alpha(b) = \alpha(a \wedge b) = \emptyset$. Therefore $Spec_I^E(R)$ is Hausdorff. □

**Theorem 4.2.** *For any $E$-ideal $G$ of an ADL $R$, $(G, E) = \bigcap\{P \in Spec_I^E(R) \mid G \nsubseteq P\}$.*

*Proof.* Let $G$ be an $E$-ideal of $L$. Consider $K = \bigcap\{P \in Spec_I^E(R) \mid G \nsubseteq P\}$. Let $P \in \alpha(G)$. Then $G \nsubseteq P$. Since $G \cap (G, E) = E \subseteq P$ and $P$ is prime, we get $(G, E) \subseteq P$. Hence every prime $E$-ideal

$P$ of $R$ such that $G \nsubseteq P$ contains $(G, E)$. Therefore $(G, E) \subseteq K$. Let $x \notin (G, E)$. Then there exists $y \in G$ such that $x \wedge y \notin E$. Let $\mathcal{K} = \{G \mid G \text{ is an } E-\text{ideal of } L \text{ and } x \wedge y \notin G\}$. Clearly, $E \in \mathcal{K}$ and so $P = \emptyset$. Clearly, $(\mathcal{K}, \subseteq)$ is a partially ordered set and it satisfies the hypothesis of the Zorn's lemma, $\mathcal{K}$ has a maximal element, say $N$. Then $N$ is an $E$-ideal of $R$ and $x \wedge y \notin N$. Therefore $x \notin N$ and $y \notin N$. Since $y \in G$, we get $G \nsubseteq N$. We now show that $N$ is prime. Let $a, b \in R$ with $a \notin N$ and $b \notin N$. Then $N \subsetneq N \vee (a)^E$ and $N \subsetneq N \vee (b)^E$. By the maximality of $N$, we get $x \wedge y \in N \vee (a)^E$ and $x \wedge y \in N \vee (b)^E$. Hence, $x \wedge y \in \{N \vee (a)^E\} \cap \{N \vee (b)^E\} = N \vee \{(a)^E \cap (b)^E\} = N \vee (a \wedge b)^E$. If $a \wedge b \in N$, then $x \wedge y \in N$ which is a contradiction. Thus $N$ is a prime $E$-ideal of $R$ such that $G \nsubseteq N$ and $x \notin N$. Therefore $x \notin K$. Hence $K \subseteq (G, E)$. □

**Corollary 4.1.** *For any ADL $R$ and $a \in R, (a, E) = \bigcap\{P \in Spec_I^E(R) \mid a \notin P\}$.*

Let $Min_I^E(R)$ denote the set of all minimal prime $E$-ideals of ADL $R$. For any $a \in R$, write $\alpha_m(x) = \alpha(x) \cap Min_I^E(R)$.

**Theorem 4.3.** *For any ADL $R$, the following conditions hold in $R$ :*

1. *Every prime $E$-ideals contains a minimal prime $E$-ideal*
2. $\bigcap_{P \in Min_I^E(R)} P = E$
3. *for any subset $A$ with $E \subseteq A, (A, E) = \bigcap_{P \in \alpha_m(A)} (P)$.*

*Proof.* (1) Let $P$ be a prime $E$-ideal of $R$. Consider $X = \{N \in Spec_I^E(R) \mid N \subseteq P\}$. Clearly $X$ is a partially ordered set under set inclusion and hence it satisfies the hypothesis of the Zorn's lemma, $X$ has a minimal element say $M$. Clearly $M$ will be the required minimal prime $E$-ideal of $R$.
(2) Since $E$ is contained in every minimal prime $E$-ideal of $R$ and so contained in the intersection of all minimal prime $E$-ideals. Let $x \notin E$. Then there exists a prime $E$-ideal $P$ of $L$ such that $x \notin P$. By (1), there exists a minimal prime $E$-ideal of $R$ such that $M \subseteq P$. Since $x \notin P$, we get $x \notin M$. That implies $M$ is a minimal prime $E$-ideal of $R$ such that $x \notin M$. Hence $x$ is not in the intersection of all minimal prime. Thus intersection of all minimal prime $E$-ideals of $R$ is equal to $E$.
(3) Let $P \in Min_I^E(R)$ such that $A \nsubseteq P$. Choose $x \in A$ such that $x \notin P$. Then $(A, E) \subseteq (x, E) \subseteq P$. That implies $(A, E)$ is contained in every minimal prime $E$-ideal of $R$ such that $A \nsubseteq P$. Hence $(A, E) \subseteq \bigcap_{P \in \alpha_m(A)} (P)$. Let $x \notin (A, E)$. Then $x \wedge y \notin E$, for some $y \in A$. By the condition (2), there exists a minimal prime $E$-ideal $P$ of $R$ such that $x \wedge y \notin P$. That implies $x \notin P$ and $y \notin P$. Therefore $x \notin \bigcap_{P \in \alpha_m(A)} P$ and hence $(A, E) = \bigcap_{P \in \alpha_m(A)} P$. □

