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Expectile-Based Capital Allocation

Khalil Said*

National Institute of Statistics and Applied Economics, Rabat, Morocco

**Corresponding author:* ksaid@insea.ac.ma

Abstract. This paper focuses on capital allocation using the Euler principle with Expectiles as risk measures. We delve into the allocation composition across various actuarial models, examining the influence of dependence through copulas, and studying the case of comonotonicity. Additionally, we provide expressions for marginal contributions related to some of the models under investigation.

Introduction

Capital allocation is a crucial issue for insurance groups due to its significant impact on financial results. Once the solvency capital is determined using risk aggregation methods, it needs to be allocated across different business lines. In the context of dependent risk processes $\mathbf{X} = (X_1, \ldots, X_d)$, the determination of solvency capital is based on studying the stochastic behavior of the aggregated claim amount $S = X_1 + \cdots + X_d$. Capital allocation involves determining the portion of the obtained economic capital that will be assigned to each risk X_i, where $i = 1, \ldots, d$. The choices made for modeling dependence will inevitably influence the allocation contributions. This process typically follows a top-down approach. We assume a multivariate model for the risk vector $\mathbf{X}$, select a risk measure to assess the solvency capital based on the distribution of the sum S and employ an allocation method to determine the marginal contribution of each risk to this capital.

Several methods for allocating economic capital have been proposed in the literature. One of the most well-known principles is Euler's method, also referred to as the *gradient method*. Based on Euler's principle, allocation rules can be derived using any homogeneous risk measure. There is a wealth of literature on VaR and TVaR allocation rules, which includes expressions for contributions in

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various cases. These studies examine the impact of dependence on allocation and provide concrete examples. For further reading, refer to Tasche (2000) [20], Bargès et al. (2009) [2], or Cossette et al. (2012) [8].

The significance of VaR and TVaR-based allocation rules stems directly from the practical interest in VaR and TVaR as commonly used risk measures. However, VaR is a non-coherent measure with respect to coherence, as defined by Artzner et al. (1999) [1], making VaR-based capital allocation subject to criticism for the same reason. On the other hand, the coherence of TVaR naturally lends greater importance to allocation rules based on it. Nevertheless, recent works in risk theory highlight the non-elicitability of TVaR, making direct backtesting of TVaR a challenging task (Gneiting, 2011 [13]; Bellini and Bignozzi, 2015 [3]). This verdict inevitably affects the performance of an allocation constructed using the TVaR rule.

Bellini and Bignozzi (2015) [3] show that expectiles of level $\alpha \in [1/2, 1[$ are the only law-invariant risk measures that are both elicitable and coherent. This property makes expectiles a perfect candidate for constructing capital allocation. Emmer et al. (2015) [11] derived a general formula for contributions in a capital allocation based on expectiles. In this paper, we focus on expectile-based capital allocation. Our main objective is to closely examine the allocation composition for some common risk models and analyze its differences with the TVaR allocation rule.

The paper is organized into 6 sections. Section 1 provides a review of Euler's allocation principle and its application to derive allocation rules from homogeneous risk measures, particularly the Wang measures family. It also provides a brief introduction to expectiles as risk measures and recalls the expectile-based allocation rule. We offer an economic interpretation of this rule and compare it to TVaR allocation. In Section 2, we examine capital allocation for various independent models. Sections 3 and 4 focus on the allocation composition for exponential combinations and mixture models, respectively. Section 5 examines the case of perfect dependence. The final section presents numerical illustrations.

1. Expectile-based capital allocation

This section is dedicated to the presentation of the allocation method. Firstly, we provide a reminder of the Euler allocation principle, followed by the presentation of the allocation rule derived from expectile risk measures. An economic interpretation of the resulting rule is provided, along with an initial comparison to the TVaR-based allocation rule. Euler's capital allocation method is studied in Tasche (2007) [22] and Tasche (2008) [23]. This technique is based on the concept of allocating capital based on the *infinitesimal marginal impact* of each risk, which represents the reduction in overall risk resulting from an infinitely small decrement in risk X_i.

We denote the contribution of risk X_i to the overall risk as $\rho(X_i|S)$. This contribution can be obtained using Euler's principle:

$$\rho(X_i|S) = \lim_{h\to 0} \frac{\rho(S) - \rho(S - hX_i)}{h}.$$

Using Euler's allocation principle, it is possible to construct an allocation rule with any homogeneous risk measure. We provide a reminder of the definitions of the commonly used risk measures: Value at Risk (VaR) and Tail Value at Risk (TVaR).

The VaR risk measure of level α is defined for any random variable X as:

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\} = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\} = F_X^{-1}(\alpha),$$

where F_X denotes the cumulative distribution function (CDF) of X. This represents the quantile of the same level.

The TVaR of level α is defined as the mean of VaRs exceeding $VaR_\alpha(X)$:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\mu(X)\mathrm{d}^-.$$

The well-known VaR-based and TVaR-based allocations are examples of rules obtained from this method. The risk contribution of each risk in the overall risk using the VaR-based allocation rule is given by:

$$VaR(X_i|S) = \mathbb{E}[X_i|S = VaR_\alpha(S)].$$

For continuous distributions, the expression for the risk contribution using the TVaR risk measure and Euler's method is:

$$TVaR\,(X_i|S) = \frac{\mathbb{E}[X_i|S > VaR_\alpha(S)]}{1-\alpha}.$$

Euler's method has been extensively studied in the literature on capital allocation in the past decade. Its properties, such as coherence and compatibility with Risk-Adjusted Return on Capital (RORAC), have been analyzed in numerous works under various assumptions. Examples include Balog (2011) [26], Tasche (2000) [20], and Tasche (2004) [21]. The economic interpretation of Euler's method provides a relevant solution to the capital allocation problem and explains its popularity in actuarial practice.

The composition of VaR-based capital allocation has been studied for several risk models in Marceau (2013) [28]. The TVaR-based allocation rule has been explored in Bargès et al. (2009) [2] and Cossette et al. (2012) [8].

Elicitability is a desirable statistical property for risk measures. According to Bellini and Bignozzi (2015) [3], a risk measure ρ is said to be elicitable in respect to the class $\mathcal{P}$ if there exists a scoring function $S : \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$\rho(\mathbb{P}) = \arg\min_{x\in\mathbb{R}} \int S(x,y)d\mathbb{P}(y),\ \forall \mathbb{P} \in \mathcal{P}.$$

They demonstrate in the same paper that expectiles are the only risk measures that are both coherent and elicitable.

Expectiles were introduced in the context of statistical regression models by Newey and Powell (1987) [15]. For a random variable X with finite second moment, the expectile of level α is defined as follows:

$$e_\alpha(X) = \arg\min_{x\in\mathbb{R}} \mathbb{E}[\alpha(X-x)_+^2 + (1-\alpha)(x-X)_+^2], \tag{1.1}$$

where $(x)_+ = \max(x, 0)$.

Bellini et al. (2014) [5] introduced generalized quantile risk measures, which encompass expectiles and are defined as the minimizers of an asymmetric error given by:

$$x_\alpha(X) = \arg\min_{x\in\mathbb{R}}\{\alpha\mathbb{E}[\Phi_+((X-x)_+)] + (1-\alpha)\mathbb{E}[\Phi_-((X-x)_-)]\},$$

where Φ_+ and Φ_- are convex scoring functions. Expectiles correspond to the case when $\Phi_+(x) = \Phi_-(x) = x^2$. Maume-Deschamps et al. (2017) [17] introduced multivariate extensions of expectile risk measures.

Expectiles are inherently elicitable. They are coherent for all $\alpha \geqslant 1/2$. Expectiles can also be defined equivalently for any random variable with a finite first-order moment as the unique solution to the following equation:

$$\alpha\mathbb{E}[(X-x)_+] = (1-\alpha)\mathbb{E}[(x-X)_+]. \tag{1.2}$$

The properties of expectile risk measures have been studied in several papers, including [11] and [4]. The asymptotic behavior of expectiles is examined in [4], and the second-order behavior is analyzed in [16]. Extremes for multivariate expectiles are investigated in [18].

In this paper, our focus is on Euler's capital allocation rule based on expectiles. Emmer et al. (2015) [11] showed that the contribution of risk X_i to the sum $S = \sum_{\ell=1}^d X_\ell$ is given by Definition 1.1.

Definition 1.1 (Expectile-based capital allocation)**.** *The marginal contribution of a risk X_i to an aggregated risk $S = \sum_{\ell=1}^d X_\ell$ using expectiles is given by*

$$e_\alpha(X_i|S) = \frac{\alpha\mathbb{E}\left[X_i\mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[X_i\mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{P}\left(S>e_\alpha(S)\right) + (1-\alpha)\mathbb{P}\left(S<e_\alpha(S)\right)}, \tag{1.3}$$

for $\alpha \in [1/2, 1[$.

To provide an economic interpretation of the capital allocation rule defined in Definition 1.1, let α_s denote the percentage given by

$$\alpha_s = \frac{\alpha\mathbb{P}\left(S>e_\alpha(S)\right)}{\alpha\mathbb{P}\left(S>e_\alpha(S)\right) + (1-\alpha)\mathbb{P}\left(S<e_\alpha(S)\right)}.$$

The contribution $e_\alpha(X_i|S)$ can then be expressed as

$$e_\alpha(X_i|S) = \alpha_s\underbrace{\mathbb{E}\left[X_i|S>e_\alpha(S)\right]}_{T_-} + (1-\alpha_s)\underbrace{\mathbb{E}\left[X_i|S<e_\alpha(S)\right]}_{T_+}.$$

Hence, the allocation can be interpreted as a linear combination of the marginal contribution in exceeding the overall expectile in a ruin scenario (T_-) and the marginal contribution in achieving overall solvency from an expectile perspective (T_+). This allocation rule takes into account not only the marginal participation in negative global scenarios, as in the case of TVaR allocation, but also the participation in overall performance.

In order to clarify the relationship between the expectile-based allocation and the TVaR-based allocation, we can express equation (1.3) in the following form:

$$e_\alpha(X_i|S) = \frac{(2\alpha-1)(1-\beta)}{(2\alpha-1)(1-\beta)+(1-\alpha)} TVaR_\beta\left(X_i|S\right) + \frac{(1-\alpha)}{(2\alpha-1)(1-\beta)+(1-\alpha)} \mathbb{E}\left[X_i\right],$$

where $\beta = F_S(e_\alpha(S))$. The allocation using expectiles can be seen as a transformation of the TVaR-based rule with a safety margin percentage. This transformation involves adjusting the TVaR level ($e_\alpha \longrightarrow TVaR_\beta$) as well as the composition, using a linear convex combination between the contribution based on $TVaR_\beta$ and the contribution based on $\mathbb{E}\left[X_i\right]$.

In a financial context, when the random variables represent P&L (Profit and Loss), an economic interpretation of the allocation contributions can be derived from the following expression:

$$e_\alpha(X_i|S) = \frac{\alpha(1-\beta)}{\alpha(1-\beta)+(1-\alpha)\beta} TVaR_\beta\left(X_i|S\right) - \frac{(1-\alpha)\beta}{\alpha(1-\beta)+(1-\alpha)\beta} TVaR_{1-\beta}\left(-X_i|-S\right).$$

This expression corresponds to a linear combination of the marginal participation in the global profits, measured by $TVaR_\beta$, and the marginal participation in the global losses, measured by $TVaR_{1-\beta}$. Thus, the allocation rule considers both the positive and negative aspects of the P&L, taking into account the contributions to the overall profitability and loss.

