Int. J. Anal. Appl. (2023), 21:90

International Journal of Analysis and Applications

Upper and Lower Weakly α-$\star$-Continuous Multifunctions

Chawalit Boonpok, Prapart Pue-on*

Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

**Corresponding author:* prapart.p@msu.ac.th

Abstract. This paper deals with the concepts of upper and lower weakly α-$\star$-continuous multifunctions. Moreover, some characterizations of upper and lower weakly α-$\star$-continuous multifunctions are investigated. Furthermore, the relationships between almost α-$\star$-continuity and weak α-$\star$-continuity are discussed.

1. Introduction

Topology is concerned with all questions directly or indirectly related to continuity. Weaker and stronger forms of open sets play an important role in the researches of generalizations of continuity for functions and multifunctions. In 1965, Njåstad [18] introduced a weak form of open sets called α-sets. Mashhour et al. [17] defined a function to be α-continuous if the inverse image of each open set is an α-set and obtained several characterizations of such functions. Noiri [20] investigated the relationships between α-continuous functions and several known functions, for example, almost continuous functions, η-continuous functions, δ-continuous functions or irresolute functions. In [21], the present author introduced the concept of almost α-continuity in topological spaces as a generalization of α-continuity and almost continuity. Neubrunn [19] introduced the notion of upper (resp. lower) α-continuous multifunctions. These multifunctions are further investigated by the present authors [24]. In 1996, Popa and Noiri [23] introduced the notion of upper (resp. lower) almost α-continuous multifunctions and investigated several characterizations and some basic properties concerning upper (resp. lower) almost α-continuous multifunctions. Moreover, some characterizations of weakly α-continuous multifunctions were investigated in [11], [22] and [23]. Topological ideals have played an important

Received: Jun. 13, 2023.

2020 *Mathematics Subject Classification.* 54C08, 54C60.

Key words and phrases. upper weakly α-$\star$-continuous multifunction; lower weakly α-$\star$-continuous multifunction.

https://doi.org/10.28924/2291-8639-21-2023-90
ISSN: 2291-8639

role in topology. Kuratowski [16] and Vaidyanathswamy [25] introduced and studied the concept of ideals in topological spaces. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. In 1990, Janković and Hamlett [15] introduced the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [4] studied the concepts of α-$\mathscr{I}$-continuity and α-$\mathscr{I}$-openness in ideal topological spaces and obtained several characterizations of these functions. Hatir and Noiri [14] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. Moreover, Açikgöz et al. [3] introduced and studied the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. In [10], the present author introduced and investigated the concepts of upper and lower weakly $\star$-continuous multifunctions. In this paper, we introduce the concepts of upper and lower weakly α-$\star$-continuous multifunctions. In particular, several characterizations of upper and lower weakly α-$\star$-continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [16], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ. A subset A is said to be $\star$-closed [15] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi***-$\mathscr{I}$-open* [13] (resp. *semi-$\mathscr{I}$-open* [14]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}^\star(A))$ (resp. $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$). The complement of a semi*-$\mathscr{I}$-open (resp. semi-$\mathscr{I}$-open) set is said to be *semi***-$\mathscr{I}$-closed* [13] (resp. semi-$\mathscr{I}$-closed [14]). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-closed (resp. semi*-$\mathscr{I}$-closed) sets containing A is called the *semi-$\mathscr{I}$-closure* [14] (resp. *semi***-$\mathscr{I}$-closure* [12]) of A and is denoted by $s\mathrm{Cl}_{\mathscr{I}}(A)$ (resp. $s^\star\mathrm{Cl}_{\mathscr{I}}(A)$). The union of all semi-$\mathscr{I}$-open (resp. semi*-$\mathscr{I}$-open) sets contained in A is called the *semi-$\mathscr{I}$-interior* (resp. *semi*-$\mathscr{I}$-interior*) of A and is denoted by $s\mathrm{Int}_{\mathscr{I}}(A)$ (resp. $s^\star\mathrm{Int}_{\mathscr{I}}(A)$).

Lemma 2.1. [6] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *If A is an open set, then $s^\star Cl_{\mathscr{I}}(A) = Int(Cl^\star(A))$.*
(2) *If A is a $\star$-open set, then $sCl_{\mathscr{I}}(A) = Int^\star(Cl(A))$.*

Recall that a subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *α-$\star$-closed* [2] if $\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A))) \subseteq A$. The complement of an α-$\star$-closed set is said to be *α-$\star$-open*.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\star$-closed sets containing A is called the *α-$\star$-closure* [6] of A and is denoted by $\star\alpha\mathrm{Cl}(A)$. The *$\alpha$-$\star$-interior* [6] of A is defined by the union of all α-$\star$-open sets contained in A and is denoted by $\star\alpha\mathrm{Int}(A)$.

