Int. J. Anal. Appl. (2023), 21:84

International Journal of Analysis and Applications

On M^*-Irresolute Topological Rings

Shallu Sharma[1,*], Naresh Digra[1], Pooja Saproo[1], Tsering Landol[2]

[1] *Department of Mathematics, University of Jammu, Jammu, India*

[2] *Department of Mathematics, Cluster University of Jammu, Jammu, India*

[*] *Corresponding author:* shallujamwal09@gmail.com

Abstract. The main aim of this paper is to introduce and study the new notions namely M^*-irresolute topological rings and M^*-irresolute topological $\mathcal{R}$-modules by virtue of M^*-open sets. Examples of an M^*-irresolute topological ring and module have been put forth. Further, we provide several fundamental properties and characterizations of M^*-irresolute topological rings and M^*-irresolute topological $\mathcal{R}$-modules. In addition, we shall define boundedness in these two structures and present several results on them.

1. Introduction

Although topological rings are useful in many branches of mathematics, they are also fascinating on their own. Since the 1940s, the theory of topological rings has been extensively developed, but primarily within the broader concept of a topological module. L.S. Pontryagin obtained one of the first fundamental results in the theory of topological rings in the classification of locally compact skew fields, which was included in his famous book [14] on topological groups. Some topological ring and module properties have also been noted in books [2, 11]. In-depth study has also been done in the last 50 years in the area of normed and Banach algebras, which constitute one of the most significant classes of topological rings (see, for example [5–8, 13]).

Besides, the theory of topological rings have been thoroughly investigated in a number of review papers and monographs (see, for example [1, 9, 10, 15–18]). In 2016, A. Devika and A. Thilagavathi [4] introduced a new class of sets in topological spaces called M^*-open sets and studied some of its properties. By continuing the study of M^*-open sets and topological rings, in this paper we introduce

Received: Jun. 12, 2023.

2020 *Mathematics Subject Classification.* 54H13, 16W80, 16W99.

Key words and phrases. M^*-open sets; M^*-irresolute topological rings; M^*-irresolute topological $\mathcal{R}$-modules.

https://doi.org/10.28924/2291-8639-21-2023-84

ISSN: 2291-8639

a new class of topological rings and modules called M^*-irresolute topological rings and M^*-irresolute topological $\mathcal{R}$-modules.

2. Preliminaries

This section covers some fundamental definitions that will be utilized in the next sections. Throughout this section, $\mathcal{A}$ and $\mathcal{A}'$ will represent two topological spaces with topologies τ and τ' respectively, on which no separation axioms are imposed. The notations $Int(U)$ and $Cl(U)$ stand for the interior and closure of a subset U of topological space $\mathcal{A}$, respectively.

Definition 2.1. *[3] Let $(\mathcal{A}, \tau)$ be a topological space and $U \subseteq \mathcal{A}$. A point $x \in \mathcal{A}$ is said to be a θ-interior point of U if there exists an open set O containing x such that $O \subseteq Cl(O) \subseteq U$. The set of all θ-interior points of U is said to be the θ-interior of U and is denoted by $Int_\theta(U)$.*

Lemma 2.1. *[3] For a subset U of a topological space $(\mathcal{A}, \tau)$, the following statements are true:*

(1) *$Int_\theta(U)$ is the union of all open sets of $\mathcal{A}$ whose closures are contained in U.*
(2) *U is θ-open if and only if $U = Int_\theta(U)$.*

Definition 2.2. *[4] Let $(\mathcal{A}, \tau)$ be a topological space. Then a subset U of a space $(\mathcal{A}, \tau)$ is said to be,*

(1) *an M^*-open set if $U \subseteq Int(Cl(Int_\theta(U)))$.*
(2) *an M^*-closed set if $U \supseteq Cl(Int(Cl_\theta(U)))$.*

The complement of an M^*-open set is called an M^*-closed. We denote the family of all M^*-open subsets (closed subsets) of $\mathcal{A}$ is denoted by $\tau_{M^*} = M^*\text{-}O(\mathcal{A})$ $(M^*\text{-}C(\mathcal{A}))$.

Lemma 2.2. *[4] Let $(\mathcal{A}, \tau)$ be a topological space. Then the following assertions are true:*

(1) *The arbitrary union of M^*-open sets is M^*-open.*
(2) *The arbitrary intersection of M^*-closed sets is M^*-closed.*

Lemma 2.3. *[4] For a topological space $(\mathcal{A}, \tau)$ the family of all M^*-open sets of $\mathcal{A}$ forms a topology for $\mathcal{A}$.*

Definition 2.3. *[4] Let U be a subset of a space $(\mathcal{A}, \tau)$. Then,*

(1) *the intersection of all M^*-closed sets containing U is called the M^*-closure of U and is denoted by $Cl_{M^*}(U)$ (sometimes by $\langle U \rangle_{\mathcal{A}}$).*
(2) *the union of all M^*-open sets contained in U is called the M^*-interior of U and is denoted by $Int_{M^*}(U)$.*

Theorem 2.1. *[4] Let U be a subset of a topological space $(\mathcal{A}, \tau)$. Then the following statements hold:*

(1) *U is an M^*-open set if and only if $U = Int_{M^*}(U)$.*
(2) *U is an M^*-closed set if and only if $U = Cl_{M^*}(U)$.*

Theorem 2.2. *[4] Let U and V be subsets of a topological space $(\mathcal{A}, \tau)$. Then the following statements are true:*

(1) *$Cl_{M^*}(\mathcal{A} - U) = \mathcal{A} - Int_{M^*}(U)$.*
(2) *$Int_{M^*}(\mathcal{A} - U) = \mathcal{A} - Cl_{M^*}(U)$.*
(3) *If $U \subseteq V$ then $Cl_{M^*}(U) \subseteq Cl_{M^*}(V)$ and $Int_{M^*}(U) \subseteq Int_{M^*}(V)$.*
(4) *$Cl_{M^*}(Cl_{M^*}(U)) = Cl_{M^*}(U)$ and $Int_{M^*}(Int_{M^*}(U)) = Int_{M^*}(U)$.*

Definition 2.4. *[4] A subset U of a topological space $(\mathcal{A}, \tau)$ is said to be a M^*-neighborhood (Briefly M^*-nbd) of a point $x \in \mathcal{A}$ if there exists an M^*-open set V such that $x \in V \subseteq U$. The family of M^*-neighborhoods of $x \in \mathcal{A}$ is called the M^*-neighborhood system of x and denoted by M^*-N_x.*

Theorem 2.3. *[4] A subset U of a topological space $(\mathcal{A}, \tau)$ is said to be M^*-open if and only if it is M^*-neighborhood for every point $x \in U$.*

Definition 2.5. *[12] A mapping $g : (\mathcal{A}, \tau) \to (\mathcal{A}', \tau')$ is called M^*-irresolute at element $x \in \mathcal{A}$ if for any M^*-open set U in $\mathcal{A}'$ containing $g(x)$ there exists an M^*-open set V in $\mathcal{A}$ containing x such that $g(V) \subseteq U$.*

Theorem 2.4. *[12] Let $g : \mathcal{A} \to \mathcal{A}'$ be a function. Then the following statements are equivalent.*

(1) *g is M^*-irresolute.*
(2) *For each $x \in \mathcal{A}$ and each M^*-neighborhood U of $g(x)$ in $\mathcal{A}'$, there is an M^*-neighborhood V of x in $\mathcal{A}$ such that $g(V) \subseteq U$.*
(3) *The inverse image of every M^*-closed subset of $\mathcal{A}'$ is an M^*-closed subset of $\mathcal{A}$.*
(4) *The inverse image of every M^*-open subset of $\mathcal{A}'$ is an M^*-open subset of $\mathcal{A}$.*

Definition 2.6. *[12] Consider two topological spaces $(\mathcal{A}, \tau)$ and $(\mathcal{A}', \tau')$. A function $g : \mathcal{A} \to \mathcal{A}'$ is called:*