We can also express the contribution $e_\alpha(X_i|S)$ as follows:

$$e_\alpha(X_i|S) = \frac{\alpha\mathbb{E}\left[X_i\mathbb{1}_{\{S>e_\alpha(S)\}}\right]+(1-\alpha)\mathbb{E}\left[X_i\mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{E}\left[S\mathbb{1}_{\{S>e_\alpha(S)\}}\right]+(1-\alpha)\mathbb{E}\left[S\mathbb{1}_{\{S<e_\alpha(S)\}}\right]} e_\alpha(S), \tag{1.4}$$

since the expectile $e_\alpha(S)$ satisfies

$$e_\alpha(S) = \frac{\alpha\mathbb{E}\left[S\mathbb{1}_{\{S>e_\alpha(S)\}}\right]+(1-\alpha)\mathbb{E}\left[S\mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{P}\left(S>e_\alpha(S)\right)+(1-\alpha)\mathbb{P}\left(S<e_\alpha(S)\right)}.$$

The allocation percentage $e_\alpha(X_i|S)/e_\alpha(S)$ can be directly obtained from (1.4).
The expectile capital allocation is trivially additive, as stated in (1.4):

$$\sum_{i=1}^{d} e_\alpha(X_i|S) = e_\alpha\left(\sum_{i=1}^{d} X_i\right).$$

Moreover, it is a neutral allocation, since

$$\exists C_i \in \mathbb{R},\ X_i = C_i\ a.s\ \Rightarrow\ e_\alpha(X_i|S) = C_i.$$

The allocation is sub-additive, as for any subsets $A \subseteq 1, \ldots, d$, we have

$$e_\alpha(\sum_{\ell\in A} X_\ell|S) = \sum_{\ell\in A} e_\alpha(X_\ell|S),$$

which is also the case for VaR and TVaR-based allocation rules.

In the rest of this article, we will focus on analyzing the behavior of the contributions provided by the expectile-based allocation rule. We will examine the impact of dependence using different families of models.

2. Some bivariate independent models

This section presents a study of the expectile-based allocation rule in the case of independence. The main objective of this part is to highlight the impact of the nature of the marginal distributions on the allocation contributions.

2.1. **Bivariate independent exponential model.** We consider a bivariate independent exponential random vector (X_1, X_2) with $X_i \in \mathcal{E}(\beta_i), i \in \{1,2\}$. We denote by S the aggregated sum of risks $X_1 + X_2$. In the case where $\beta_1 = \beta_2$, the allocation is trivial $e_\alpha(X_1|S) = e_\alpha(X_2|S) = e_\alpha(S)/2$. Proposition 2.1 provides the expressions for the allocation contributions.

Proposition 2.1 (Expectile-based allocation, EI Model)**.** *According to the expectile allocation rule, the contribution from the risk X_i is*

$$e_\alpha\left(X_i|S\right) = \frac{(2\alpha-1)\beta_i \xi\left(s^*;\beta_i,\beta_{3-i}\right) + (1-\alpha)}{(2\alpha-1)\bar{H}\left(s^*;\beta_1,\beta_2\right) + (1-\alpha)} \frac{1}{\beta_i},$$

where s^* is the unique solution to the following equation

$$(2\alpha-1)\left[\zeta\left(s;\beta_1,\beta_2\right) - s\bar{H}\left(s;\beta_1,\beta_2\right)\right] = (1-\alpha)\left[s - \frac{1}{\beta_1} - \frac{1}{\beta_2}\right],$$

where $\bar{H},\zeta,\xi$ are defined as follows

$$\bar{H}\left(x;\beta_i,\beta_j\right) = \begin{cases} \mathrm{e}^{-\beta x}\sum_{\ell=0}^{2-1}\frac{(\beta x)^\ell}{\ell!}, & \beta_i=\beta_j=\beta \\ \sum_{k=1}^{2}\left(\prod_{\ell=1,\ell\neq k}^{2}\frac{\beta_\ell}{\beta_\ell-\beta_k}\right)\mathrm{e}^{-\beta_k x}, & \beta_i\neq\beta_j \end{cases},$$

$$\zeta\left(x;\beta_1,\beta_2\right) = \begin{cases} \frac{2}{\beta}\left(\mathrm{e}^{-\beta x}\sum_{\ell=0}^{2}\frac{(\beta x)^\ell}{\ell!}\right), & \beta_i=\beta_j=\beta \\ \sum_{k=1}^{2}\left(\prod_{\ell=1,\ell\neq k}^{2}\frac{\beta_\ell}{\beta_\ell-\beta_k}\right)\left(x\mathrm{e}^{-\beta_k x} + \frac{\mathrm{e}^{-\beta_k x}}{\beta_k}\right), & \beta_i\neq\beta_j \end{cases},$$

and

$$\xi\left(x;\beta_i,\beta_j\right) = \begin{cases} \frac{1}{\beta}\bar{H}\left(x;3,\beta\right), & \beta_i=\beta_j=\beta \\ \frac{\beta_j e^{-\beta_i x}\left(x+\frac{1}{\beta_i}\right)}{(\beta_j-\beta_i)} - \left(\frac{\beta_j e^{-\beta_i x}}{\left(\beta_i-\beta_j\right)^2} - \frac{\beta_i e^{-\beta_j x}}{\left(\beta_i-\beta_j\right)^2}\right), & \beta_i\neq\beta_j \end{cases}.$$

Proof. From the expectile definition (1.2), $e_\alpha(S)$ is the unique solution to the equation:

$$\alpha\mathbb{E}[(S-s)_+] = (1-\alpha)\mathbb{E}[(s-S)_+],$$

which can be written as:

$$(2\alpha-1)\mathbb{E}[(S-s)_+] = (1-\alpha)\left(s-\mathbb{E}[S]\right),$$

and from Equation 1.3, the contribution $e_\alpha(X_i|S)$ can be written as:

$$e_\alpha(X_i|S) = \frac{(2\alpha-1)\mathbb{E}\left[X_i \mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}[X_i]}{(2\alpha-1)\mathbb{P}(S>e_\alpha(S)) + (1-\alpha)},$$

then, the expressions are obtained straightforwardly from their definition using

$$F_S(x) = H(x;\beta_1,\beta_2) = \begin{cases} 1 - \mathrm{e}^{-\beta x}\sum_{j=0}^{2-1}\frac{(\beta x)^j}{j!}, & \beta_1=\beta_2=\beta \\ \sum_{i=1}^{2}\left(\prod\limits_{j=1,j\neq i}^{2}\frac{\beta_j}{\beta_j-\beta_i}\right)\left(1-\mathrm{e}^{-\beta_i x}\right), & \beta_1\neq\beta_2 \end{cases},$$

$$\mathbb{E}\left[S\times\mathbb{1}_{\{S>x\}}\right] = \zeta(x;\beta_1,\beta_2) = \begin{cases} \frac{2}{\beta}\left(\mathrm{e}^{-\beta x}\sum_{j=0}^{2}\frac{(\beta x)^j}{j!}\right), & \beta_1=\beta_2=\beta \\ \sum_{i=1}^{2}\left(\prod\limits_{j=1,j\neq i}^{2}\frac{\beta_j}{\beta_j-\beta_i}\right)\left(x\mathrm{e}^{-\beta_i x}+\frac{\mathrm{e}^{-\beta_i x}}{\beta_i}\right), & \beta_1\neq\beta_2 \end{cases},$$

and

$$\mathbb{E}\left[X_1\times\mathbb{1}_{\{S>x\}}\right] = \xi(x;\beta_1,\beta_2) = \begin{cases} \frac{1}{\beta}\bar{H}(x;3,\beta), & \beta_1=\beta_2=\beta \\ \frac{\beta_2 e^{-\beta_1 x}\left(x+\frac{1}{\beta_1}\right)}{(\beta_2-\beta_1)} - \left(\frac{\beta_2 e^{-\beta_1 x}}{(\beta_1-\beta_2)^2} - \frac{\beta_1 e^{-\beta_2 x}}{(\beta_1-\beta_2)^2}\right), & \beta_1\neq\beta_2 \end{cases}.$$

□

In this model, the random variable S follows an Erlang-2 distribution if $\beta_1=\beta_2=\beta$ and a generalized Erlang distribution if $\beta_1\neq\beta_2$.

Proposition 2.1 can be generalized in the case of higher dimension $d>2$ and different distribution parameters. In fact, let $X_1, X_2, \ldots, X_d$ be independent exponential random variables with respective parameters $0<\beta_1<\beta_2<\cdots<\beta_d$. We denote by S the aggregated sum of risks X_i, $i=1,\ldots,d$. Since $X_i\sim\mathcal{E}(\beta_i)$ for all $i\in 1,\ldots,d$, we have

$$\bar{F}_S(s) = \bar{H}_S(s,\beta_1,\ldots,\beta_d) = \sum_{\ell=1}^{d}\left(\prod_{j=1,j\neq\ell}^{d}\frac{\beta_j}{\beta_j-\beta_\ell}\right)e^{-\beta_\ell s},\ \forall\ s\in\mathbb{R}^+,$$

which represents the distribution function of the Generalized Erlang distribution.

On the other hand, the sum's expectile $e_\alpha(S)$ is the unique solution to the equation:

$$\alpha\mathbb{E}[(S-s)_+] = (1-\alpha)\mathbb{E}[(S-s)_-],$$

This equation can be rewritten as:

$$s = \mathbb{E}[S] + \frac{2\alpha-1}{1-\alpha}\mathbb{E}[(S-s)_+].$$

Since

$$\mathbb{E}[(S-s)_+] = \int_s^{+\infty}\bar{F}_S(s)dt = \sum_{\ell=1}^{d}\frac{A_\ell}{\beta_\ell}e^{-\beta_\ell s},$$

where $A_\ell = \prod_{j=1, j\neq \ell}^{d} \frac{\beta_j}{\beta_j - \beta_\ell}$, $\forall \ell \in \{1, \ldots, d\}$, we can express $e_\alpha(S)$ as the unique solution to the equation:

$$s = \sum_{\ell=1}^{d} \frac{1}{\beta_\ell}\left(1 + \frac{2\alpha - 1}{1-\alpha} A_\ell e^{-\beta_\ell s}\right). \tag{2.1}$$

We observe that X_i and $S^{(-i)} = \sum_{\ell=1, \ell\neq 1}^{d} X_\ell$ are independent. Since $S^{(-i)}$ is also the sum of exponentially independent random variables, its probability density function can be expressed as:

$$f_S^{(-i)}(s) = \sum_{\ell=1, \ell\neq i}^{d}\left(\prod_{j=1, j\neq\ell, j\neq i}^{d} \frac{\beta_j}{\beta_j - \beta_\ell}\right)\beta_\ell e^{-\beta_\ell s} = \sum_{\ell=1, \ell\neq i}^{d} A_\ell \frac{\beta_\ell}{\beta_i}(\beta_i - \beta_\ell) e^{-\beta_\ell s}, \ \forall\, s \in \mathbb{R}^+.$$

This expression is used to calculate:

$$\mathbb{E}\left[X_i \times \mathbb{1}_{\{S>x\}}\right] = \xi_i\left(x; \beta_1, \ldots, \beta_d\right) = \sum_{\ell=1, \ell\neq i}^{d} \frac{A_\ell}{\beta_i - \beta_\ell}\left[e^{-\beta_\ell x} - e^{-\beta_i x}\left(1 + x + \frac{1}{\beta_i}\right)\right].$$

Finally, the allocation contributions in this case are given by

$$e_\alpha\left(X_i|S\right) = \frac{(2\alpha-1)\beta_i \xi_i\left(s^*; \beta_1, \ldots, \beta_d\right) + (1-\alpha)}{(2\alpha-1)\bar{H}\left(s^*; \beta_1, \ldots, \beta_d\right) + (1-\alpha)} \frac{1}{\beta_i},$$

where s^* is the unique solution to Equation (2.1).
Note that in the particular case where $\beta_1 = \beta_2 = \cdots = \beta_d$, we have

$$e_\alpha\left(X_i|S\right) = e_\alpha(S)/d, \ \forall i \in \{1, \ldots, d\}.$$

2.2. **Bivariate independent Gamma model.** In this subsection, we consider two random variables following the gamma distribution: $X_i \sim \text{Gamma}(\alpha_i, \beta)$ for $i = 1, 2$. The rate parameter is the same for both distributions. If X_1 and X_2 are independent, then the sum $S = X_1 + X_2$ follows the gamma distribution with parameters $\alpha_1 + \alpha_2$ and β.