**Lemma 4.2.** *For any $a, b \in R$, we have following:*

1. $(a, E) \subseteq (b, E)$ *if and only if* $\alpha_m(b) \subseteq \alpha_m(a)$
2. $\alpha_m(a) = \emptyset$ *if and only if* $a \in E$
3. $\alpha_m(a) = Min_I^E(R)$ *if and only if* $(a, E) = E$.

*Proof.* (1) Let $a, b \in R$. Assume that $(a, E) \subseteq (b, E)$. Let $P \in \alpha_m(b)$ Then $b \notin P$. That implies $(a, E) \subseteq (b, E) \subseteq P$. Therefore $a \notin P$ and hence $P \in \alpha_m(a)$. Thus $\alpha_m(b) \subseteq \alpha_m(a)$. Conversely, assume that $\alpha_m(b) \subseteq \alpha_m(a)$. Now, $(a, E) = \bigcap_{P\in\alpha_m(a)} P \subseteq \bigcap_{P\in\alpha_m(b)} P = (b, E)$. Hence $(a, E) \subseteq (b, E)$.
(2) Suppose $Min_I^E(R) = \emptyset$. Then $a \in P$ for all $P \in Min_I^E(R)$. That implies $a \in \bigcap_{P\in Min_I^E(R)} P$. Since $a \in \bigcap_{P\in Min_I^E(R)} P = E$, we get $a \in E$. The converse is clear.
(3) Assume $\alpha_m(a) = Min_I^E(R)$. Then $(a, E) = \bigcap_{P\in\alpha_m(a)} P = \bigcap_{P\in Min_I^E(R)} P = E$. Therefore$(a, E) = E$. Conversely, assume $(a, E) = E$. Then $(a, E) = E \subseteq P$. That implies $a \notin P$, for all $P \in Min_I^E(R)$. Therefore $\alpha_m(a) = Min_I^E(R)$. □

For any $E$-ideal $G$ of an ADL $R$, define $\beta_m(G) = \{P \in Min_I^E(R) \mid G \subseteq P\}$.

**Lemma 4.3.** *Let $G$ be an $E$-ideal of an ADL $R$. If $\beta_m(G) = \emptyset$, then $(G, E) = E$.*

*Proof.* Let $\beta_m(G) = \emptyset$. Then $\beta_m(G) = Min_I^E(R)$. That implies $(G, E) = \bigcap_{P\in\alpha_m(F)} P \subseteq \bigcap_{P\in Min_I^E(R)} P = E$. Therefore $(G, E) = E$. □

For any ADL $R$, define $K = \{x \in R \mid (x, E) = E\}$.

**Lemma 4.4.** *For any ADL $R, K$ is a filter of $R$.*

*Proof.* Clearly, we have that for any $m \in \mathcal{M}$, $m \in K$. Let $x, y \in K$. Then $((x\wedge y, E), E) = ((x, E), E)\cap((y, E), E) = (E, E) \cap (E, E) = R \cap R = R$. That implies $((x \wedge y), E) = (R, E) = E$. Therefore $x\wedge y \in K$. Let $x \in K$. Then $(x, E) = E$. Let $y \in R$. Now, $(x\vee y, E) = (x, E)\cap(y, E) = E\cap(y, E) = E$. Therefore $x \vee y \in K$. Hence $K$ is a filter of $R$. □

**Theorem 4.4.** *Let $G$ be an $E$-ideal of an ADL $R$. Then $Min_I^E(R)$ is compact if and only if $\beta_m(G) = \emptyset$ implies $G \cap K \neq \emptyset$.*