(1) *M^*-continuous if $g^{-1}(U)$ is M^*-open set in $\mathcal{A}$ for every open set U in $\mathcal{A}'$.*
(2) *pre-M^*-open if $g(U)$ is M^*-open set in $\mathcal{A}'$ for every M^*-open set U in $\mathcal{A}$.*
(3) *M^*-homeomorphism if g is bijective, M^*-irresolute and pre-M^*-open.*

Definition 2.7. *[12] Let $(\mathcal{A}, \tau)$ be a topological space. Then $\mathcal{A}$ is said to be M^*-compact if each of its cover by M^*-open sets has finite subcover. A subset $S \subseteq \mathcal{A}$ is said to be M^*-compact if each cover of S by M^*-open sets of $\mathcal{A}$ has a finite subcover.*

Definition 2.8. *[1] Let $\mathcal{R}$ be a ring and $\mathcal{E}$ be an $\mathcal{R}$-module, $S \subseteq \mathcal{E}$ and $Q \subseteq \mathcal{R}$. If $Q.S \subseteq S$, then the subset S is called Q-stable.*

3. M^*-Irresolute topological rings

Definition 3.1. *An M^*-irresolute topological ring $(\mathcal{R}, +, \cdot, \tau)$ is a ring $(\mathcal{R}, +, \cdot)$ endowed with some topology τ such that the following conditions are satisfied:*

- (C1) *for each $r_1, r_2 \in \mathcal{R}$ and M^*-open set U in $\mathcal{R}$ containing $r_1 + r_2$, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing r_1 and r_2 respectively, such that $U_1 + U_2 \subseteq U$;*
- (C2) *for each $r \in \mathcal{R}$ and M^*-open set U in $\mathcal{R}$ containing $-r$, there exists M^*-open set U_1 in $\mathcal{R}$ containing r such that $-U_1 \subseteq U$;*
- (C3) *for each $r_1, r_2 \in \mathcal{R}$ and M^*-open set U in $\mathcal{R}$ containing $r_1.r_2$, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing r_1 and r_2 respectively, such that $U_1.U_2 \subseteq U$.*

Remark 3.1. *It is easy to verify that conditions (C1) and (C2) are equivalent to the following condition: For each $r_1, r_2 \in \mathcal{R}$ and each M^*-open set U in $\mathcal{R}$ containing $r_1 - r_2$, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing r_1 and r_2, respectively, such that $U_1 - U_2 \subseteq U$.*

Example 3.1. *Consider $\mathbb{Z}_4$, the ring of integers modulo 4. Define topology τ on $\mathbb{Z}_4$ as $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{1,3\}, \{0,1,3\}, \{0,2\}, \{0,1,2\}, \mathbb{Z}_4\}$, $\tau_{M^*} = \{\emptyset, \{0,2\}, \{1,3\}, \mathbb{Z}_4\}$. Then, $(\mathbb{Z}_4, \oplus_4, \odot_4, \tau)$ is an M^*-irresolute topological ring.*

Example 3.2. *Consider the ring $\mathcal{R} = (\mathbb{Z}_3, \oplus_3, \odot_3)$. Let $\tau = \{\emptyset, \{0,1\}, \mathbb{Z}_3\}$, then $\tau_{M^*} = \{\emptyset, \mathbb{Z}_3\}$. Thus, $(\mathbb{Z}_3, \oplus_3, \odot_3, \tau)$ is an M^*-irresolute topological ring.*

Example 3.3. *Let $(\mathcal{R}, +, \cdot)$ be any ring and suppose τ be a discrete or indiscrete topology on $\mathcal{R}$. Then $(\mathcal{R}, +, \cdot, \tau)$ is always an M^*-irresolute topological ring.*

Definition 3.2. *Let $(\mathcal{R}, +, \cdot, \tau)$ be an M^*-irresolute topological ring. A left $\mathcal{R}$-module $\mathcal{E}$ is called an M^*-irresolute topological left $\mathcal{R}$-module if on $\mathcal{E}$ is specified a topology such that the following conditions are satified:*

- (M1) *for every $m, n \in \mathcal{E}$ and M^*-open set U in $\mathcal{E}$ containing $m + n$, there exist M^*-open sets U_1 and U_2 in $\mathcal{E}$ containing m and n respectively, such that $U_1 + U_2 \subseteq U$;*
- (M2) *for every $m \in \mathcal{E}$ and M^*-open set U in $\mathcal{E}$ containing $-m$, there exists an M^*-open set U_1 in $\mathcal{E}$ containing m such that $-U_1 \subseteq U$;*
- (M3) *for every $r \in \mathcal{R}$ and $m \in \mathcal{E}$ and M^*-open set U in $\mathcal{E}$ containing $r.m$, there exist M^*-open set U_1 in $\mathcal{R}$ containing r and M^*-open set U_2 in $\mathcal{E}$ containing m such that $U_1.U_2 \subseteq U$.*

Remark 3.2. *In a similar manner, M^*-irresolute topological right $\mathcal{R}$-modules over an M^*-irresolute topological ring can be investigated. Any M^*-irresolute topological ring $\mathcal{R}$ is both an M^*-irresolute topological left $\mathcal{R}$-module and an M^*-irresolute topological right $\mathcal{R}$-Module.*

Remark 3.3. *Hereafter, by an M^*-irresolute topological $\mathcal{R}$-module (unless otherwise asserted), we mean an M^*-irresolute topological left $\mathcal{R}$-module.*

Example 3.4. *Consider the ring* $\mathcal{R} = (\mathbb{Z}_4, \oplus_4, \odot_4)$ *with topology* $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{1,3\}, \{0,1,3\}, \{0,2\}, \{0,1,2\}, \mathbb{Z}_4\}$, *then* $\tau_{M^*} = \{\emptyset, \{0,2\}, \{1,3\}, \mathbb{Z}_4\}$. *Therefore,* $\mathcal{R} = (\mathbb{Z}_4, \oplus_4, \odot_4, \tau)$ *is an* M^**-irresolute topological ring (see Example 3.1). Let* τ' *be the indiscrete topology on the ring* $(\{0,2\}, \oplus_4, \odot_4)$, $\tau'_{M^*} = \{\emptyset, \{0,2\}\}$. *Then, left* $\mathcal{R}$*-module* $\{0,2\}$ *with* τ' *is an* M^**-irresolute topological left* $\mathcal{R}$*-module.*

Note 3.1. *By* $\mathcal{I}(\mathcal{R})$, *we denote the set of all invertible elements in an* M^**-irresolute topological ring* $\mathcal{R}$.

Theorem 3.1. *Consider an* M^**-irresolute topological ring* $\mathcal{R}$, *an* M^**-irresolute topological* $\mathcal{R}$*-module* $\mathcal{E}$, $a \in \mathcal{R}$, $m \in \mathcal{E}$, *and* S *be a subset in* $\mathcal{R}$, T *a subset in* $\mathcal{E}$. *Then the following statements are true:*

(1) *the mapping* $\Omega_a : \mathcal{E} \to \mathcal{E}$, *where* $\Omega_a(x) = a.x$, $x \in \mathcal{E}$, *is an* M^**-irresolute mapping of the topological space* $\mathcal{E}$ *into itself.*
(2) *the mapping* $\Omega_m : \mathcal{R} \to \mathcal{E}$, *where* $\Omega_m(x) = x.m$, $x \in \mathcal{R}$ *is an* M^**-irresolute mapping of the topological space* $\mathcal{R}$ *to the topological space* $\mathcal{E}$.
(3) $\langle S.T\rangle_{\mathcal{E}} \supseteq \langle S\rangle_{\mathcal{R}}.\langle T\rangle_{\mathcal{E}}$.

Proof. (1) Let U be any M^*-open set in $\mathcal{E}$ containing $\Omega_a(x) = a.x$, where $x \in \mathcal{E}$ is arbitrary. Then, by Definition 3.2, there exist M^*-open set U_1 in $\mathcal{R}$ containing a and M^*-open set U_2 in $\mathcal{E}$ containing x, such that $U_1.U_2 \subseteq U$. Thus, $\Omega_a(U_2) = a.U_2 \subseteq U_1.U_2 \subseteq U$. This proves that Ω_a is an M^*-irresolute mapping.