Proposition 2.2 (Expectile-allocation, IG-Model)**.** *According to the expectile allocation rule, the contribution from the risk X_i is given by:*

$$e_\alpha\left(X_i|S\right) = \frac{(2\alpha-1)\bar{G}\left(s^*; \alpha_1+\alpha_2+1, \beta\right) + (1-\alpha)}{2\alpha-1)\bar{G}\left(s^*; \alpha_1+\alpha_2, \beta\right) + (1-\alpha)} \frac{\alpha_i}{\beta}, \ i \in \{1, 2\},$$

where s^* is the unique solution to the following equation:

$$(2\alpha-1)\left[\frac{\alpha_1+\alpha_2}{\beta}\bar{G}\left(s^*; \alpha_1+\alpha_2+1, \beta\right) - s^*\bar{G}\left(s^*; \alpha_1+\alpha_2, \beta\right)\right] = (1-\alpha)\left[s - \frac{\alpha_1+\alpha_2}{\beta}\right],$$

and $\bar{G}\left(., \alpha, \beta\right)$ is the survival function of the gamma distribution with parameters α and β.

Proposition 2.2 can be generalized for dimensions higher than 2. For d independent random variables following the gamma distribution, $X_i \sim G(\alpha_i, \beta)$, where $i = 1, \ldots, d$, the allocation of X_i given the sum $S = \sum_{\ell=1}^{d} X_\ell$ is given by:

$$e_\alpha \left(X_i \,\middle|\, S = \sum_{\ell=1}^{d} X_\ell \right) = \frac{(2\alpha - 1)\bar{G}\left(s^*; 1 + \sum_{\ell=1}^{d} \alpha_\ell, \beta\right) + (1 - \alpha)}{2\alpha - 1)\bar{G}\left(s^*; \sum_{\ell=1}^{d} \alpha_\ell, \beta\right) + (1 - \alpha)} \frac{\alpha_i}{\beta}, \ i \in \{1, \ldots, d\},$$

where s^* is the unique solution to the following equation:

$$(2\alpha - 1) \left[\frac{\sum_{\ell=1}^{d} \alpha_\ell}{\beta} \bar{G}\left(s^*; 1 + \sum_{\ell=1}^{d} \alpha_\ell, \beta\right) - s^* \bar{G}\left(s^*; \sum_{\ell=1}^{d} \alpha_\ell, \beta\right) \right] = (1 - \alpha) \left[s - \frac{\sum_{\ell=1}^{d} \alpha_\ell}{\beta} \right].$$

Proof. The result is obtained directly using Equation 1.3. □

3. Bivariate combinations of exponentials

Bivariate distributions with exponential marginals are well-known in the actuarial science literature, and extensive discussions can be found in Kotz et al. (2004) [27] and Balakrishnan and Lai (2009) [24]. In Cossette et al. (2015) [9], the TVaR-based allocation rule was investigated for this family of bivariate models, providing explicit formulas for contributions.

In this section, we begin by presenting the general expression for the expectile-based allocation contributions in the family of bivariate combinations of exponentials with exponential marginals. Subsequently, we illustrate these expressions through several examples of models.

3.1. **Bivariate combinations of exponential distributions.** A random vector (X_1, X_2) follows a bivariate combination of exponential distributions if its joint density can be expressed as:

$$f_{X_1,X_2}(x_1, x_2) = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \gamma_i \mathrm{e}^{-\gamma_i x_1} \lambda_j \mathrm{e}^{-\lambda_j x_2}, \tag{3.1}$$

where $c_{i,j} \in \mathbb{R}$ with $\sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} = 1$.

We assume that $0 < \gamma_1 < \ldots < \gamma_m$ and $0 < \lambda_1 < \ldots < \lambda_m$. We denote $c_{i,*} = \sum_{j=1}^{m} c_{i,j}$ and $c_{*,j} = \sum_{i=1}^{m} c_{i,j}$ where $\left\{c_{i,j}, i = 1, \ldots, m, j = 1, \ldots, m\right\}$ are such that $f_{X_1,X_2}(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R}^2$.

This class includes the family of bivariate mixed exponential distributions, where $0 \leq c_{i,j} \leq 1$. It is important to note that the class of bivariate combinations of exponential distributions is a subset within the family of bivariate matrix exponential distributions studied in Bladt and Nielsen (2010) [6].The marginal distributions are univariate combinations of exponentials, given by:

$$F_{X_1}(x_1) = \sum_{i=1}^{m} c_{i,*} \left(1 - \mathrm{e}^{-\gamma_i x_1}\right) \text{ and } F_{X_2}(x_2) = \sum_{j=1}^{m} c_{*,j} \left(1 - \mathrm{e}^{-\lambda_j x_2}\right).$$

Proposition 3.1 provides the general expressions of marginal contributions in solvency capital using the expectile-based allocation method.

Proposition 3.1 (Expectile-allocation for bivariate combinations of exponentials)**.** *Let (X_1, X_2) follow a bivariate combination of exponentials. Then, for $S = X_1 + X_2$, we have*

$$e_\alpha\left(X_1|S\right) = \frac{(2\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\xi\left(s^*;\gamma_i,\lambda_j\right) + (1-\alpha)\sum_{i=1}^{m}\frac{c_{i,*}}{\gamma_i}}{(2\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\bar{H}\left(s^*;\gamma_i,\lambda_j\right) + 1-\alpha},$$

where ξ, ζ and $\bar{H}$ are the same functions defined in Proposition 2.1, and s^* is the unique solution to the following equation

$$(2\alpha-1)\left[\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\left(\zeta\left(s;\gamma_i,\lambda_j\right) - s\bar{H}\left(s;\gamma_i,\lambda_j\right)\right)\right] = (1-\alpha)\left[s - \sum_{i=1}^{m}\left(\frac{c_{i,*}}{\gamma_i} + \frac{c_{*,i}}{\lambda_i}\right)\right]. \tag{3.2}$$

The contribution of X_2 is given directly from

$$e_\alpha\left(X_2|S\right) = s^* - e_\alpha\left(X_1|S\right),$$

and it can also be obtained directly by

$$e_\alpha\left(X_2|S\right) = \frac{(2\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{m} c_{j,i}\xi\left(s^*;\lambda_j,\gamma_i\right) + (1-\alpha)\sum_{i=1}^{m}\frac{c_{*,j}}{\lambda_j}}{(2\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\bar{H}\left(s^*;\gamma_i,\lambda_j\right) + 1-\alpha}.$$

Proof. Since the marginals are $F_{X_1}(x_1) = \sum_{i=1}^{m} c_{i,*}\left(1-\mathrm{e}^{-\gamma_i x_1}\right)$ and $F_{X_2}(x_2) = \sum_{j=1}^{m} c_{*,j}\left(1-\mathrm{e}^{-\lambda_j x_2}\right)$ respectively, then

$$\mathbb{E}[X_1] = \sum_{i=1}^{m}\frac{c_{i,*}}{\gamma_i} \text{ and } \mathbb{E}[X_2] = \sum_{j=1}^{m}\frac{c_{*,j}}{\lambda_j}.$$

In this model, the joint distribution of (X_1, X_2) is a linear combination of $m \times m$ terms. By a direct calculation, we get

$$\bar{F}_S(s) = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\bar{H}\left(s;\gamma_i,\lambda_j\right), \tag{3.3}$$

$$\mathbb{E}\left[S \times \mathbb{1}_{\{S>s\}}\right] = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\zeta\left(s;\gamma_i,\lambda_j\right), \tag{3.4}$$

and

$$\mathbb{E}\left[X_1 \times \mathbb{1}_{\{S>s\}}\right] = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\xi\left(s;\gamma_i,\lambda_j\right). \tag{3.5}$$

It also follows that

$$\mathbb{E}[(S-s)_+] = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\left(\zeta\left(s;\gamma_i,\lambda_j\right) - s\bar{H}\left(s;\gamma_i,\lambda_j\right)\right). \tag{3.6}$$

Combining expressions (3.3), (3.5) and (3.6), we obtain the announced result. □

Note that in this model, S follows a combination of Erlang-2 and/or generalized Erlang distributions. The value of $e_\alpha(S)$ is obtained by solving Equation 3.2 using numerical methods. Subsequently, we compute $e_\alpha(X_1|S)$ and $e_\alpha(X_2|S)$.

3.2. **Specific models.** We consider some well-known bivariate exponential distributions that belong to the class presented in the previous subsection.

3.2.1. *Bivariate FGM-exponential Model.* Let the joint distribution of (X_1, X_2) be defined with a Farlie-Gumbel-Morgenstern (FGM) copula, given by

$$C_\theta(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1-u_1)(1-u_2), \ -1 \le \theta \le 1,$$

(see e.g., Nelsen (2007) [29], Example 3.12, Section 3.2.5). The marginal distributions are exponential with parameters β_1 and β_2, respectively. This leads to the joint cumulative distribution function:

$$F_{X_1,X_2}(x_1,x_2) = \left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right) + \theta\left(1-\mathrm{e}^{-\beta_1 x_1}\right)\left(1-\mathrm{e}^{-\beta_2 x_2}\right)\mathrm{e}^{-\beta_1 x_1}\mathrm{e}^{-\beta_2 x_2}.$$

It is important to note that the FGM construction is considered a weak dependence model. The Pearson correlation coefficient is $\rho_P(X1, X_2) = \frac{\theta}{4}$, which implies $\rho_P(X1, X_2) \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. The Spearman's correlation coefficient, denoted as ρ_S, is given by $\rho_S = \frac{\theta}{3} \in \left[-\frac{1}{3}, \frac{1}{3}\right]$. We recall that Spearman's rho is a concordance measure defined for continuous bivariate distributions with copula C as the dependence structure. It can be calculated as:

$$\rho_S = 12\int\int_{[0,1]^2} uv dC(u,v) - 3 = 12\int\int_{[0,1]^2} C(u,v) du dv - 3.$$

The FGM construction is also considered as an asymptotic independent model since its upper tail dependence coefficient is $\lambda_U = 0$. We recall the definition of the upper tail dependence coefficient as presented in Joe (1997) [25], for bivariate random variables (X, Y) of a continuous marginal distributions

$$\lambda_U = \lim_{u \longrightarrow 1^-} \mathbb{P}(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)).$$

The upper tail dependence coefficient can be expressed in terms of copula as:

$$\lambda_U = \lim_{u \longrightarrow 1^-} \frac{1-2u+C(u,u)}{1-u},$$

when the limit exists.