*Proof.* Assume that $Min_I^E(R)$ is compact. Let $G$ be an $E$-ideal $R$ such that $\beta_m(G) = \emptyset$. Then $\alpha_m(G) = Min_I^E(R)$. Since $Min_I^E(R)$ is compact, there exists $a \in G$ such that $\alpha_m(a) = Min_I^E(R)$. That implies $(a, E) = E$. Therefore $a \in K$ and hence $G \cap K \neq \emptyset$. Conversely, assume that for any $E$-ideal $G$ of $R, \beta_m(G) = \emptyset$ implies $G \cap K \neq \emptyset$. Let $A \subseteq R$ be such that $Min_I^E(R) = \bigcup_{a\in A} \alpha_m(A) = \alpha_m(A) = \alpha_m(G)$ where $G = A^E$. Since $Min_I^E(R) = \alpha_m(G)$, we get $\beta_m(G) = \emptyset$. By the assumption, we get $G \cap E \neq \emptyset$. Choose $d \in G \cap K$. Since $d \in G$ and $G = A^E$, there exists $a_1, a_2, \ldots, a_n \in A$ such that $d = (a_1 \vee a_2 \vee \ldots \vee a_n) \wedge d$. Since $d \in E$, $Min_I^E(R) = \alpha_m(d) \subseteq \alpha_m\Big(\bigvee_{i=1}^{n} a_i\Big) = \bigcup_{i=1}^{n} \alpha_m(a_i)$. Hence $Min_I^E(R)$ is compact. □

**Theorem 4.5.** *Let $R$ be an ADL. For any $Y \subseteq Min_I^E(R)$, the closure of $Y$ in $Min_I^E(R)$ is $\beta_m(\bigcap_{P\in Y} P)$ and, in particular, $\overline{\alpha_m(F)} = \beta_m((G, E))$, for any $E \subseteq G \subseteq R$.*

*Proof.* Let $Y \subseteq Min_I^E(R)$. Then $\overline{Y}$ in $Min_I^E(R) = \{\overline{Y}$ in $Spec_I^E(R)\} \cap Min_I^E(R) = H(\bigcap_{P \in Y} P) \cap Min_I^E(R) = \beta_m(\bigcap_{P \in Y} P)$. In particular, for any $E \subseteq G \subseteq R$, we have $\overline{\alpha_m(G)} = \beta_m(\bigcap_{P \in \alpha_m(G)} P) = \beta_m(\bigcap_{I \nsubseteq P,\ P \in Min_I^E(R)} P) = \beta_m((F, E))$. □

**Proposition 4.2.** *Let $F, G$ be two $E$-ideals of an ADL $R$. Then the following are equivalent:*

(1) $G \subseteq (F, E)$
(2) $G \cap F = E$
(3) $\alpha_m(G) \cap \alpha_m(F) = \emptyset$.

*Proof.* (1) ⇒ (2) Assume that $G \subseteq (F, E)$. Then $G \cap F \subseteq (F, E) \cap F = E$. Therefore $G \cap F = E$.
(2) ⇒ (3) Assume that $G \cap F = E$. Let $P \in \alpha_m(G) \cap \alpha_m(F) = \alpha_m(G \cap F)$. Then $E = G \cap F \nsubseteq P$, which is a contradiction. Therefore $\alpha_m(G) \cap \alpha_m(F) = \emptyset$.
(3) ⇒ (1) Assume that $\alpha_m(G) \cap \alpha_m(F) = \emptyset$. Let $x \in G$. Suppose $x \notin (F, E)$. Then there exists $y \in F$ such that $x \wedge y \notin E$. Then there exists $P \in Min_I^E(R)$ such that $x \wedge y \notin P$. That implies $x \notin P$ and $y \notin P$. Hence $G \nsubseteq P$ and $F \nsubseteq P$. Therefore $P \in \alpha_m(G)$ and $P \in \alpha_m(F)$. Therefore $P \in \alpha_m(G) \cap \alpha_m(F)$, which is a contradiction. So $x \in (F, E)$. Therefore $G \subseteq (F, E)$. □