(2) Let $x \in \mathcal{R}$ and let U be any M^*-open set in $\mathcal{E}$ containing $\Omega_m(x) = x.m$. Then, by Definition 3.2, there exist M^*-open set U_1 in $\mathcal{R}$ containing x and M^*-open set U_2 in $\mathcal{E}$ containing m, such that $U_1.U_2 \subseteq U$. Thus, $\Omega_m(U_1) = U_1.m \subseteq U_1.U_2 \subseteq U$. This proves that Ω_m is an M^*-irresolute mapping.

(3) Let $x \in \langle S\rangle_{\mathcal{R}}.\langle T\rangle_{\mathcal{E}}$ and let U be an M^*-open set in $\mathcal{E}$ containing x. Then $x = s.t$, where $s \in \langle S\rangle_{\mathcal{R}}$ and $t \in \langle T\rangle_{\mathcal{E}}$, and hence, there exist M^*-open set U_1 in $\mathcal{R}$ containing s and M^*-open set U_2 in $\mathcal{E}$ containing t, such that $U_1.U_2 \subseteq U$. By virtue of the fact that $U_1 \cap S \neq \emptyset$ and $U_2 \cap T \neq \emptyset$, elements $s_1 \in U_1 \cap S$ and $t_1 \in U_2 \cap T$ can be found. Thus, $s_1.t_1 \in S.T$ and $s_1.t_1 \in U_1.U_2 \subseteq U$, that is $(S.T) \cap U \neq \emptyset$. Consequently, $\langle S.T\rangle_{\mathcal{E}} \supseteq \langle S\rangle_{\mathcal{R}}.\langle T\rangle_{\mathcal{E}}$.

□

Theorem 3.2. *Let* $\mathcal{R}$ *be an* M^**-irresolute topological ring,* $a \in \mathcal{R}$, *and let* S *and* T *be subsets in* $\mathcal{R}$. *Then the following statements are true:*

(1) *the mappings* $\Sigma_a : \mathcal{R} \to \mathcal{R}$ *and* $\Sigma'_a : \mathcal{R} \to \mathcal{R}$, *where* $\Sigma_a(x) = x.a$ *and* $\Sigma'_a(x) = a.x$ *for* $x \in \mathcal{R}$, *are* M^**-irresolute mappings of the topological space* $\mathcal{R}$ *into itself;*
(2) *the mappings* $\Gamma_a : \mathcal{R} \to \mathcal{R}$ *and* $\Gamma'_a : \mathcal{R} \to \mathcal{R}$, *where* $\Gamma_a(x) = x + a$ *and* $\Gamma'_a(x) = a + x$, *for* $x \in \mathcal{R}$, *are* M^**-irresolute mappings of the topological space* $\mathcal{R}$ *into itself;*
(3) $\langle S.T\rangle_{\mathcal{R}} \supseteq \langle S\rangle_{\mathcal{R}}.\langle T\rangle_{\mathcal{R}}$.

Proof. The proof of statements (1) and (2) directly follows from the definition of an M^*-irresolute topological ring $\mathcal{R}$ (see Definition 3.1). Statement (3) is proved in the same manner as statement (3) of Theorem 3.1. $\square$

Theorem 3.3. *Let S be an M^*-open set in an M^*-irresolute topological ring $\mathcal{R}$. Then $S.a$(respectively $a.S$) is M^*-open set in $\mathcal{R}$ for every $a \in \mathcal{I}(\mathcal{R})$.*

Proof. Let $x \in S.a$ be an arbitrary element. Then $x = s.a$ for some $s \in S$. Now $s = (s.a).a^{-1} = x.a^{-1} = \Sigma_{a^{-1}}(x)$. By Theorem 3.2(1), $\Sigma_{a^{-1}} : \mathcal{R} \to \mathcal{R}$ is M^*-irresolute. Thus for the M^*-open set S containing $s = \sum_{a^{-1}}(x)$, there exists an M^*-open set U_x in $\mathcal{R}$ containing x such that $\Sigma_{a^{-1}}(U_x) \subseteq S$. This results in $U_x.a^{-1} \subseteq S$ and hence $U_x \subseteq S.a$. Thus, $S.a$ is an M^*-open set in $\mathcal{R}$. $\square$

Corollary 3.1. *Let S be an M^*-open set in an M^*-irresolute topological ring $\mathcal{R}$. Then $S+a$(respectively $a+S$) is M^*-open set in $\mathcal{R}$ for every $a \in \mathcal{R}$.*

Remark 3.4. *Theorem 3.3 and Corollary 3.1 do not hold for the choice of openness of S instead of M^*-openness of S.*

For instance, let $\mathcal{R} = (\mathbb{Z}_4, \oplus_4, \odot_4)$ with $\tau = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{1,3\}, \{0,1,3\}, \{0,2\}, \{0,1,2\}, \mathbb{Z}_4\}$. Then, by Example 3.1, $\mathcal{R}$ is an M^*-irresolute topological ring. Now, consider $S = \{0,1,2\} \in \tau$ and $3 \in \mathcal{R}$. Then $S \odot_4 3 = \{0,2,3\} \notin \tau_{M^*}$. Similarly, $S \oplus_4 3 = \{0,1,3\} \notin \tau_{M^*}$.

Theorem 3.4. *Let H be an M^*-closed set in an M^*-irresolute topological ring $\mathcal{R}$. Then $H.a$ (respectively $a.H$) is M^*-closed in $\mathcal{R}$ for every $a \in \mathcal{I}(\mathcal{R})$.*

Proof. Let $x \in \langle H.a\rangle_{\mathcal{R}}$ and U be an M^*-open set in $\mathcal{R}$ containing y, where $y = x.a^{-1}$. Then, by Definition 3.1, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing x and a^{-1} respectively such that $U_1.U_2 \subseteq U$. Since $x \in \langle H.a\rangle_{\mathcal{R}}$, $H.a \cap U_1 \neq \emptyset$. Consider $r \in H.a \cap U_1$, then $r.a^{-1} \in H \cap (U_1.U_2) \subseteq H \cap U$. Consequently, $H \cap U \neq \emptyset$, and, hence $y \in \langle H\rangle_{\mathcal{R}}$. Further, since H is M^*-closed, we have $y \in H$. Therefore $x \in H.a$. Thus $\langle H.a\rangle_{\mathcal{R}} \subseteq H.a$ and since $H.a \subseteq \langle H.a\rangle_{\mathcal{R}}$ is obvious. Hence $H.a$ is M^*-closed. $\square$

Corollary 3.2. *Let H be an M^*-closed set in an M^*-irresolute topological ring $\mathcal{R}$. Then $H + a$ (respectively $a + H$) is M^*-closed in $\mathcal{R}$ for every $a \in \mathcal{R}$.*

Remark 3.5. *Theorem 3.4 and Corollary 3.2 do not hold for the choice of closedness of H instead of M^*-closedness of H.*

For example, let $\mathcal{R} = (\mathbb{Z}_3, \oplus_3, \odot_3)$ with $\tau = \{\emptyset, \{0,1\}, \mathbb{Z}_3\}$. Then, by Example 3.2, $\mathcal{R}$ is an M^*-irresolute topological ring. Now, consider a closed set $H = \{2\}$ in $\mathcal{R}$ and $2 \in \mathcal{R}$. Then $H \odot_3 2 = H \oplus_3 2 = \{1\} \notin M^*\text{-}C(\mathcal{R})$.