The joint density is given by:

$$f_{X_1,X_2}(x_1,x_2) = \beta_1 \mathrm{e}^{-\beta_1 x_1}\beta_2 \mathrm{e}^{-\beta_2 x_2} + \theta\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j} \times i\beta_1 \mathrm{e}^{-i\beta_1 x_1} \times j\beta_2 \mathrm{e}^{-j\beta_2 x_2}. \tag{3.7}$$

Given (3.7), with $m = 2$, $\gamma_i = i\beta_1$ $(i = 1, 2)$, and $\lambda_j = j\beta_2$ $(j = 1, 2)$, the bivariate distribution defined with the FGM copula and exponential marginals is a bivariate combination of exponentials. The specific values for the coefficients are: $c_{1,1} = 1+\theta$, $c_{1,2} = c_{2,1} = -\theta$, and $c_{2,2} = \theta$.

Lemma 3.1 presents the expressions of marginal contributions obtained using the expectile-based allocation rule. The expressions for the contributions in the TVaR allocation are given in Bargès et al. (2009) [2].

Lemma 3.1 (Expectile-Allocation, FGM Model)**.** *Let (X_1, X_2) follow a bivariate FGM model. Then, for $S = X_1 + X_2$, we have for all $(k, \ell) \in \{(1, 2), (2, 1)\}$*

$$e_\alpha (X_k|S) = \frac{(2\alpha - 1)\beta_k \left[\xi (s^*; \beta_k, \beta_\ell) + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} \xi (s^*; i\beta_k, j\beta_\ell)\right] + 1 - \alpha}{(2\alpha - 1) \left[\bar{H} (s^*; \beta_1, \beta_2) + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} \bar{H} (s^*; i\beta_1, j\beta_2)\right] + 1 - \alpha} \frac{1}{\beta_k},$$

where s^* is the unique solution to the following equation

$$(2\alpha - 1) \left[T (s; \beta_1, \beta_2) + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} T (s; i\beta_1, j\beta_2)\right] = (1 - \alpha) \left[s - \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)\right],$$

and ξ, ζ and $\bar{H}$ are the same function defined in Proposition 2.1, and T is the function defined by

$$T (s; i\beta_1, j\beta_2) = \zeta (s; i\beta_1, j\beta_2) - s\bar{H} (s; i\beta_1, j\beta_2), \ \forall s\mathbb{R}^+, \ \forall (i, j) \in \{1, 2\}^2.$$

Proof. The allocation contributions are directly obtained using 1.3 and the results of Bargès et al. (2009) [2] without the constraints on β_1 and β_2 i.e.

$$F_S (x) = H (x; \beta_1, \beta_2) + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} H (x; i\beta_1, j\beta_2),$$

$$\mathbb{E} \left[S \times \mathbb{1}_{\{S>x\}}\right] = \zeta (x; \beta_1, \beta_2) + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} \zeta (x; i\beta_1, j\beta_2)$$

and

$$\mathbb{E} \left[X_k \times \mathbb{1}_{\{S>x\}}\right] = \xi (x; \beta_k, \beta_{3-k}) + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} \xi_1 (x; i\beta_k, j\beta_{3-k}).$$

□

3.2.2. *Bivariate AMH-exponential Model.* Let the joint distribution of (X_1, X_2) be defined by a bivariate Ali-Mikhail-Haq (AMH) copula, given by

$$C_\theta (u_1, u_2) = \frac{u_1 u_2}{1 - \theta (1 - u_1) (1 - u_2)} = u_1 u_2 + u_1 u_2 \sum_{k=1}^{\infty} \theta^k (1 - u_1)^k (1 - u_2)^k,$$

with dependence parameter $\theta \in [-1, 1]$. As a special case, $C_0 (u_1, u_2) = u_1 u_2$ represents the independence copula. The AMH copula is also an Archimedean copula associated with the following generator:

$$\phi(t) = \frac{\ln (1 - \theta(1 - t))}{t}.$$

It introduces a moderate positive or negative dependence relation and is considered a perturbation of the independence copula. The first-degree approximation of the AMH copula corresponds to the FGM copula (see e.g., Nelsen (2007) [29]).

The Pearson correlation coefficient is given by

$$\rho_P(X_1, X_2) = \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \frac{1}{(k+i)(k+j)} \in \left[4\ln(2) - 3, \frac{\pi^2}{3} - 3 \right].$$

The upper extremes are asymptotically independent since $\lambda_U = 0$.

The joint density of (X_1, X_2) is given by

$$f_{X_1,X_2}(x_1, x_2) = \beta_1 e^{-\beta_1 x_1} \beta_2 e^{-\beta_2 x_2} + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} (k+i) \beta_1 e^{-(k+i)\beta_1 x_1} (k+j) \beta_2 e^{-(k+j)\beta_2 x_2},$$

which can be seen as a bivariate combination of exponentials by taking $m = \infty$, $\gamma_i = i\beta_1$ $\left(i \in \mathbb{N}^+\right)$, $\lambda_j = j\beta_2$ $\left(i \in \mathbb{N}^+\right)$, $c_{1,1} = (1+\theta)$, $c_{1,2} = c_{2,1} = -\theta$, $c_{1,j} = 0$ for $j = 2, 3, \ldots$, and $c_{i,1} = 0$ for $i = 2, 3, \ldots$. Additionally,

$$\begin{aligned} c_{k,k} &= c_{k+1,k+1} = \theta, \qquad c_{k,k+1} = c_{k+1,k} = -\theta, \\ c_{k,j} &= c_{k+1,j} = 0, \ \left(j \in \mathbb{N}^+ \backslash \{k, k+1\}\right), \\ c_{i,k} &= c_{i,k+1} = 0, \ \left(i \in \mathbb{N}^+ \backslash \{k, k+1\}\right), \end{aligned}$$

for $k = 2, 3, \ldots$. By Proposition 3.1, we obtain the expressions of marginal contributions in expectile allocation as presented in Lemma 3.2.

Lemma 3.2 (Expectile-Allocation, AHM Model)**.** *Let (X_1, X_2) follow a bivariate FGM model. Then, for $S = X_1 + X_2$, we have for $(k, \ell) \in \{(1,2), (2,1)\}$*

$$e_\alpha(X_k|S) = \frac{(2\alpha-1)\beta_k \left[\xi(s^*; \beta_k, \beta_\ell) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \xi(s^*; (k+i)\beta_k, (k+j)\beta_\ell)\right] + 1 - \alpha}{(2\alpha-1)\left[\bar{H}(s^*; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \bar{H}(s^*; (k+i)\beta_1, (k+j)\beta_2)\right] + 1 - \alpha} \frac{1}{\beta_k},$$

where s^* is the unique solution to the following equation

$$(2\alpha-1)\left[T(x; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} T(x; (k+i)\beta_1, (k+j)\beta_2)\right] = (1-\alpha)\left[s - \left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)\right],$$

and ξ, ζ and $\bar{H}$ are the same function defined in Proposition 2.1, and T is the function defined by

$$T(s; a_1, a_2) = \zeta(s; a_1, a_2) - s\bar{H}(s; a_1, a_2).$$

Proof. The allocation contributions are obtained using 1.3 and the following expressions :

$$F_S(x) = H(x; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} H(x; (k+i)\beta_1, (k+j)\beta_2),$$

$$\mathbb{E}\left[S \times \mathbb{1}_{\{S>x\}}\right] = \zeta(x; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \zeta(x; (k+i)\beta_1, (k+j)\beta_2),$$

and

$$\mathbb{E}\left[X_{\ell} \times \mathbb{1}_{\{S>x\}}\right] = \xi\left(x; \beta_{\ell}, \beta_{3-\ell}\right) + \sum_{k=1}^{\infty} \theta^{k} \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \xi\left(x; (k+i)\beta_{\ell}, (k+j)\beta_{3-\ell}\right), \ \ell \in \{1, 2\}.$$

□

Another interesting example is Sarmanov's bivariate exponential distribution introduced by Sarmanov (1966) [19]. The bivariate density is given by

$$f_{X_1,X_2}(x_1, x_2) = \beta_1\beta_2 e^{-(\beta_1 x_1+\beta_2 x_2)} + \frac{\theta\beta_1\beta_2}{(\beta_1+1)(\beta_2+1)} \sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}(\beta_1+i)e^{-(\beta_1+i)x_1}(\beta_2+j)e^{-(\beta_2+j)x_2}.$$

where $\frac{-(1+\beta_1)(1+\beta_2)}{max(\beta_1,\beta_2,1)} \leq \theta \leq \frac{(1+\beta_1)(1+\beta_2)}{max(\beta_1,\beta_2)}$. The correlation coefficient is

$$\rho_P\left(X_1, X_2\right) = \frac{\theta\beta_1\beta_2}{(1+\beta_1)^2(1+\beta_2)^2} \in \left[-\frac{1}{4}, +\frac{1}{4}\right].$$

The expectile-based allocation contributions can be found directly using Proposition 3.1 by letting $m = 3$, $\gamma_1 = \beta_1$, $\lambda_1 = \beta_2$, $\gamma_i = \beta_1 + i - 2$ $(i = 2, 3)$, $\lambda_j = \beta_2 + i - 2$ $(j = 2, 3)$, $c_{1,1} = 1$, $c_{1,2} = c_{1,3} = c_{2,1} = c_{3,1} = 0$, $c_{2,2} = c_{3,3} = \frac{\theta\beta_1\beta_2}{(\beta_1+1)(\beta_2+1)}$, and $c_{2,3} = c_{3,2} = -\frac{\theta\beta_1\beta_2}{(\beta_1+1)(\beta_2+1)}$.

The main limitation of the three previous examples is the narrow range of correlation that is considered. To overcome this issue, Bladt and Nielsen (2010) [6] employed multivariate phase-type distributions to define a class of bivariate exponential distributions that encompass any feasible Pearson correlation coefficient $\rho_P\left(X_1, X_2\right) \in [\rho_{\min}, \rho_{\max}]$. The joint density expression for Bladt-Nielsen's bivariate exponential distribution, denoted by (X_1, X_2), is given by:

$$f_{X_1,X_2}(x_1, x_2) = \sum_{l=1}^{m}\sum_{k=1}^{m} c_{l,k}\, l\lambda e^{-l\lambda x_1} k\mu e^{-k\mu x_2},$$

where

$$c_{l,k} = \frac{(-1)^{l+k-(m+1)}}{m}\binom{m}{l}\binom{m}{k}\sum_{i=m+1-l}^{m}\sum_{j=1}^{k} p_{i,j}(-1)^{-i-j}\binom{l-1}{m-i}\binom{k-1}{k-j}$$

and

$$p_{i,j} = \begin{cases} \frac{\rho}{\rho_{\max}^{(m)}}\delta_{i+j-n-1} + \frac{1}{m}\left(1 - \frac{\rho}{\rho_{\max}^{(m)}}\right), & \rho > 0 \\ \frac{\rho}{\rho_{\min}^{(m)}}\delta_{i-j} + \frac{1}{m}\left(1 - \frac{\rho}{\rho_{\min}^{(m)}}\right), & \rho < 0 \end{cases},$$

with $\delta_x = 1$, if $x = 0$. From this expression and taking $\gamma_i = i\beta_1$ $(i = 1, 2, ..., m)$ and $\lambda_j = j\beta_2$ $(j = 1, 2, ..., m)$, this construction can be seen as a bivariate combination of exponentials. Then, using Proposition 3.1, we can find the expectile-based allocation contributions.