Lemma 2.2. [6] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *$\star\alpha Cl(A)$ is α-$\star$-closed.*
(2) *A is α-$\star$-closed if and only if $A = \star\alpha Cl(A)$.*

Lemma 2.3. [6] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) *A is α-$\star$-open in X;*
(2) *$G \subseteq A \subseteq Int^\star(Cl(G))$ for some $\star$-open set G;*
(3) *$G \subseteq A \subseteq sCl_{\mathscr{I}}(G)$ for some $\star$-open set G;*
(4) *$A \subseteq sCl_{\mathscr{I}}(Int^\star(A))$.*

Lemma 2.4. [6] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *A is α-$\star$-closed in X if and only if $sInt_{\mathscr{I}}(Cl^\star(A)) \subseteq A$.*
(2) *$sInt_{\mathscr{I}}(Cl^\star(A)) = Cl^\star(Int(Cl^\star(A)))$.*
(3) *$\star\alpha Cl(A) = A \cup Cl^\star(Int(Cl^\star(A)))$.*
(4) *$\star\alpha Int(A) = A \cap Int^\star(Cl(Int^\star(A)))$.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [5] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A}F(x)$.

3. Upper and lower weakly α-$\star$-continuous multifunctions

We begin this section by introducing the concepts of upper and lower weakly α-$\star$-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) *upper weakly α-$\star$-continuous at a point $x \in X$ if, for each $\star$-open set V of Y such that $F(x) \subseteq V$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq Cl^\star(V)$;*

(2) *lower weakly α-$\star$-continuous at a point $x \in X$ if, for each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\star$-open set U of X containing x such that $F(z) \cap Cl^\star(V) \neq \emptyset$ for every $z \in U$;*

(3) *upper (resp. lower) weakly α-$\star$-continuous if F is upper (resp. lower) weakly α-$\star$-continuous at each point of X.*

Theorem 3.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper weakly α-$\star$-continuous at $x \in X$;*

(2) *$x \in \star\alpha Int(F^+(Cl^\star(V)))$ for every $\star$-open set V of Y containing $F(x)$;*

(3) *$x \in Int^\star(Cl(Int^\star(F^+(Cl^\star(V)))))$ for every $\star$-open set V of Y containing $F(x)$.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y containing $F(x)$. Then, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq \mathrm{Cl}^\star(V)$; hence $U \subseteq F^+(\mathrm{Cl}^\star(V))$. Thus, $x \in \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$.

(2) $\Rightarrow$ (3): Let V be any $\star$-open set of Y containing $F(x)$. Then by (2), we have $x \in \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$ and by Lemma 2.4(4), $x \in \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))))$.

(3) $\Rightarrow$ (1): Let V be any $\star$-open set of Y containing $F(x)$. By (3), $x \in \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))))$ and by Lemma 2.4(4), $x \in \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$. Then, there exists an α-$\star$-open set U of X containing x such that $U \subseteq F^+(\mathrm{Cl}^\star(V))$; hence $F(U) \subseteq \mathrm{Cl}^\star(V)$. This shows that F is upper weakly α-$\star$-continuous at x. $\square$

Theorem 3.2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower weakly α-$\star$-continuous at $x \in X$;*

(2) *$x \in \star\alpha Int(F^-(Cl^\star(V)))$ for every $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*

(3) *$x \in Int^\star(Cl(Int^\star(F^-(Cl^\star(V)))))$ for every $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 3.1. $\square$

Definition 3.2. [10] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *R-$\mathscr{I}^\star$-open if $A = Int^\star(Cl^\star(A))$;*

(2) *R-$\mathscr{I}^\star$-closed if its complement is R-$\mathscr{I}^\star$-open.*

Definition 3.3. [9] *A point x in an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\star_\theta$-cluster point of A if $Cl^\star(U) \cap A \neq \emptyset$ for every $\star$-open set U of X containing x. The set of all $\star_\theta$-cluster points of A is called the $\star_\theta$-closure of A and is denoted by $\star_\theta Cl(A)$.*

Definition 3.4. [9] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *$\star_\theta$-closed if $\star_\theta Cl(A) = A$;*