Corollary 3.3. *Let S be an M^*-open set in an M^*-irresolute topological ring $\mathcal{R}$. Then $-S \in \tau_{M^*}$.*

Theorem 3.5. *Let $\mathcal{R}$ be an M^*-irresolute topological ring with unity and $a \in \mathcal{I}(\mathcal{R})$. Let $\mathcal{E}$ be an M^*-irresolute topological $\mathcal{R}$-module, then*

(1) *mapping $\Omega_a : \mathcal{E} \to \mathcal{E}$ (see Theorem 3.1(1)) is M^*-homeomorphism of the topological space $\mathcal{E}$ to itself;*
(2) *mappings $\Sigma_a : \mathcal{R} \to \mathcal{R}$ and $\Sigma'_a : \mathcal{R} \to \mathcal{R}$ (see Theorem 3.2(1)) are M^*-homeomorphism of the topological space $\mathcal{R}$ into itself.*

Proof. It is clearly evident that mappings Ω_a, Σ_a and Σ'_a are bijective mappings. Now, consider an M^*-open subset U of $\mathcal{E}$, and $x \in \Omega_a(U)$. Then $x = \Omega_a(u) = a.u$ for some $u \in U$. From $u = a^{-1}.x$ and Definition 3.2, follows the existence of an M^*-open set U_1 in $\mathcal{E}$ containing x such that $a^{-1}.U_1 \subseteq U$. Then, $U_1 \subseteq a.U = \Omega_a(U)$. This shows that $\Omega_a(U)$ is an M^*-neighborhood of each of its points and, hence, $\Omega_a(U)$ is an M^*-open subset of $\mathcal{E}$. Thus, Ω_a is pre-M^*-open mapping.

Next, consider an M^*-open set U in $\mathcal{R}$. Then $\Sigma_a(U) = U.a$. By Theorem 3.3, $U.a$ is an M^*-open set in $\mathcal{R}$. Hence, Σ_a is an pre-M^*-open mapping. In a similar manner, it can be proved that Σ'_a is also an pre-M^*-open mapping.

In view of the fact that mappings Ω_a, Σ_a and Σ'_a are M^*-irresolute (see Theorem 3.1 and Theorem 3.2), the mappings Ω_a, Σ_a and Σ'_a are M^*-homeomorphism. □

Corollary 3.4. *Let $\mathcal{R}$ be an M^*-irresolute topological ring and $a \in \mathcal{R}$. Then, the following statements are true:*

(1) *the mappings $\Gamma_a : \mathcal{R} \to \mathcal{R}$ and $\Gamma'_a : \mathcal{R} \to \mathcal{R}$ (see Theorem 3.2) are M^*-homeomorhisms of the topological space $\mathcal{R}$ into itself.*
(2) *the mapping $\Gamma : \mathcal{R} \to \mathcal{R}$, where $\Gamma(x) = -x$, are M^*-homeomorphisms of the topological space $\mathcal{R}$ into itself.*

Theorem 3.6. *For any subset S of an M^*-irresolute topological ring $\mathcal{R}$, $\langle a.S\rangle_{\mathcal{R}} = a.\langle S\rangle_{\mathcal{R}}$ for every $a \in \mathcal{I}(\mathcal{R})$.*

Proof. Consider $x \in \langle a.S\rangle_{\mathcal{R}}$ and $y = a^{-1}.x$. Let U be an M^*-open set in $\mathcal{R}$ containing y. Then, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing a^{-1} and x respectively such that $U_1.U_2 \subseteq U$. Since $x \in \langle a.S\rangle_{\mathcal{R}}$, $a.S \cap U_2 \neq \emptyset$. Suppose $r \in a.S \cap U_2$, then $a^{-1}.r \in S \cap (U_1.U_2) \subseteq S \cap U$. Thus each M^*-open set containing y intersects S. Hence, $y \in \langle S\rangle_{\mathcal{R}}$ and so $x \in a.\langle S\rangle_{\mathcal{R}}$. Conversely, let $x \in a.\langle S\rangle_{\mathcal{R}}$, then $x = a.y$ for some $y \in \langle S\rangle_{\mathcal{R}}$. Consider an M^*-open set U in $\mathcal{R}$ containing x. By hypothesis, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing a and y respectively such that $U_1.U_2 \subseteq U$. As $y \in \langle S\rangle_{\mathcal{R}}$, there exists some element $r \in S \cap U_2$. Then $a.r \in (a.S) \cap (U_1.U_2) \subseteq (a.S) \cap U$. Thus, $(a.S) \cap U$ is non empty which implies each M^*-open set containing x intersects $a.S$. Therefore, $x \in \langle a.S\rangle_{\mathcal{R}}$. Hence $\langle a.S\rangle_{\mathcal{R}} = a.\langle S\rangle_{\mathcal{R}}$. □

Corollary 3.5. *For any subset S of an M^*-irresolute topological ring $\mathcal{R}$, $\langle a + S\rangle_{\mathcal{R}} = a + \langle S\rangle_{\mathcal{R}}$ for every $a \in \mathcal{R}$.*

Lemma 3.1. *Let (A, τ) and (A', τ') be topological spaces, let $g : A \to A'$ be an M^*-homeomorphism, and let $U \subseteq A$. Then $g(\langle U\rangle_A) = \langle g(U)\rangle_{A'}$.*

Theorem 3.7. *Let S and T be subsets of an M^*-irresolute topological ring $\mathcal{R}$. Then the following statements are true:*

(1) $\langle S + T\rangle_{\mathcal{R}} \supseteq \langle S\rangle_{\mathcal{R}} + \langle T\rangle_{\mathcal{R}}$;
(2) $\langle -S\rangle_{\mathcal{R}} = -\langle S\rangle_{\mathcal{R}}$;
(3) $\langle S - T\rangle_{\mathcal{R}} \supseteq \langle S\rangle_{\mathcal{R}} - \langle T\rangle_{\mathcal{R}}$.

Proof. (1) Let $x \in \langle S\rangle_{\mathcal{R}} + \langle T\rangle_{\mathcal{R}}$ and let U be an M^*-open set in $\mathcal{R}$ containing x. Then $x = s + t$, where $s \in \langle S\rangle_{\mathcal{R}}$ and $t \in \langle T\rangle_{\mathcal{R}}$, and, hence, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing s and t respectively, such that $U_1 + U_2 \subseteq U$. By virtue of the fact that $U_1 \cap S \neq \emptyset$ and $U_2 \cap T \neq \emptyset$, elements $s_1 \in U_1 \cap S$ and $t_1 \in U_2 \cap B$ can be found. Then, $s_1 + t_1 \in S + T$ and $s_1 + t_1 \in U_1 + U_2 \subseteq U$, that is $(S + T) \cap U \neq \emptyset$. Consequently, $\langle S + T\rangle_{\mathcal{R}} \supseteq \langle S\rangle_{\mathcal{R}} + \langle T\rangle_{\mathcal{R}}$.
(2) Since the mapping $x \mapsto -x$ is an M^*-homeomorphism of the topological space $\mathcal{R}$ onto itself (see Corollary 3.4), and in view of Lemma 3.1, $\langle -S\rangle_{\mathcal{R}} = -\langle S\rangle_{\mathcal{R}}$ is valid.
(3) Inclusion $\langle S - T\rangle_{\mathcal{R}} \supseteq \langle S\rangle_{\mathcal{R}} - \langle T\rangle_{\mathcal{R}}$ results from (1) and (2). □

4. Characterizations of M^*-irresolute topological rings

Theorem 4.1. *Let $\mathcal{R}$ be an M^*-irresolute topological ring with unity, $a \in \mathcal{I}(\mathcal{R})$ and $x \in \mathcal{R}$. Then the following statements are equivalent:*

(1) *V is an M^*-neighborhood of element x of $\mathcal{R}$;*
(2) *$V.a$ is an M^*-neighborhood of element $x.a$ of $\mathcal{R}$;*
(3) *$a.V$ is an M^*-neighborhood of element $a.x$ of $\mathcal{R}$.*

Proof. Consider the M^*-homeomorphism $\Sigma_a : \mathcal{R} \to \mathcal{R}$ (see Theorem 3.5). Since $x.a = \Sigma_a(x)$ and $V.a = \Sigma_a(V)$, then (1) $\Rightarrow$ (2).