4. Bivariate exponentials mixture models

This section is devoted to stronger dependence models, in the sense of the presence of extreme dependence ($\lambda_U > 0$). The first subsection focuses on studying the Marshall-Olkin model. The second subsection presents the contributions made by expectile allocation in the case of a common mixture model.

4.1. **Marshall-Olkin model.** Let $Y_i \sim \exp(\lambda_i)$, with $i = 0, 1, 2$, be three independent random variables. We construct two random variables with a common shock: $X_i = \min(Y_i, Y_0)$ for $i = 1, 2$. The obtained random variables X_i have exponential marginal distributions with parameters $\beta_i = \lambda_i + \lambda_0$ (see, e.g., Nelsen [29], section 3.1.1).
The joint distribution function is given by:

$$\begin{aligned}\bar{F}_{X_1,X2}(x_1, x_2) &= \mathbb{P}(X_1 > x_1, X_2 > x_2) = \mathbb{P}(Y_1 > x_1, Y_2 > x_2, Y_0 > \max(x_1, x_2)) \\ &= e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} e^{-\lambda_0 \max(x_1,x_2)} \\ &= e^{-(\lambda_0+\lambda_1)x_1} e^{-(\lambda_0+\lambda_2)x_2} e^{\lambda_0 \min(x_1,x_2)} \\ &= \bar{F}_{X_1}(x_1)\bar{F}_{X_2}(x_2)e^{\lambda_0 \min(x_1,x_2)}.\end{aligned}$$

This construction leads to a copula given by:

$$C(u_1, u_2) = \min\left(u_1^{1-\lambda_0/\beta_1} u_2, u_1 u_2^{1-\lambda_0/\beta_2}\right).$$

The joint density is:

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f^1_{X_1,X_2}(x_1, x_2) = \beta_1 e^{-\beta_1 x_1}(\beta_2 - \lambda_0)e^{-(\beta_2-\lambda_0)x_2} & si \quad x_1 > x_2 \\ f^2_{X_1,X_2}(x_1, x_2) = (\beta_1 - \lambda_0)e^{-(\beta_1-\lambda_0)x_1}\beta_2 e^{-\beta_2 x_2} & si \quad x_1 < x_2 \\ f^0_{X_1,X_2}(x_1, x_2) = \lambda_0 e^{-\beta_1 x} e^{-\beta_2 x} e^{\lambda_0 x} & si \quad x_1 = x_2 = x \end{cases}.$$

This model has as Pearson correlation coefficient $\rho_P = \frac{\lambda_0}{\lambda_s}$, where $\lambda_s = \lambda_0 + \lambda_1 + \lambda_2$.
The Spearman's rho for Marshall-Olkin copulas is given by:

$$\rho_S = \frac{1}{1 + \frac{2}{3}\frac{\lambda_1+\lambda_2}{\lambda_0}}.$$

Since $\rho_S \in]0, 1[$, the Marshall-Olkin copulas model only positive dependence.
On the other hand, they have upper tail dependence, given by:

$$\lambda_U = \min\left(\frac{\lambda_0}{\beta_1}, \frac{\lambda_0}{\beta_2}\right) = \frac{\lambda_0}{\max(\lambda_1, \lambda_2) + \lambda_0},$$

The Marshall-Olkin model considers the presence of asymptotic dependence.

The density of $S = X_1 + X_2$ can be expressed as follows:

$$\begin{aligned} f_S(s) &= f^0_{X_1,X_2}(s/2, s/2) + \int_0^{s/2} f^2_{X_1,X_2}(x, s-x)dx + \int_{s/2}^{s} f^1_{X_1,X_2}(x, s-x)dx \\ &= \lambda_0 e^{-\lambda_s \frac{s}{2}} + \frac{\lambda_1\beta_2}{\lambda_1-\beta_2}\left(e^{-\beta_2 s} - e^{-\lambda_s \frac{s}{2}}\right) + \frac{\lambda_2\beta_1}{\lambda_2-\beta_1}\left(e^{-\beta_1 s} - e^{-\lambda_s \frac{s}{2}}\right) \\ &= \left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2\beta_1}{\beta_1-\lambda_1}\right) e^{-\lambda_s \frac{s}{2}} + \frac{\lambda_1\beta_2}{\lambda_1-\beta_2} e^{-\beta_2 s} + \frac{\lambda_2\beta_1}{\lambda_2-\beta_1} e^{-\beta_1 s}. \end{aligned}$$

From this, we can deduce the cumulative distribution function:

$$\bar{F}^{MO}_S(s, \lambda_0, \lambda_1, \lambda_2) = \frac{2}{\lambda_s}\left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2\beta_1}{\beta_1-\lambda_1}\right) e^{-\lambda_s \frac{s}{2}} + \frac{\lambda_1}{\lambda_1-\beta_2} e^{-\beta_2 s} + \frac{\lambda_2}{\lambda_2-\beta_1} e^{-\beta_1 s}.$$

Proposition 4.1 provides the allocation contributions based on expectiles for the Marshall-Olkin Model.

Proposition 4.1 (Expectile-Allocation, MO Model)**.** *Let (X_1, X_2) follow a bivariate Marshall-Olkin model. Then, for $S = X_1 + X_2$, we have for $(k, \ell) \in \{(1,2),(2,1)\}$*

$$e_\alpha(X_k|S) = \frac{(2\alpha-1)\xi_{MO}(s^*, \lambda_0, \lambda_k, \lambda_\ell) + (1-\alpha)\frac{1}{\lambda_0+\lambda_k}}{(2\alpha-1)\bar{F}^{MO}_S(s^*, \lambda_0, \lambda_k, \lambda_\ell) + (1-\alpha)},$$

where s^* is the unique solution to the following equation

$$\begin{aligned} (1-\alpha)s &= \left(\frac{2}{\lambda_s}\right)^2 \left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2\beta_1}{\beta_1-\lambda_1}\right)\left((2\alpha-1)e^{-\lambda_s \frac{s}{2}} + 1 - \alpha\right) \\ &\quad + \frac{\lambda_1/\beta_2}{\lambda_1-\beta_2}\left((2\alpha-1)e^{-\beta_2 s} + 1 - \alpha\right) + \frac{\lambda_2/\beta_1}{\lambda_2-\beta_1}\left((2\alpha-1)e^{-\beta_1 s} + 1 - \alpha\right), \end{aligned}$$

and ξ_{MO} is defined by

$$\begin{aligned} \xi_{MO}(s, \lambda_0, \lambda_1, \lambda_2) &= \left(\frac{\lambda_0}{\lambda_s} + \frac{\lambda_1}{\lambda_s}\frac{\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2}{\lambda_s}\frac{\beta_1}{\beta_1-\lambda_1}\right) e^{-\lambda_s \frac{s}{2}}\left(s + \frac{2}{\lambda_s}\right) + \frac{\lambda_2}{\lambda_2-\beta_1} e^{-\beta_1 s}\left(s + \frac{1}{\beta_1}\right) \\ &\quad + \frac{\lambda_1\beta_2}{(\lambda_1-\beta_2)^2}\left(\frac{1}{\beta_2}e^{-\beta_2 s} - \frac{2}{\lambda_s}e^{-\lambda_s \frac{s}{2}}\right) - \frac{\lambda_2\beta_1}{(\lambda_2-\beta_1)^2}\left(\frac{1}{\beta_1}e^{-\beta_1 s} - \frac{2}{\lambda_s}e^{-\lambda_s \frac{s}{2}}\right). \end{aligned}$$

Proof. Using the expression of $\bar{F}^{MO}_S$, we obtain

$$\mathbb{E}[(S-s)_+] = \left(\frac{2}{\lambda_s}\right)^2 \left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2\beta_1}{\beta_1-\lambda_1}\right) e^{-\lambda_s \frac{s}{2}} + \frac{\lambda_1/\beta_2}{\lambda_1-\beta_2} e^{-\beta_2 s} + \frac{\lambda_2/\beta_1}{\lambda_2-\beta_1} e^{-\beta_1 s},$$

in particular

$$\mathbb{E}[S] = \left(\frac{2}{\lambda_s}\right)^2 \left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2\beta_1}{\beta_1-\lambda_1}\right) + \frac{\lambda_1}{\lambda_1-\beta_2}\frac{1}{\beta_2} + \frac{\lambda_2}{\lambda_2-\beta_1}\frac{1}{\beta_1}.$$

So, the expectile $e_\alpha(S)$ is the unique solution to the following equation

$$\begin{aligned} (1-\alpha)s &= \left(\frac{2}{\lambda_s}\right)^2 \left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2-\lambda_1} + \frac{\lambda_2\beta_1}{\beta_1-\lambda_1}\right)\left((2\alpha-1)e^{-\lambda_s \frac{s}{2}} + 1 - \alpha\right) \\ &\quad + \frac{\lambda_1/\beta_2}{\lambda_1-\beta_2}\left((2\alpha-1)e^{-\beta_2 s} + 1 - \alpha\right) + \frac{\lambda_2/\beta_1}{\lambda_2-\beta_1}\left((2\alpha-1)e^{-\beta_1 s} + 1 - \alpha\right). \end{aligned}$$

And using the bivariate distribution, we get

$$\mathbb{E}\left[X_1 \times \mathbb{1}_{\{S=s\}}\right] = \left(\lambda_0 + \frac{\lambda_1\beta_2}{\beta_2 - \lambda_1} + \frac{\lambda_2\beta_1}{\beta_1 - \lambda_1}\right)\frac{s}{2}e^{-\lambda_s \frac{s}{2}} + \frac{\beta_1\lambda_2}{\lambda_2 - \beta_1}se^{-\beta_1 s} + \frac{\lambda_1\beta_2}{(\lambda_1 - \beta_2)^2}\left(e^{-\beta_2 s} - e^{-\lambda_s \frac{s}{2}}\right) - \frac{\lambda_2\beta_1}{(\lambda_2 - \beta_1)^2}\left(e^{-\beta_1 s} - e^{-\lambda_s \frac{s}{2}}\right),$$

and

$$\begin{aligned}\mathbb{E}\left[X_1 \times \mathbb{1}_{\{S>s\}}\right] &= \xi_{MO}\left(s, \lambda_0, \lambda_1, \lambda_2\right)\\ &= \left(\frac{\lambda_0}{\lambda_s} + \frac{\lambda_1}{\lambda_s}\frac{\beta_2}{\beta_2 - \lambda_1} + \frac{\lambda_2}{\lambda_s}\frac{\beta_1}{\beta_1 - \lambda_1}\right)e^{-\lambda_s \frac{s}{2}}\left(s + \frac{2}{\lambda_s}\right) + \frac{\lambda_2}{\lambda_2 - \beta_1}e^{-\beta_1 s}\left(s + \frac{1}{\beta_1}\right)\\ &+ \frac{\lambda_1\beta_2}{(\lambda_1 - \beta_2)^2}\left(\frac{1}{\beta_2}e^{-\beta_2 s} - \frac{2}{\lambda_s}e^{-\lambda_s \frac{s}{2}}\right) - \frac{\lambda_2\beta_1}{(\lambda_2 - \beta_1)^2}\left(\frac{1}{\beta_1}e^{-\beta_1 s} - \frac{2}{\lambda_s}e^{-\lambda_s \frac{s}{2}}\right).\end{aligned}$$

That is sufficient to obtain the expressions for the allocation contributions. □

Note that in the Marshall-Olkin model, the dependence construction alters the marginal distributions, unlike the FGM model, for example, where the marginals remain the same throughout, and the dependence effect is confined to the copula.