**Corollary 4.2.** *Let $G$ be an $E$-ideal of an ADL $R$ and $x \in R$. Then $x \in (G, E)$ if and only if $\alpha_m(x) \cap \alpha_m(G) = \emptyset$.*

*Proof.* By taking $G = \{x\}$, in the above proposition. □

**Theorem 4.6.** *Every open subset of $Min_I^E(R)$ is closed if and only if for any $E$-ideal of $R$, $(G, E) = E$ implies $\beta_m(G) = \emptyset$.*

*Proof.* Assume that every open set of $Min_I^E(R)$ is closed. Let $G$ be an $E$-ideal of $R$. Then $\beta_m(G)$ is an open set in $Min_I^E(R)$. Now, $\beta_m(G) \neq \emptyset$. Then there exists $x \in R \setminus E$ such that $\alpha_m(x) \subseteq \beta_m(G)$. That implies $\alpha_m(x) \cap \alpha_m(G) = \emptyset$. Therefore $x \in (G, E)$ and $x \notin E$. Hence $(G, E) \neq E$. Thus $(G, E) = E$, which gives $\beta_m(G) = \emptyset$. Conversely, assume that the condition holds. Let $H$ be an open subset of $Min_I^E(R)$. Then $H = \alpha_m(G)$, for some $E$-ideal $G$ of $L$. By Theorem 4.5, we have $\overline{\alpha_m(G)} = \beta_m((G, E))$. It is enough to show that $\beta_m((G, E)) = \alpha_m(G)$. Since $((G \vee (G, E)), E) = E$, by the assumption, we get $\beta_m(G \vee (G, E)) = \emptyset$. Now, for any $P \in Min_I^E(R)$, we have $P \in \alpha_m(G) \Leftrightarrow G \nsubseteq P \Leftrightarrow (G, E) \subseteq P \Leftrightarrow P \in \beta_m(G)$. Hence $\alpha_m(G) = \beta_m(G)$. Therefore $H$ is closed in $Min_I^E(R)$. □

**Theorem 4.7.** *In an ADL $R$, $Min_I^E(R)$ is a Hausdorff space.*

*Proof.* Let $P$ and $Q$ be distinct elements of $Min_I^E(R)$. Then there exists $a \in P$ such that $a \notin Q$. Since $P$ is minimal, we get $(a, E) \nsubseteq P$. Then there exists $b \in (a, E)$ such that $b \notin P$. That implies $a \wedge b \in E$ and hence $\alpha_m(a) \cap \alpha_m(b) = \emptyset$. Since $a \notin Q$ and $b \notin P$, we get $Q \in \alpha_m(a)$ and $P \in \alpha_m(b)$. Therefore $Min_I^E(R)$ is a Hausdorff space. □

**Acknowledgment:** This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF66-UoE017).

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

[1] G. Birkhoff, Lattice theory, Colloq. Publ. XXV, Amer. Math. Soc. Providence, 1967.

[2] W.H. Cornish, Normal lattices, J. Aust. Math. Soc. 14 (1972), 200-215. https://doi.org/10.1017/s1446788700010041.

[3] G. Grätzer, General lattice theory, Birkhäuser, Basel, 1978. https://doi.org/10.1007/978-3-0348-7633-9.

[4] A.P. Phaneendra Kumar, M. Sambasiva Rao, K. Sobhan Babu, Generalized prime $D$-filters of distributive lattices, Arch. Math. 57 (2021), 157-174. https://doi.org/10.5817/AM2021-3-157.

[5] G.C. Rao, Almost distributive lattices, Doctoral Thesis, Department of Mathematics, Andhra University, Visakhapatnam, 1980.

[6] G.C. Rao, S. Ravi Kumar, Minimal prime ideals in an ADL, Int. J. Contemp. Math. Sci. 4 (2009), 475-484

[7] G.C. Rao, S. Ravi Kumar, Normal almost distributive lattices, Southeast Asian Bull. Math. 32 (2008), 831-841.

[8] G.C. Rao, M. Sambasiva Rao, Annulets in almost distributive lattices, Eur. J. Pure Appl. Math. 2 (2009), 58-72.

[9] U.M. Swamy, G.C. Rao, Almost distributive lattices, J. Aust. Math. Soc. 31 (1981), 77-91. https://doi.org/10.1017/s1446788700018498.