The mapping $\psi_a : \mathcal{R} \to \mathcal{R}$, where $\psi_a(y) = a.(y.a^{-1})$ for $y \in \mathcal{R}$, is the composition of the M^*-homeomorphisms $\Sigma_{a^{-1}} : \mathcal{R} \to \mathcal{R}$ and $\Sigma'_a : \mathcal{R} \to \mathcal{R}$ (see Theorem 3.5), hence, it is M^*-homeomorphism. Since $\psi_a(x.a) = a.x$ and $\psi_a(V.a) = a.V$, then (2) $\Rightarrow$ (3)

By taking into consideration the M^*-homeomorphism $\Sigma'_{a^{-1}} : \mathcal{R} \to \mathcal{R}$, equalities $\Sigma'_{a^{-1}}(a.x) = x$ and $\Sigma'_{a^{-1}}(a.V) = V$ are obtained. Then from Theorem 3.5, it follows that V is an M^*-neighborhood of the element x. Thus, (3) $\Rightarrow$ (1). □

Corollary 4.1. *Let $\mathcal{R}$ be an M^*-irresolute topological ring with unity and $a \in \mathcal{I}(\mathcal{R})$. Then the following statements are equivalent:*

(1) *V is an M^*-neighborhood of element 0 of $\mathcal{R}$;*
(2) *$V.a$ is an M^*-neighborhood of element 0 of $\mathcal{R}$;*
(3) *$a.V$ is an M^*-neighborhood of element 0 of $\mathcal{R}$.*

Corollary 4.2. *Let $\mathcal{R}$ be an M^*-irresolute topological ring and $a \in \mathcal{R}$. Then a subset $V \subseteq \mathcal{R}$ is an M^*-neighborhood of a if and only if $V - a$ is an M^*-neighborhood of 0.*

Proof. Consider an M^*-neighborhood V of a. Then there exists $V' \in \tau_{M^*}$ such that $a \in V' \subseteq V$. Since the mapping $\Gamma_{-a} : \mathcal{R} \to \mathcal{R}$ is an M^*-homeomorphism (see Corollary 3.4(1)), then $\Gamma_{-a}(V') = V' - a$ is an M^*-open set containing zero. Furthermore, clearly $V' - a \subseteq V - a$. Hence $V - a$ is an M^*-neighborhood of zero.

By virtue of the fact that the mapping $\Gamma_a : \mathcal{R} \to \mathcal{R}$, where $\Gamma_a(x) = x + a$, is an M^*-homeomorphism (see Corollary 3.4(1)), the converse can be proved similarly. □

Definition 4.1. *A collection $\mathfrak{U}_r$ of subsets of an M^*-irresolute topological ring $\mathcal{R}$ is called a basis of M^*-neighborhoods of $r \in \mathcal{R}$ if any subset of $\mathfrak{U}_r$ is an M^*-neighborhood of r and any M^*-neighborhood of the element r contains some subset from $\mathfrak{U}_r$.*

Theorem 4.2. *Let $\mathfrak{U}_0$ be a basis of M^*-neighborhoods of zero of an M^*-irresolute topological ring $\mathcal{R}$. Then the following conditions are satisfied:*

- $(N1)$ $0 \in \bigcap_{V \in \mathfrak{U}_0} V$;
- $(N2)$ *for any subsets U and W from $\mathfrak{U}_0$ there exists a subset $V \in \mathfrak{U}_0$ such that $V \subseteq U \cap W$;*
- $(N3)$ *for any subset $V \in \mathfrak{U}_0$ there exists a subset $W \in \mathfrak{U}_0$ such that $W + W \subseteq V$;*
- $(N4)$ *for any subset $V \in \mathfrak{U}_0$ there exists a subset $W \in \mathfrak{U}_0$ such that $-W \subseteq V$;*
- $(N5)$ *for any subset $V \in \mathfrak{U}_0$ there exists a subset $W \in \mathfrak{U}_0$ such that $W.W \subseteq V$;*
- $(N6)$ *for any subset $V \in \mathfrak{U}_0$ and any element $a \in \mathcal{R}$ there exists a subset $W \in \mathfrak{U}_0$ such that $a.W \subseteq V$ and $W.a \subseteq V$.*

Besides, if $a \in \mathcal{R}$, then $\mathfrak{U}_a = \{a + U \mid U \in \mathfrak{U}_0\}$ is a basis of M^*-neighborhoods of the element a.

Proof. The definition of the basis of M^*-neighborhoods of an element in a topological space results in the fulfillment of conditions $(N1)$ and $(N2)$.

Consider an M^*-neighborhood V of $0 = 0 + 0$. Since $\mathcal{R}$ is an M^*-irresolute topological ring, then there are M^*-open sets V_1 and V_2 containing zero such that $V_1 + V_2 \subseteq V$. Suppose that $W = V_1 \cap V_2$, then W is M^*-neighborhood of zero and $W + W \subseteq V_1 + V_2 \subseteq V$.

In a similar way, the fulfillment of conditions $(N4)$-$(N6)$ results from the fact that $\mathfrak{U}_0$ is a basis of M^*-neighborhoods of zero in an M^*-irresolute topological ring $\mathcal{R}$, from conditions $(C2)$ and $(C3)$ (see Definition 3.1) with regard to $-0 = 0$, $0.0 = 0$ and $0.a = 0$ for any $a \in \mathcal{R}$.

If $a \in \mathcal{R}$, then the mapping $\Gamma'_a : \mathcal{R} \to \mathcal{R}$ is an M^*-homeomorphism in view of Corollary 3.4, and, hence $\mathfrak{U}_a = \{a + U \mid U \in \mathfrak{U}_0\} = \{\Gamma'_a(U) \mid U \in \mathfrak{U}_0\}$ is a basis of M^*-neighborhoods of the element a. □

Theorem 4.3. *Let $\mathcal{R}$ be an M^*-irresolute topological ring and $\mathfrak{U}_0$ be a basis of M^*-neighborhoods of zero of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$. Then conditions $(N1)$ to $(N4)$ of Theorem 4.2, are satisfied together with the following conditions:*

($N5'$) *for any subset* $V \in \mathfrak{U}_0$ *there exist a subset* $W \in \mathfrak{U}_0$ *and an* M^**-neighborhood* U *of zero in* $\mathcal{R}$ *such that* $U.W \subseteq V$*;*

($N6'$) *for any subset* $V \in \mathfrak{U}_0$ *and any element* $r \in \mathcal{R}$ *there exists a subset* $W \in \mathfrak{U}_0$ *such that* $r.W \subseteq V$*;*

($N7'$) *for any subset* $V \in \mathfrak{U}_0$ *and any element* $a \in \mathcal{E}$ *there exists an* M^**-neighborhood* U *of zero in* $\mathcal{R}$ *such that* $U.a \subseteq V$*.*

Proof. To prove these conditions, it is necessary to use Theorem 4.2, condition ($M3$) (see Definition 3.2), and to take account of $0.a = r.0 = 0$ for any $r \in \mathcal{R}$ and $a \in \mathcal{E}$. □

Theorem 4.4. *Let* S *be a subset of an* M^**-irresolute topological ring* $\mathcal{R}$ *and* $\mathfrak{U}_0$ *be a basis of* M^**-neighborhoods of zero. Then* $\langle S\rangle_{\mathcal{R}} = \bigcap_{U\in\mathfrak{U}_0}(S+U)$*.*

Proof. Let $x \in \langle S\rangle_{\mathcal{R}}$ and $U \in \mathfrak{U}_0$. Thus, by condition $N3$ (see Theorem 4.2), there exists $W \in \mathfrak{U}_0$ such that $-W \subseteq U$. Since $x \in \langle S\rangle_{\mathcal{R}}$, then $(x+W)\cap S \neq \emptyset$. This gives $x \in S - W \subseteq S + U$. Therefore, $\langle S\rangle_{\mathcal{R}} \subseteq S+U$, and, hence $\langle S\rangle_{\mathcal{R}} \subseteq \bigcap_{U\in\mathfrak{U}_0}(S+U)$.