4.2. **Common Mixture Model.** This method of constructing multivariate models is presented in detail by Joe (1997) [25]. It is based on choosing a random variable Θ with support S_Θ and independent random variables Y_i to construct random variables X_i that are conditionally independent given Θ. This construction ensures that the conditional distribution function of X_i given $\Theta = \theta$ is given by

$$\bar{F}_{X_i|\Theta=\theta}(x_i) = (\bar{F}_{Y_i}(x_i))^\theta.$$

This construction provides the marginal distributions and the joint distribution by integrating with respect to the law of Θ, as described in Marceau (2013) [28].

Here, we are specifically interested in the case of a bivariate exponential mixture model. We assume that the moment-generating function of Θ, denoted by M_Θ, exists. The joint density function of X_1 and X_2 is then given by:

$$f_{X_1,X_2}(x_1, x_2) = \int_{\theta\in S_\Theta} \beta_1\theta e^{-\beta_1\theta x_1}\beta_2\theta e^{-\beta_2\theta x_2}\, dF_\Theta(\theta) = \beta_1\beta_2\frac{d^2 M_\Theta(t)}{dt^2}|_{t=-(\beta_1 x_1+\beta_2 x_2)}.$$

Let (X_1, X_2) be a pair of continuous random variables following a mixture of exponential distributions. For all $i \in 1, 2$, we have $X_i \sim \mathcal{E}(\beta_i\theta)$, with $\beta_1 < \beta_2$, and $\theta \sim Ga(\gamma, b)$. Therefore, the survival functions of X_i are given by:

$$\bar{F}_{X_i}(x) = \int_0^\infty \bar{F}_{X_i|\Theta=\theta} f_\Theta(\theta)d\theta = \int_0^\infty e^{-\beta_i\theta x} f_\Theta(\theta)d\theta = \left(1 + \frac{\beta_i x}{b}\right)^{-\gamma}.$$

Consequently, X_i follows a Pareto distribution with parameters $\left(\gamma, \frac{b}{\beta_i}\right)$. The risks X_1 and X_2 are conditionally independent. The survival bivariate distribution is given by:

$$\bar{F}_{X_1,X_2}(x_1,x_2) = \left(\frac{1}{1+\frac{\beta_1}{b}x_1+\frac{\beta_2}{b}x_2}\right)^{\gamma} = \left(\bar{F}_{X_1}(x_1)^{-1/\gamma}+\bar{F}_{X_1}(x_1)^{-1/\gamma}-1\right)^{-\gamma},$$

which represents the survival Clayton copula with a dependence parameter $\theta = 1/\gamma$. Therefore, the upper tail dependence coefficient is:

$$\lambda_U = \lambda_L^{Clayton} = 2^{-\gamma},$$

where $\lambda_L^{Clayton}$ is the lower tail dependence coefficient of the Clayton copula. This dependence model exhibits upper tail dependence.

The density of S is given by:

$$f_S(s) = \frac{\beta_1\beta_2\gamma}{(\beta_1-\beta_2)b}\left[\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma+1} - \left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma+1}\right],$$

and its distribution function is given by:

$$\bar{F}_S^{CM}(s) = \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma} + \frac{\beta_2}{\beta_2-\beta_1}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma}.$$

Proposition 4.2 (Expectile-Allocation, CM Model)**.** *Let (X_1, X_2) follow a bivariate common Gamma mixture model. Then, for $S = X_1 + X_2$, we have for $(k,\ell) \in \{(1,2),(2,1)\}$*

$$e_\alpha\left(X_k|S\right) = \frac{(2\alpha-1)\xi_{CM}\left(s^*,\beta_k,\beta_\ell,\gamma,b\right) + (1-\alpha)\frac{b}{(\gamma-1)\beta_k}}{(2\alpha-1)\bar{F}_S^{CM}\left(s^*,\beta_1,\beta_2,\gamma,b\right)+(1-\alpha)},$$

where s^* is the unique solution to the following equation

$$(2\alpha-1)\left[\frac{\beta_1/\beta_2}{(\beta_1-\beta_2)}\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma-1} + \frac{\beta_2/\beta_1}{(\beta_2-\beta_1)}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma-1}\right] = (1-\alpha)\left[\frac{s}{b}(\gamma-1)-\frac{1}{\beta_1}-\frac{1}{\beta_1}\right],$$

and ξ_{CM} is defined by

$$\begin{aligned}\xi_{CM}\left(s^*,\beta_k,\beta_\ell,\gamma,b\right) &= \frac{\beta_\ell b}{(\beta_\ell-\beta_k)\beta_k(\gamma-1)}\left(\frac{1}{1+\frac{\beta_k}{b}s}\right)^{\gamma}\left(1+\gamma\frac{\beta_k}{b}s\right)\\ &\quad+\frac{1}{(\beta_k-\beta_\ell)^2(\gamma-1)}\left[\beta_k b\left(\frac{1}{1+\frac{\beta_\ell}{b}s}\right)^{\gamma-1} - \beta_\ell b\left(\frac{1}{1+\frac{\beta_k}{b}s}\right)^{\gamma-1}\right],\end{aligned}$$

$\forall (k,\ell) \in \{(1,2),(2,1)\}$.

Proof. Firstly, we have

$$\mathbb{E}[(S-s)_+] = \frac{\frac{\beta_1}{\beta_2}b}{(\beta_1-\beta_2)(\gamma-1)}\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma-1} + \frac{\frac{\beta_2}{\beta_1}b}{(\beta_2-\beta_1)(\gamma-1)}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma-1},$$

the expectile $e_\alpha(S)$ is then the unique solution to the following equation

$$(2\alpha-1)\left[\frac{\beta_1/\beta_2}{(\beta_1-\beta_2)}\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma-1}+\frac{\beta_2/\beta_1}{(\beta_2-\beta_1)}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma-1}\right]=(1-\alpha)\left[\frac{s}{b}(\gamma-1)-\frac{1}{\beta_1}-\frac{1}{\beta_1}\right].$$

Now, using

$$\mathbb{E}\left[X_1\times \mathbb{1}_{\{S=s\}}\right]=\frac{\beta_1\beta_2\gamma}{(\beta_2-\beta_1)b}s\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma+1}+\frac{\beta_1\beta_2}{(\beta_1-\beta_2)^2}\left[\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma}-\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma}\right],$$

we calculate the truncated expectation $\mathbb{E}\left[X_1\times \mathbb{1}_{\{S>s\}}\right]$

$$\begin{aligned}\mathbb{E}\left[X_1\times \mathbb{1}_{\{S\geq s\}}\right]&=\frac{\beta_2 b}{(\beta_2-\beta_1)\beta_1(\gamma-1)}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma}\left(1+\gamma\frac{\beta_1}{b}s\right)\\&\quad+\frac{1}{(\beta_1-\beta_2)^2(\gamma-1)}\left[\beta_1 b\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma-1}-\beta_2 b\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma-1}\right],\end{aligned}$$

which gives us the announced expressions of the allocation contributions.

□

Remark: Computations can also be performed by conditioning on the random variable θ and then integrating the formulas derived for the case of independent exponential distributions.

5. Comonotonic case for positive distributions

In this section, we investigate the case of comonotonic risks, which correspond to perfect dependence. The concept of comonotonic random variables is related to the studies of Hoeffding (1940) [14] and Fréchet (1951) [12]. Here, we adopt the definition of comonotonic risks as first introduced in the actuarial literature by Borch (1962) [7].

A vector of random variables $(X_1, X_2, \ldots, X_n)$ is said to be comonotonic if and only if there exists a random variable Y and non-decreasing functions $\varphi_1, \ldots, \varphi_n$ such that:

$$(X_1,\ldots,X_n)\stackrel{d}{=}(\varphi_1(Y),\ldots,\varphi_n(Y)).$$

In the case where the risks $X_1, \ldots, X_d$ are comonotonic, there exists a uniform random variable U such that $X_i = F_{X_i}^{-1}(U)$ for all $i \in 1, \ldots, d$, and $S = \sum_{i=1}^{d} F_{X_i}^{-1}(U) = \varphi(U)$, where $\varphi(t) = \sum_{i=1}^{d} F_{X_i}^{-1}(t)$ and φ is a non-decreasing function. Proposition 5.1 provides a general expression for marginal contributions using the expectile-based capital allocation rule for comonotonic risk vectors. Two applications in the case of exponential and Pareto distributions are presented respectively in Lemmas 5.1 and 5.2.

Proposition 5.1 (Expectile-based allocation for comonotonic risks). *Let $X_1, \ldots, X_d$ be continuous risks with increasing distribution functions and comonotonicity. The marginal contributions using the* *expectile allocation rule are given by:*

$$e_\alpha\left(X_i|S\right) = \frac{(2\alpha-1)\left[(1-\varphi^{-1}(s^*))F_{X_i}^{-1}\left(\varphi^{-1}(s^*)\right) + \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}\left(\varphi^{-1}(s^*)\right)\right)_+\right]\right] + (1-\alpha)\mathbb{E}[X_i]}{(2\alpha-1)(1-\varphi^{-1}(s^*)) + 1 - \alpha},$$

for all $i \in \{1,\dots,d\}$, where s^* is the unique solution to the following equation

$$(2\alpha-1)\left(\sum_{\ell=1}^{d}\mathbb{E}\left[\left(X_\ell - F_{X_\ell}^{-1}\left(\varphi^{-1}(s)\right)\right)_+\right]\right) = (1-\alpha)\left(s - \sum_{\ell=1}^{d}\mathbb{E}[X_\ell]\right).$$

Proof. Since the risks $X_1,\dots,X_d$ are comonotonic, X_i and S are also comonotonic for all $i \in 1,\dots,d$. Assuming that the distributions are positive and continuous, we have:

$$\begin{aligned}\mathbb{E}\left[X_i \times \mathbb{1}_{\{S>s\}}\right] &= \int_0^{+\infty} \min\left(\bar{F}_{X_i}(t), \bar{F}_S(s)\right) dt \\ &= \int_0^{F_{X_i}^{-1}(F_S(s))} \bar{F}_S(s) dt + \int_{F_{X_i}^{-1}(F_S(s))}^{+\infty} \bar{F}_{X_i}(t) dt \\ &= \bar{F}_S(s) \times F_{X_i}^{-1}\left(F_S(s)\right) + \mathbb{E}\left[\left(X_i - F_{X_i}^{-1}\left(F_S(s)\right)\right)_+\right].\end{aligned}$$

From Equation 1.3, we directly obtain the corresponding contributions. The equation satisfied by the sum's expectile is rewritten using Theorem 7 of Dhaene et al. (2002) [10]. □

Lemma 5.1 (Comonotonic Exponential distributions)**.** *Let $X_1,\dots,X_d$ be comonotonic risks with exponential marginal distributions, where $X_i \sim \mathcal{E}(\beta_i)$ for $i = 1,\dots,d$. The marginal contributions using the expectile allocation rule are given by:*

$$e_\alpha\left(X_i|S\right) = \frac{\beta_s}{\beta_i}s^*,$$

for all $i \in \{1,\dots,d\}$, where s^* is the unique solution to the following equation

$$(2\alpha-1)\frac{e^{-\beta_s s}}{\beta_s} = (1-\alpha)\left(s - \frac{1}{\beta_s}\right),$$

and $\beta_s = 1/\sum_{\ell=1}^{d}\frac{1}{\beta_\ell}$.