(2) $\star_\theta$-*open if its complement is* $\star_\theta$-*closed.*

Lemma 3.1. [9] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *If A is $\star$-open in X, then $Cl^\star(A) = \star_\theta Cl(A)$.*
(2) *$\star_\theta Cl(A)$ is $\star$-closed in X.*

Theorem 3.3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper weakly α-$\star$-continuous;*
(2) *$F^+(V) \subseteq Int^\star(Cl(Int^\star(F^+(Cl^\star(V)))))$ for every $\star$-open set V of Y;*
(3) *$Cl^\star(Int(Cl^\star(F^-(Int^\star(K))))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(4) *$\star\alpha Cl(F^-(Int^\star(K))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(5) *$\star\alpha Cl(F^-(Int^\star(Cl^\star(B)))) \subseteq F^-(Cl^\star(B))$ for every subset B of Y;*
(6) *$F^+(Int^\star(B)) \subseteq \star\alpha Int(F^+(Cl^\star(Int^\star(B))))$ for every subset B of Y;*
(7) *$F^+(V) \subseteq \star\alpha Int(F^+(Cl^\star(V)))$ for every $\star$-open set V of Y;*
(8) *$\star\alpha Cl(F^-(Int^\star(K))) \subseteq F^-(K)$ for every R-$\mathscr{J}^\star$-closed set K of Y;*
(9) *$\star\alpha Cl(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every $\star$-open set V of Y;*
(10) *$\star\alpha Cl(F^-(Int^\star(\star_\theta Cl(B)))) \subseteq F^-(\star_\theta Cl(B))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq \mathrm{Cl}^\star(V)$; hence $U \subseteq F^+(\mathrm{Cl}^\star(V))$ and so $x \in U \subseteq \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))))$. Thus, $F^+(V) \subseteq \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))))$.

(2) $\Rightarrow$ (3): Let K be any $\star$-closed set of Y. Then, $Y - K$ is $\star$-open in Y and by (2), we have

$$\begin{aligned}
X - F^-(K) &= F^+(Y - K) \\
&\subseteq \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(Y - K))))) \\
&= \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(Y - \mathrm{Int}^\star(K))))) \\
&= \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(X - F^-(\mathrm{Int}^\star(K))))) \\
&= \mathrm{Int}^\star(\mathrm{Cl}(X - \mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))))) \\
&= \mathrm{Int}^\star(X - \mathrm{Int}(\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))))) \\
&= X - \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K)))))
\end{aligned}$$

and hence $\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))))) \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let K be any $\star$-closed set of Y. By (3), we have $\mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(F^-(\mathrm{Int}^\star(K))))) \subseteq F^-(K)$ and hence $\star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$ by Lemma 2.4(3).

(4) $\Rightarrow$ (5): Let B be any subset of Y. Then, $\mathrm{Cl}^\star(B)$ is $\star$-closed in Y and by (4), $\star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(B)))) \subseteq F^-(\mathrm{Cl}^\star(B))$.

(5) ⇒ (6): Let B be any subset of Y. By (5),

$$\begin{aligned}F^+(\mathrm{Int}^\star(B)) &= X - F^-(\mathrm{Cl}^\star(Y - B))\\ &\subseteq X - \star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(Y - B))))\\ &= X - \star\alpha\mathrm{Cl}(F^-(Y - \mathrm{Cl}^\star(\mathrm{Int}^\star(B))))\\ &= X - \star\alpha\mathrm{Cl}(X - F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B))))\\ &= \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(B)))).\end{aligned}$$

(6) ⇒ (7): The proof is obvious.

(7) ⇒ (1): Let $x \in X$ and V be any $\star$-open set of Y containing $F(x)$. It follows from Lemma 2.4(4) that $x \in F^+(V) \subseteq \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(V))) \subseteq \mathrm{Int}^\star(\mathrm{Cl}(\mathrm{Int}^\star(F^+(\mathrm{Cl}^\star(V)))))$ and hence F is upper weakly α-$\star$-continuous at x by Theorem 3.1. This shows that F is upper weakly α-$\star$-continuous.

(4) ⇒ (8): The proof is obvious.