Now, let $y \in \bigcap_{U\in\mathfrak{U}_0}(S+U)$ and let U_1 be an M^*-neighborhood of zero in $\mathcal{R}$. Let's choose an M^*-neighborhood $U \in \mathfrak{U}_0$ of zero such that $-U \subseteq U_1$. Since $y \in S+U$, then $S\cap(y+U_1) \supseteq S\cap(y-U) \neq \emptyset$. Thus $\bigcap_{U\in\mathfrak{U}_0}(S+U) \subseteq \langle S\rangle_{\mathcal{R}}$. □

Corollary 4.3. *Let* $\mathfrak{U}_0$ *be a basis of* M^**-neighborhoods of zero of an* M^**-irresolute topological ring* $\mathcal{R}$*. Then* $\bigcap_{U\in\mathfrak{U}_0} U$ *is an* M^**-closed set.*

Proof. By Theorem 4.4, $\bigcap_{U\in\mathfrak{U}_0} U$ is an M^*-closure of a subset $\{0\}$ in $\mathcal{R}$, and hence is M^*-closed set in $\mathcal{R}$. □

Corollary 4.4. *Let* U *and* W *be* M^**-neighborhoods of zero of an* M^**-irresolute topological ring* $\mathcal{R}$ *such that* $W + W \subseteq U$*. Then* $\langle W\rangle_{\mathcal{R}} \subseteq U$*.*

Proof. It is obvious that the family $\mathfrak{U}_0$ of all M^*-neighborhoods of zero in $\mathcal{R}$ is a basis of M^*-neighborhoods of zero of $\mathcal{R}$. By Definition 4.1, for M^*-neighborhood W of zero, there exists $V \in \mathfrak{U}_0$ such that $V \subseteq W$. In the view of Theorem 4.4,

$$\langle W\rangle_{\mathcal{R}} = \bigcap_{V\in\mathfrak{U}_0}(W+V) \subseteq W+V \subseteq W+W \subseteq U.$$

Thus, $\langle W\rangle_{\mathcal{R}} \subseteq U$. □

Corollary 4.5. *Let* $\mathcal{R}$ *be an* M^**-irresolute topological ring,* $\mathcal{E}$ *be an* M^**-irresolute topological* $\mathcal{R}$*-module,* $N \subseteq \mathcal{E}$ *and let* $\mathfrak{U}_0(\mathcal{E})$ *be a basis of* M^**-neighborhoods of zero in* $\mathcal{E}$*. Then following statements are true:*

(1) $\langle N\rangle_{\mathcal{E}} = \bigcap_{U\in\mathfrak{U}_0(\mathcal{E})} (N+U)$.

(2) $\bigcap_{U\in\mathfrak{U}_0(\mathcal{E})} U$ *is* M^**-closed set in* $\mathcal{E}$.

Theorem 4.5. *Let* $\mathcal{R}$ *be an* M^**-irresolute topological ring with unity, and let* S *be an* M^**-compact subset in* $\mathcal{R}$*. Then the following assertions hold:*

(1) $a.S$ *is* M^**-compact for each* $a \in \mathcal{I}(\mathcal{R})$;

(2) $a+S$ *is* M^**-compact for each* $a \in \mathcal{R}$;

Proof. Let Λ be an indexing set and let $\{U_\beta : \beta \in \Lambda\}$ be an M^*-open cover of $a.S$. Then, $a.S \subseteq \bigcup_{\beta\in\Lambda} U_\beta$ implies that $S \subseteq \bigcup_{\beta\in\Lambda} (a^{-1}.U_\beta)$. Since each U_β is M^*-open subset of an M^*-irresolute topological ring $\mathcal{R}$, then by Theorem 3.3, $a^{-1}.U_\beta \in M^*\text{-}O(\mathcal{R})$ for each $\beta \in \Lambda$. Further, since S is M^*-compact, therefore, $S \subseteq \bigcup_{\beta\in\Lambda'} (a^{-1}.U_\beta)$ for some finite subset $\Lambda' \subseteq \Lambda$. Consequently, $a.S \subseteq \bigcup_{\beta\in\Lambda'} U_\beta$. Hence $a.S$ is M^*-compact.

Analogously, the second part of the theorem is proved. □

Definition 4.2. *Let* $\mathcal{R}$ *and* $\mathcal{S}$ *be an* M^**-irresolute topological rings. A mapping* $\Omega : \mathcal{R} \to \mathcal{S}$ *is said to be an* M^**-irresolute (pre-*M^**-open) homomorphism if* Ω *is a homomorphism of rings and an* M^**-irresolute (pre-*M^**-open) mapping of the topological spaces. A homomorphism of rings, which is at the same time* M^**-irresolute and pre-*M^**-open, is called an* M^**-topological homomorphism.*

Proposition 4.1. *Let* $\mathcal{R}$ *and* $\mathcal{S}$ *be an* M^**-irresolute topological rings, and* $\Omega : \mathcal{R} \to \mathcal{S}$ *be a homomorphic mapping of* $\mathcal{R}$ *to* $\mathcal{S}$*. Then*

(1) Ω *is an* M^**-irresolute homomorphism if and only if* $\Omega^{-1}(V)$ *is an* M^**-neighborhood of zero in* $\mathcal{R}$ *for any* M^**-neighborhood* V *of zero in* $\mathcal{S}$;

(2) Ω *is a pre-*M^**-open homomorphism if and only if* $\Omega(W)$ *is an* M^**-neighborhood of zero in* $\mathcal{S}$ *for any* M^**-neighborhood* W *of zero in* $\mathcal{R}$;

(3) Ω *is a* M^**-topological homomorphism if and only if for any* M^**-neighborhoods* U *and* U_1 *of zero in* $\mathcal{R}$ *and* $\mathcal{S}$*, correspondingly,* $\Omega(U)$ *and* $\Omega^{-1}(U_1)$ *are* M^**-neighborhoods of zero in* $\mathcal{S}$ *and* $\mathcal{R}$ *respectively.*

Proof. (1) Let V be an M^*-neighborhood of zero in $\mathcal{S}$. Then, there exists M^*-open set V' in $\mathcal{S}$ containing zero such that $0 \in V' \subseteq V$. Since Ω is an M^*-irresolute homomorphism, then it is, in particular, M^*-irresolute at $0 \in \mathcal{R}$. Further since $\Omega(0) = 0$, then by Definition 2.5, there exists M^*-open set U in $\mathcal{R}$ containing zero such that $0 \in \Omega(U) \subseteq V'$. Consequently, $0 \in U \subseteq \Omega^{-1}(V)$. This proves that $\Omega^{-1}(V)$ is an M^*-neighborhood of zero in $\mathcal{R}$.

Now let $a \in \mathcal{R}$ and U' be an M^*-neighborhood of $\Omega(a)$ in $\mathcal{S}$. Then, due to Corollary 4.2, $U' - \Omega(a)$ is an M^*-neighborhood of zero in $\mathcal{S}$. Therefore, $W = \Omega^{-1}(U' - \Omega(a))$ is an M^*-neighborhood of zero in $\mathcal{R}$. Consequently, $W + a$ is an M^*-neighborhood of $a \in \mathcal{R}$, besides,

$$\Omega(W+a)=\Omega(W)+\Omega(a)=\Omega\left(\Omega^{-1}(U'-\Omega(a))\right)+\Omega(a)\subseteq U'-\Omega(a)+\Omega(a)\subseteq U'.$$

Therefore, the mapping $\Omega:\mathcal{R}\to\mathcal{S}$ is M^*-irresolute.

(2) Let W be an M^*-neighborhood of zero in $\mathcal{R}$. Then, there exists an M^*-open set U in $\mathcal{R}$ contaning zero such that $0\in U\subseteq W$. Since Ω is pre-M^*-open mapping, then $\Omega(U)$ is an M^*-open set in $\mathcal{S}$. Besides, $0=\Omega(0)\in\Omega(U)\subseteq\Omega(W)$. Thus, $\Omega(W)$ is an M^*-neighborhood of zero in $\mathcal{S}$.