Proof. In this case, $S \sim \mathcal{E}\left(\beta_s\right)$, where $\beta_s = 1/\sum_{\ell=1}^{d}\frac{1}{\beta_\ell}$. According to Proposition 5.1, the marginal contributions are given by:

$$e_\alpha\left(X_i|S\right) = \frac{(2\alpha-1)e^{-\beta_s s}}{(2\alpha-1)e^{-\beta_s s} + 1 - \alpha}\frac{\beta_s}{\beta_i}e_\alpha(S) + \frac{1}{\beta_i},$$

which directly yields the expressions for the obtained contributions. □

Remark: In this case, the allocation percentages can be written as follows:

$$e_\alpha\left(X_i|S\right)/e_\alpha(S) = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S]},\ \forall i \in \{1,\dots,d\}.$$

The allocation is proportional to the risk level, where the proportion is determined by the ratio of the expected values of X_i and S.

Lemma 5.2 (Comonotonic Pareto distributions)**.** *Let $X_1, \ldots, X_d$ be comonotonic risks following Pareto marginal distributions with the same shape parameter, i.e., $X_i \sim Pa(\beta, \lambda_i)$ for $i = 1, \ldots, d$, where $\beta > 1$. The marginal contributions using the expectile allocation rule are given by:*

$$e_\alpha\left(X_i|S\right) = \frac{\lambda_i}{\sum_{\ell=1}^d \lambda_\ell} s^*, \ \forall i \in \{1, \ldots, d\},$$

where s^* is the unique solution to the following equation:

$$s = \frac{\sum_{\ell=1}^d \lambda_\ell}{\beta - 1}\left(\frac{2\alpha - 1}{1 - \alpha}\left(\frac{\sum_{\ell=1}^d \lambda_\ell}{\sum_{\ell=1}^d \lambda_\ell + s}\right)^{\beta - 1} + 1\right).$$

Proof. We remark that in this case $S \sim Pa(\beta, \sum_{i=1}^d \lambda_i)$. By Proposition 5.1, we obtain the expressions for the marginal contributions as stated. □

It is worth noting that in this case as well, the allocation is proportional to the risk level. In fact, the allocation percentages can be expressed as follows:

$$e_\alpha\left(X_i|S\right)/e_\alpha(S) = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S]}, \ \forall i \in \{1, \ldots, d\}.$$

6. Numerical illustrations

In this section, we provide numerical illustrations to highlight the differences between contributions to the aggregate risk obtained from TVaR (Tail Value at Risk) and Expectiles-based capital allocations. Specifically, we focus on a bivariate scenario with exponential marginal distributions. Our analysis involves evaluating the allocation amounts and their respective percentages in the aggregate risk. Additionally, we investigate the influence of dependence on capital allocation using the FGM (Farlie-Gumbel-Morgenstern) model.

6.1. **Case of independence.** We consider a bivariate exponential model where X_1 represents a riskier business line compared to X_2 ($\beta_1 < \beta_2$). The expressions for the marginal contributions to the global risk can be found in Proposition 2.1. Figure 1 displays the contribution amount of X_1 (Left) and its percentage contribution to the aggregated risk (Right). Similarly, Figure 2 presents the corresponding quantities for X_2.

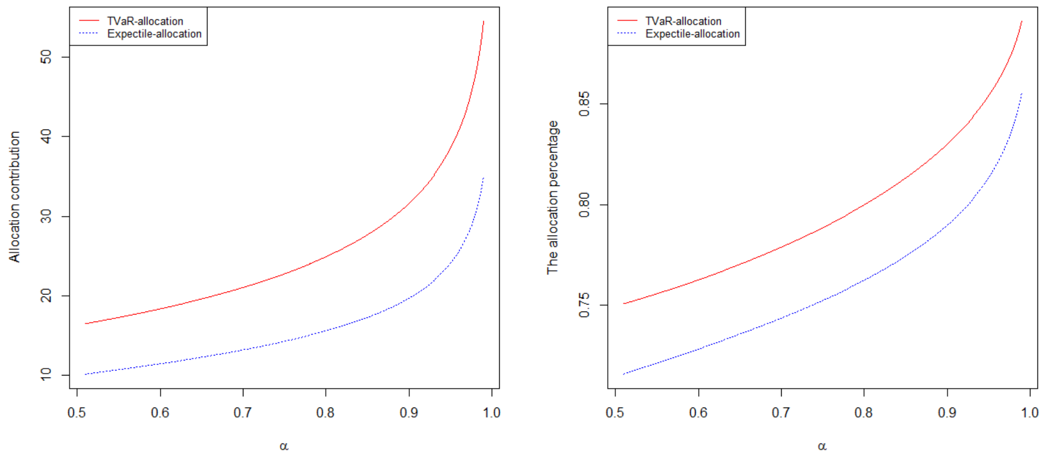

Figure 1. TVaR allocation Vs Expectile allocation, Exponential independent model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$) - X_1 contribution.

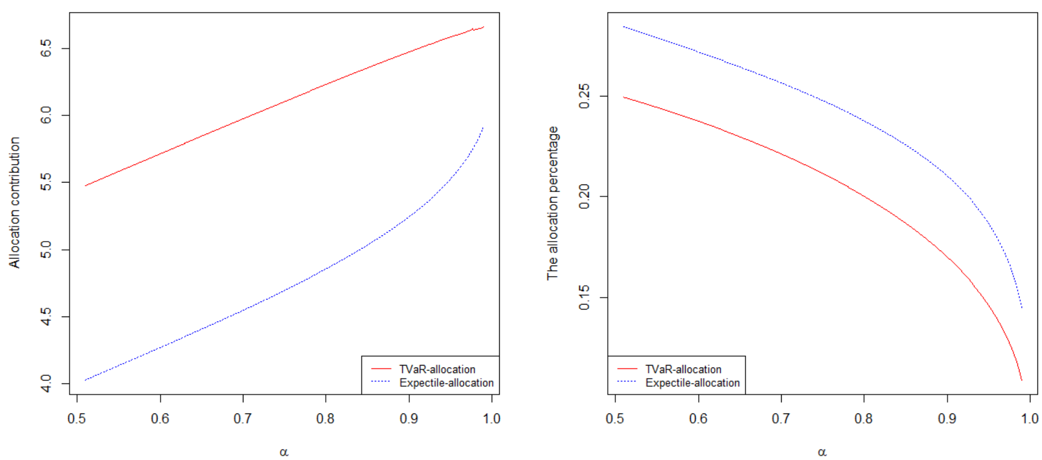

Figure 2. TVaR allocation Vs Expectile allocation, Exponential independent model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$) - X_2 contribution.

The comparison between Expectile-based and TVaR-based allocations reveals that the contributions obtained from Expectile-based allocation are consistently smaller for both risks. This discrepancy can be attributed to the nature of the Expectile risk measure, which incorporates performance considerations in its quantification of risk. By examining the percentage allocations assigned to each risk, we gain further insights into the differences between the two methods. Notably, the Expectile-based allocation assigns a relatively smaller amount of capital to the riskier branch (X_1), while maintaining an increasing allocation percentage for X_1 as the level α increases. Conversely, the allocation percentage for X_2 symmetrically decreases with increasing α, mirroring the behavior observed with TVaR allocation.

6.2. **FGM Model.** For the given marginal distributions, we now introduce a dependence structure modeled using an FGM copula with a parameter of $\theta = 1$. As a result, the correlation coefficient ρ_S is equal to $1/3$, indicating a positive dependence within the model. The expressions for marginal contributions derived from the expectile-based allocation rule can be found in Lemma 3.1.

In Figure 3, we present the contribution amount (Left) and the corresponding percentage (Right) of X_1 to the aggregated risk. Additionally, Figure 4 illustrates the variation of both the contribution amount (Left) and percentage (Right) of X_2 as a function of α.

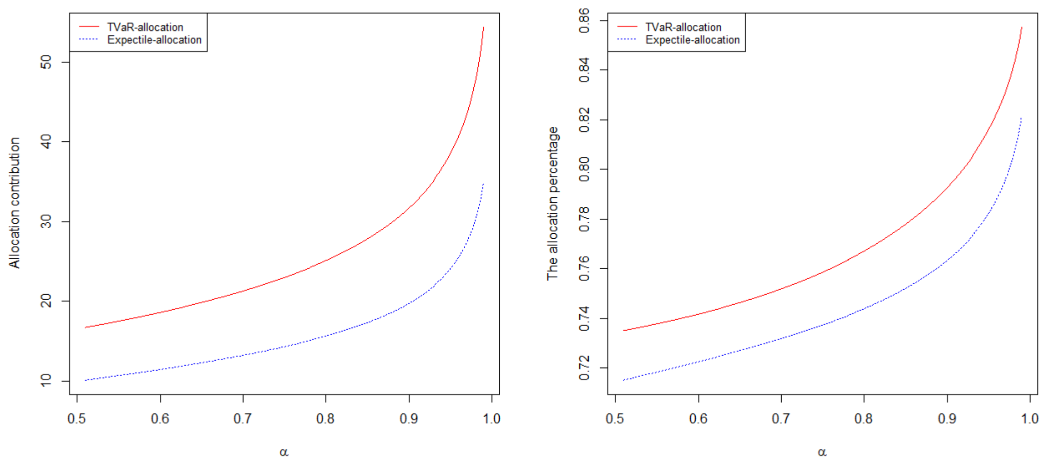

Figure 3. TVaR allocation Vs Expectile allocation, FGM model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$, $\theta = 1$) - X_1 contribution.