Conversely, consider an M^*-open subset V of $\mathcal{R}$, and $v\in\Omega(V)$. Then there exists an element $u\in V$ such that $v=\Omega(u)$. Since V is M^*-neighborhood of point u in $\mathcal{R}$, then $V-u$ is M^*-neighborhood of zero in $\mathcal{R}$. By hypothesis, $\Omega(V-u)$ is an M^*-neighborhood of zero in $\mathcal{S}$ and

$$\Omega(V-u)=\Omega(V)-\Omega(u)=\Omega(V)-v.$$

Thus, $\Omega(V)=\Omega(V-u)+v$ is an M^*-neighborhood of v in $\mathcal{S}$. Hence, $\Omega(V)$ is an M^*-neighborhood of each of its points. Therefore, $\Omega(V)$ is an M^*-open set in $\mathcal{S}$. This proves that Ω is pre-M^*-open mapping.

(3) follows from statements (1), (2) and Definition 4.2. □

Definition 4.3. *Let $\mathcal{R}$ be an M^*-irresolute topological ring, $\mathcal{E}$ be an M^*-irresolute topological $\mathcal{R}$-module. A subset K of $\mathcal{E}$ is called bounded if for any M^*-neighbourhood U of zero in $\mathcal{E}$ there exists an M^*-neighborhood U_1 of zero in $\mathcal{R}$ such that $U_1.K\subseteq U$. An M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$ is called bounded if $\mathcal{E}$ is bounded subset of the module $\mathcal{E}$.*

Definition 4.4. *Let $\mathcal{R}$ be an M^*-irresolute topological ring. A subset $K\subseteq\mathcal{R}$ is called bounded from left(right) if for any M^*-neighborhood U of zero in $\mathcal{R}$ there exists an M^*-neighborhood U_1 of zero in $\mathcal{R}$ such that $U_1.K\subseteq U$ (correspondingly, $K.U_1\subseteq U$). A subset K of an M^*-irresolute topological ring is called bounded, if it bounded from left and from right.*

Theorem 4.6. *Let $\mathcal{R}$ be an M^*-irresolute topological ring, $\mathcal{E}$ and $\mathcal{E}'$ be an M^*-irrresolute topological $\mathcal{R}$-modules. Suppose $\Omega:\mathcal{E}\to\mathcal{E}'$ is an M^*-irresolute homomorphism and a subset K is bounded in $\mathcal{E}$. Then the subset $\Omega(K)$ is bounded in $\mathcal{E}'$.*

Proof. Let U be an M^*-neighborhood of zero in M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}'$. Then, due to Proposition 4.1, $\Omega^{-1}(U)$ is an M^*-neighborhood of zero in M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$. By boundedness of K, there exists an M^*-neighborhood U_1 of zero in $\mathcal{R}$ such that $U_1.K\subseteq\Omega^{-1}(U)$. Then

$$U_1.\Omega(K)=\Omega(U_1.K)\subseteq\Omega(\Omega^{-1}(U))\subseteq U,$$

i.e $\Omega(K)$ is a bounded subset in $\mathcal{E}'$. □

Theorem 4.7. *Let $\mathcal{R}$ and $\mathcal{S}$ be M^*-irresolute topological rings, $\Omega : \mathcal{R} \to \mathcal{S}$ be an M^*-topological homomorphism. Let a subset K be bounded from left (bounded from right, bounded) in the ring $\mathcal{R}$, then the subset $\Omega(K)$ is bounded from left (bounded from right, bounded) in the ring $\mathcal{S}$.*

Proof. Let U be an M^*-neighborhood of zero in M^*-irresolute topological ring $\mathcal{S}$. Then, due to Proposition 4.1, $\Omega^{-1}(U)$ is an M^*-neighborhood of zero in M^*-irresolute topological ring $\mathcal{R}$. Since K is bounded from left in $\mathcal{R}$, then there exists an M^*-neighborhood U_1 of zero in $\mathcal{R}$ such that $U_1.K \subseteq \Omega^{-1}(U)$. By Proposition 4.1, $\Omega(U_1)$ is an M^*-neighborhood of zero in $\mathcal{S}$. Then

$$\Omega(U_1).\Omega(K) = \Omega(U_1.K) \subseteq \Omega(\Omega^{-1}(U)) \subseteq U,$$

i.e the subset $\Omega(K)$ is bounded from left in $\mathcal{S}$.

□

When the subset K of the ring $\mathcal{R}$ is bounded from right or bounded, the proof is analogous.

Theorem 4.8. *Every M^*-compact set in an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$ is bounded.*

Proof. Consider an M^*-compact subset K of $\mathcal{E}$. Let U be an M^*-neighborhood of zero in $\mathcal{E}$. Then, by condition $(M3)$ (see Definition 3.2), for any element $n \in K$ there exist an M^*-open set V_n in $\mathcal{R}$ containing zero and an M^*-open set W_n in $\mathcal{E}$ containing n such that $V_n.W_n \subseteq U$. Since $\{W_n \mid n \in K\}$ is an M^*-open cover of K, then there exist elements $n_1, n_2, \dots, n_i \in K$ such that $K \subseteq \bigcup_{j=1}^{i} W_{n_j}$. Then $V = \bigcap_{j=1}^{i} V_{n_j}$ is an M^*-neighborhood of zero in $\mathcal{R}$ and

$$V.K \subseteq V.(\bigcup_{j=1}^{i} W_{n_j}) \subseteq \bigcup_{j=1}^{i} (V_{n_j}.W_{n_j}) \subseteq U.$$

Thus, K is bounded subset of M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$. □

Theorem 4.9. *M^*-closure of any bounded subset of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$ is bounded.*

Proof. Let K be any bounded subset of $\mathcal{E}$ and U be an M^*-neighbourhood of zero in $\mathcal{E}$. Then there exists M^*-closed neighborhood U_1 of zero in $\mathcal{E}$ such that $U_1 \subseteq U$. Since K is bounded subset of $\mathcal{E}$, then there exists an M^*-neighborhood U_2 of zero in $\mathcal{R}$ such that $U_2.K \subseteq U_1$. Then

$$U_2.\langle K\rangle_{\mathcal{E}} \subseteq \langle U_2\rangle_{\mathcal{R}}.\langle K\rangle_{\mathcal{E}} \subseteq \langle U_2.K\rangle_{\mathcal{E}} \subseteq \langle U_1\rangle_{\mathcal{E}} = U_1 \subseteq U.$$

Thus, $\langle K\rangle_{\mathcal{E}}$ is a bounded set in $\mathcal{E}$.

□

Theorem 4.10. *If K is a bounded from left subset of an M^*-irresolute topological ring $\mathcal{R}$ and N is a bounded subset of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$, then $K.N$ is bounded subset of $\mathcal{E}$.*

Proof. Consider an M^*-neighborhood U of zero in $\mathcal{E}$ and M^*-neighborhood W of zero in $\mathcal{R}$ such that $W.N \subseteq U$. Since K is bounded from left subset of $\mathcal{R}$, then there exists M^*-neighborhood V of zero in $\mathcal{R}$ such that $V.K \subseteq W$. Then

$$V.(K.N) = (V.K).N \subseteq W.N \subseteq U.$$

Thus, $K.N$ is bounded subset of $\mathcal{E}$. □

Definition 4.5. *Let $(\mathcal{R}, +, \cdot, \tau)$ be an M^*-irresolute topological ring. A subset S of $\mathcal{R}$ is called a subring of an M^*-irresolute topological ring $\mathcal{R}$ if S is a subring of $\mathcal{R}$ and S is endowed with the topology $\tau|_S = \{V \cap S \mid V \in \tau\}$, induced by the topology τ.*

Definition 4.6. *Let $\mathcal{E}$ be an M^*-irresolute topological $\mathcal{R}$-module. A subset N of $\mathcal{E}$ is called a submodule of the M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$ if N is submodule of $\mathcal{R}$-module $\mathcal{E}$ and $\mathcal{R}$-module N is endowed with the topology induced by the topology of $\mathcal{R}$-module $\mathcal{E}$.*

Theorem 4.11. *Every Subring of an M^*-irresolute topological ring $\mathcal{R}$ is an M^*-irresolute topological ring.*

Proof. Consider a subring S of an M^*-irresolute topological ring $\mathcal{R}$. Let $r_1, r_2 \in S$ and U be an M^*-open set in S containing $r_1 - r_2$. Then $U = V \cap S$, where V is M^*-open set in $\mathcal{R}$ containing $r_1 - r_2$. By Definition 3.1, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing r_1 and r_2 respectively such that $U_1 - U_2 \subseteq V$. Then $W_1 = U_1 \cap S$ and $W_2 = U_2 \cap S$ are M^*-open sets in S containing r_1 and r_2 such that

$$W_1 - W_2 \subseteq (U_1 - U_2) \cap S \subseteq V \cap S = U.$$

Now, suppose $r_1, r_2 \in S$ and U be an M^*-open set in S containing $r_1.r_2$. Then $U = V \cap S$, where V is M^*-open set in $\mathcal{R}$ containing $r_1.r_2$. By hypothesis, there exist M^*-open sets U_1 and U_2 in $\mathcal{R}$ containing r_1 and r_2 respectively such that $U_1.U_2 \subseteq V$. Then $W_1 = U_1 \cap S$ and $W_2 = U_2 \cap S$ are M^*-open sets in S containing r_1 and r_2 respectively, besides

$$W_1.W_2 \subseteq (U_1.U_2) \cap S \subseteq V \cap S = U.$$

Hence, S is an M^*-irresolute topological ring. □

Remark 4.1. *A Submodule N of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$ is an M^*-irresolute topological $\mathcal{R}$-module.*

Theorem 4.12. *Let Q be a subset of an M^*-irresolute topological ring $\mathcal{R}$, and S be a Q-stable subset of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$, then $\langle S \rangle_{\mathcal{E}}$ is a $\langle Q \rangle_{\mathcal{R}}$-stable subset.*

Proof. Since S is Q-stable subset of $\mathcal{E}$, then $Q.S \subseteq S$ (see Definition 2.8). From Theorem 3.1(3), $\langle Q \rangle_{\mathcal{R}}.\langle S \rangle_{\mathcal{E}} \subseteq \langle Q.S \rangle_{\mathcal{E}} \subseteq \langle S \rangle_{\mathcal{E}}$. Thus, $\langle S \rangle_{\mathcal{E}}$ is a $\langle Q \rangle_{\mathcal{R}}$-stable subset. □

Proposition 4.2. *Let $\mathcal{R}$ be an M^*-irresolute topological ring and $\mathcal{E}$ be an M^*-irresolute topological $\mathcal{R}$-module. Let Q be a subring of $\mathcal{R}$, and N be a Q-submodule of $\mathcal{R}$-module $\mathcal{E}$, then*

(1) $\langle Q\rangle_{\mathcal{R}}$ *is a subring* $\mathcal{R}$*;*
(2) $\langle N\rangle_{\mathcal{E}}$ *is a* $\langle Q\rangle_{\mathcal{R}}$*-module.*

Proof. Let Q be a subring of an $\mathcal{R}$. Thus $Q - Q \subseteq Q$ and $Q.Q \subseteq Q$. By Theorem 3.7(3), $\langle Q\rangle_{\mathcal{R}} - \langle Q\rangle_{\mathcal{R}} \subseteq \langle Q-Q\rangle_{\mathcal{R}} \subseteq \langle Q\rangle_{\mathcal{R}}$. Since Q is a Q-stable subset of the M-irresolute topological $\mathcal{R}$-module $\mathcal{R}$, then, due to Theorem 4.12, $\langle Q\rangle_{\mathcal{R}}$ is a $\langle Q\rangle_{\mathcal{R}}$-stable subset of the M^*-irresolute topological $\mathcal{R}$-module $\mathcal{R}$, that is $\langle Q\rangle_{\mathcal{R}}.\langle Q\rangle_{\mathcal{R}} \subseteq \langle Q\rangle_{\mathcal{R}}$. Hence $\langle Q\rangle_{\mathcal{R}}$ is a subring $\mathcal{R}$.

Since N is Q-submodule of $\mathcal{R}$-module $\mathcal{E}$, then by Theorem 3.7(1) $\langle N\rangle_{\mathcal{E}} + \langle N\rangle_{\mathcal{E}} \subseteq \langle N+N\rangle_{\mathcal{E}} \subseteq \langle N\rangle_{\mathcal{E}}$. Also N is Q-stable subset of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$, then by Theorem 4.12, $\langle N\rangle_{\mathcal{E}}$ is a $\langle Q\rangle_{\mathcal{R}}$-stable subset of the M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$, and since $\langle Q\rangle_{\mathcal{R}}$ is a subring $\mathcal{R}$, then $\langle N\rangle_{\mathcal{E}}$ is a $\langle Q\rangle_{\mathcal{R}}$-module. □

Corollary 4.6. *Let Q be a subring of an M^*-irresolute topological ring $\mathcal{R}$ with $\langle Q\rangle_{\mathcal{R}} = \mathcal{R}$ and N be a Q-submodule of an M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$ Then, $\langle N\rangle_{\mathcal{E}}$ is a submodule of the M^*-irresolute topological $\mathcal{R}$-module $\mathcal{E}$.*
In particular, the M^*-closure of any submodule of an M^*-irresolute topological $\mathcal{R}$-module is also an M^*-irresolute topological $\mathcal{R}$-module.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] V.I. Arnautov, M.I. Vodichar, A.V. Mikhalev, Introduction to The Theory of Topological Rings and Modules, Kishinev, Stiintsa, 1981.
[2] N. Bourbaki, Éléments de Mathématique, Topologie Générale, Hermann, Paris, (1965).
[3] M. Caldas, S. Jafari, M.M. Kovar, Some Properties of θ-Open Sets, Divul. Mat. 12 (2004), 161-169.
[4] A. Devika, A. Thilagavathi, M^*-Open Sets in Topological Spaces, Int. J. Math. Appl. 4 (2016), 1-8.
[5] I.M. Gelfand, D.A. Raikov, G.E. Shilov, Commutative Normed Rings, Physmatizdt, 1960.
[6] A.Ya. Helemskii, Topologies in Banach and Topological Algebras, Moscow University, Moscow, 1986.
[7] A.Ya. Helemskii, Homology in Banach and Topological Algebras, Moscow University, Moscow, 1986.
[8] A.Ya. Helemskii, Banach and Normed Algebras, Nauka, 1989.
[9] I. Kaplansky, Topological Rings, Amer. J. Math. 69 (1947), 153-183. https://doi.org/10.2307/2371662.
[10] I. Kaplansky, Topological Rings, Bull. Amer. Math. Soc. 54 (1948), 809-826. https://doi.org/10.1090/s0002-9904-1948-09058-9.
[11] A.G. Kurosh, Lectures on General Algebra, Nauka, 1968.
[12] R.M. Latif, M–STAR–Irresolute Topological Vector Spaces, Int. J. Pure Math. 7 (2021), 20-36. https://doi.org/10.46300/91019.2020.7.4.
[13] M.A. Najmark, Normed Rings, Nauka, 1968.
[14] L.S. Pontryagin, Continuous Groups, Nauka, 1973.

[15] F. Szasz, On Topological Algebras and Rings, I, Mat. Zapok. 13 (1962), 256-278.
[16] F. Szasz, On Topological Algebras and Rings, II, Mat. Zapok. 14 (1963), 74-87.
[17] M. I. Ursul, Compact Rings and Their Generalizations, Kishinev, Stiinsta, 1991.
[18] S. Warner, Topological Rings, North-Holland Mathematics Studies, Vol. 178, North-Holland, (1993).