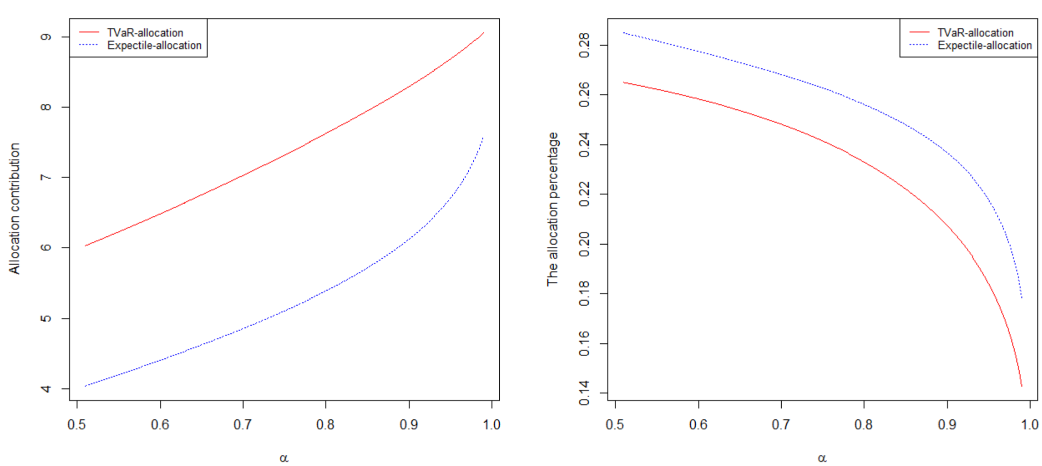

Figure 4. TVaR allocation Vs Expectile allocation, FGM model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$, $\theta = 1$) - X_2 contribution.

The inclusion of positive dependence between X_1 and X_2 resulted in an increase in the contribution of X_2. This observation aligns with the reduction in diversification gain, indicating a stronger interdependence between the two risks.

6.3. **FGM Model, Impact of dependence.** To further analyze the influence of dependence on the allocation composition, we fix the level α and vary the dependency parameter θ of the FGM copula. The outcomes for the contribution (Left) and its percentage (Right) of X_1 and X_2 are illustrated in Figures 5 and 6 respectively.

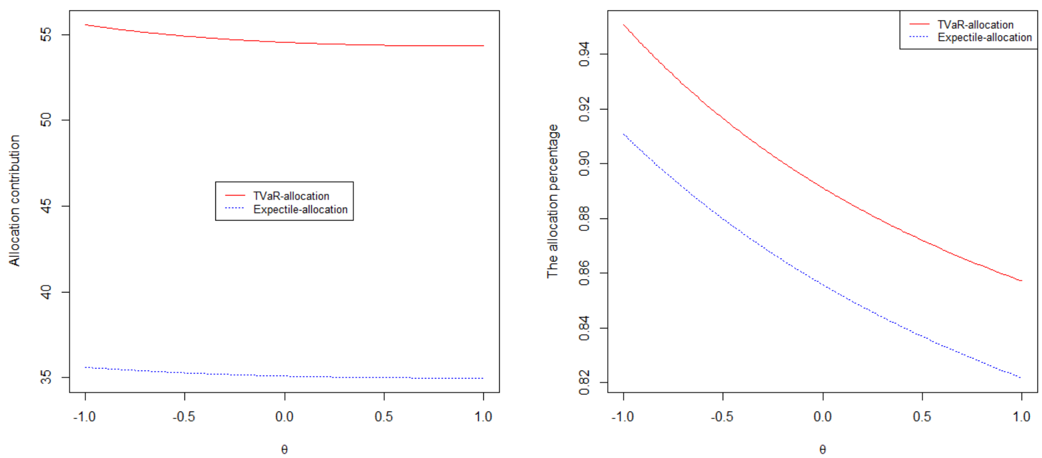

Figure 5. Impact of dependence, FGM model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$, $\alpha = 0.99$) - X_1 contribution.

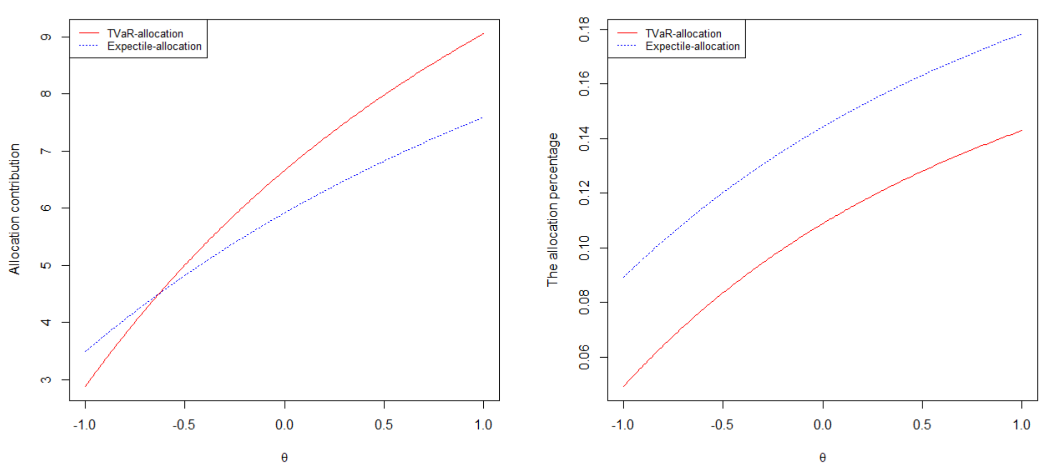

Figure 6. Impact of dependence, FGM model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$, $\alpha = 0.99$) - X_2 contribution.

As the parameter θ increases, the bivariate dependence in the FGM copula, which belongs to the family of parametric copulas, also increases. In this context, the allocation percentage assigned to the riskier branch, represented by X_1, decreases. Hence, the dependence level has a direct impact on the participation of the less risky branch, denoted as X_2, in the overall risk. Specifically, higher dependence leads to an increased involvement of X_2 in the aggregated risk.

Conclusion

The main objective of this paper was to demonstrate, using various multivariate risk models, a practical approach to constructing capital allocation based on expectile risk measures. As expectiles are the only law-invariant risk measures that are both elicitable and coherent, it is natural to focus on marginal contributions in the sum's expectile. The constructed allocation can be backtested using the elicitable nature of expectiles and it satisfies desirable properties derived from their coherence.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Artzner, F. Delbaen, J.M. Eber, D. Heath, Coherent Measures of Risk, Math. Finance. 9 (1999), 203-228. https://doi.org/10.1111/1467-9965.00068.

[2] M. Bargès, H. Cossette, É. Marceau, TVaR-Based Capital Allocation With Copulas, Insurance: Math. Econ. 45 (2009), 348-361. https://doi.org/10.1016/j.insmatheco.2009.08.002.

[3] F. Bellini, V. Bignozzi, On Elicitable Risk Measures, Quant. Finance. 15 (2015), 725-733. https://doi.org/10.1080/14697688.2014.946955.

[4] F. Bellini, E. Di Bernardino, Risk Management With Expectiles, Eur. J. Finance. 23 (2015), 487-506. https://doi.org/10.1080/1351847x.2015.1052150.

[5] F. Bellini, B. Klar, A. Müller, E.R. Gianin, Generalized Quantiles as Risk Measures, Insurance: Math. Econ. 54 (2014), 41-48. https://doi.org/10.1016/j.insmatheco.2013.10.015.

[6] M. Bladt, B.F. Nielsen, On the Construction of Bivariate Exponential Distributions with an Arbitrary Correlation Coefficient, Stoch. Models. 26 (2010), 295-308. https://doi.org/10.1080/15326341003756486.

[7] K. Borch, Equilibrium in a Reinsurance Market, Econometrica. 30 (1962), 424-444.

[8] H. Cossette, M. Mailhot, É. Marceau, TVaR-Based Capital Allocation for Multivariate Compound Distributions With Positive Continuous claim amounts, Insurance: Math. Econ. 50 (2012), 247-256. https://doi.org/10.1016/j.insmatheco.2011.11.006.

[9] H. Cossette, E. Marceau, S. Perreault, On Two Families of Bivariate Distributions With Exponential Marginals: Aggregation and Capital Allocation, Insurance: Math. Econ. 64 (2015), 214-224. https://doi.org/10.1016/j.insmatheco.2015.05.007.

[10] J. Dhaene, M. Denuit, M.J. Goovaerts, R. Kaas, D. Vyncke, The Concept of Comonotonicity in Actuarial Science and Finance: Theory, Insurance: Math. Econ. 31 (2002), 3-33. https://doi.org/10.1016/s0167-6687(02)00134-8.

[11] S. Emmer, M. Kratz, D. Tasche, What Is the Best Risk Measure in Practice? A Comparison of Standard Measures, J. Risk. 18 (2015), 31-60. https://doi.org/10.21314/jor.2015.318.

[12] M. Fréchet, Sur les Tableaux de Corrélation Dont les Marges Sont Données, Ann. l'Univ. Lyon, Ser. 3, Sect. A 14 (1951), 53-77.

[13] T. Gneiting, Making and Evaluating Point Forecasts, J. Amer. Stat. Assoc. 106 (2011), 746-762. https://doi.org/10.1198/jasa.2011.r10138.

[14] W. Hoeffding, Masstabinvariante Korrelationstheorie, Schriften des mathematischen Instituts und des Instituts für angewandte Mathematik der Universität Berlin. 5 (1940), 179-233.

[15] W.K. Newey, J.L. Powell, Asymmetric Least Squares Estimation and Testing, Econometrica. 55 (1987), 819-847. https://doi.org/10.2307/1911031.

[16] T. Mao, F. Yang, Risk Concentration Based on Expectiles for Extreme Risks Under FGM Copula, Insurance: Math. Econ. 64 (2015), 429-439. https://doi.org/10.1016/j.insmatheco.2015.06.009.

[17] V. Maume-Deschamps, D. Rullière, K. Said, Multivariate Extensions of Expectiles Risk Measures, Dependence Model. 5 (2017), 20-44. https://doi.org/10.1515/demo-2017-0002.

[18] V. Maume-Deschamps, D. Rullière, K. Said, Extremes for Multivariate Expectiles, Stat. Risk Model. 35 (2018), 111-140. https://doi.org/10.1515/strm-2017-0014.

[19] OV Sarmanov, Generalized Normal Correlation and 2-Dimensional Frechet-Classes, Dokl. Akad. Nauk SSSR. 168 (1966), 32-35.

[20] D. Tasche, Conditional Expectation as Quantile Derivative, Technical Report, TU Munchen, Germany. (2000).

[21] D. Tasche, Allocating Portfolio Economic Capital to Sub-Portfolios, Economic Capital: A Practitioner Guide, Risk Books, Risk Books, 275-302, (2004).

[22] D. Tasche, Euler Allocation: Theory and Practice, Technical Report, (2007).

[23] D. Tasche, Capital Allocation to Business Units and Sub-Portfolios: The Euler Principle, (2008). https://doi.org/10.48550/arXiv.0708.2542.

[24] C.D. Lai, N. Balakrishnan, Continuous Bivariate Distributions, Springer, New York, 2009. https://doi.org/10.1007/b101765.

[25] H. Joe, Multivariate Models and Dependence Concepts, Chapman & Hall/CRC, Boca Raton, 1997.

[26] D. Balog, Capital Allocation in Financial Institutions: The Euler Method, IEHAS Discussion Papers No. MT-DP - 2011/26, Institute of Economics, Hungarian Academy of Sciences, 2011.

[27] S. Kotz, N.L. Johnson, N. Balakrishnan, N.L. Johnson, Continuous Multivariate Distributions, Wiley, New York, 2004.

[28] É. Marceau, Modélisation et Évaluation Quantitative des Risques en Actuariat: Modèles sur une Période, Springer-Verlag France, 2013.

[29] R. Nelsen, An Introduction to Copulas, Springer, New York, 2007.