(8) ⇒ (9): Let V be any $\star$-open set of Y. Then, we have $\mathrm{Cl}^\star(V)$ is R-$\mathscr{I}^\star$-closed in Y and by (8), $\star\alpha\mathrm{Cl}(F^-(V)) \subseteq \star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(9) ⇒ (7): Let V be any $\star$-open set of Y. By (9), we have

$$\begin{aligned}X - \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(V))) &= \star\alpha\mathrm{Cl}(X - F^+(\mathrm{Cl}^\star(V)))\\ &= \star\alpha\mathrm{Cl}(F^-(Y - \mathrm{Cl}^\star(V)))\\ &\subseteq F^-(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V)))\\ &= X - F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))\end{aligned}$$

and hence $F^+(V) \subseteq F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))) \subseteq \star\alpha\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$.

(9) ⇒ (10): Let B be any subset of Y. Then, $\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B))$ is $\star$-open in Y. Thus, by (9) and Lemma 3.1(1),

$$\begin{aligned}\star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B)))) &\subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B))))\\ &\subseteq F^-(\star_\theta\mathrm{Cl}(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(B))))\\ &\subseteq F^-(\star_\theta\mathrm{Cl}(B)).\end{aligned}$$

(10) ⇒ (8): Let K be any R-$\mathscr{I}^\star$-closed set of Y. By (10) and Lemma 3.1(1), we have

$$\begin{aligned}\star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) &= \star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(K)))))\\ &= \star\alpha\mathrm{Cl}(F^-(\mathrm{Int}^\star(\star_\theta\mathrm{Cl}(\mathrm{Int}^\star(K)))))\\ &\subseteq F^-(\star_\theta\mathrm{Cl}(\mathrm{Int}^\star(K)))\\ &= F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(V)))\\ &= F^-(K).\end{aligned}$$

□

Theorem 3.4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower weakly α-$\star$-continuous;*
(2) *$F^-(V) \subseteq Int^\star(Cl(Int^\star(F^-(Cl^\star(V)))))$ for every $\star$-open set V of Y;*
(3) *$Cl^\star(Int(Cl^\star(F^+(Int^\star(K))))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(4) *$\star\alpha Cl(F^+(Int^\star(K))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(5) *$\star\alpha Cl(F^+(Int^\star(Cl^\star(B)))) \subseteq F^+(Cl^\star(B))$ for every subset B of Y;*
(6) *$F^-(Int^\star(B)) \subseteq \alpha Int_\star(F^-(Cl^\star(Int^\star(B))))$ for every subset B of Y;*
(7) *$F^-(V) \subseteq \alpha Int_\star(F^-(Cl^\star(V)))$ for every $\star$-open set V of Y;*
(8) *$\star\alpha Cl(F^+(Int^\star(K))) \subseteq F^+(K)$ for every R-$\mathscr{J}^\star$-closed set K of Y;*
(9) *$\star\alpha Cl(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every $\star$-open set V of Y;*
(10) *$\star\alpha Cl(F^+(Int^\star(\star_\theta Cl(B)))) \subseteq F^+(\star_\theta Cl(B))$ for every subset B of Y.*

Proof. The proof is similar to that of Theorem 3.3. □

Definition 3.5. [7] *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) *upper almost α-$\star$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \subseteq V$, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq Int^\star Cl(V)$;*
(2) *lower almost α-$\star$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\star$-open set U of X containing x such that $F(z) \cap Int^\star(Cl(V)) \neq \emptyset$ for every $z \in U$;*
(3) *upper (resp. lower) almost α-$\star$-continuous if F is upper (resp. lower) almost α-$\star$-continuous at each point of X.*

Remark 3.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:*

upper almost α-$\star$-continuity $\Rightarrow$ upper weakly α-$\star$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, X\}$ and an ideal $\mathscr{I} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{c\}\}$. Define $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{a, c\}$. Then, F is upper weakly α-$\star$-continuous but F is not upper almost α-$\star$-continuous.*

Definition 3.6. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be weakly α-$\star$-continuous if, for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists an α-$\star$-open set U of X containing x such that $f(U) \subseteq Cl^\star(V)$.*

Corollary 3.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly α-$\star$-continuous;*
(2) $f^{-1}(V) \subseteq \star\alpha Int(f^{-1}(Cl^{\star}(V)))$ *for every $\star$-open set V of Y;*
(3) $\star\alpha Cl(f^{-1}(Int^{\star}(K))) \subseteq f^{-1}(K)$ *for every R-$\mathscr{J}^{\star}$-closed set K of Y;*
(4) $\star\alpha Cl(f^{-1}(V)) \subseteq f^{-1}(Cl^{\star}(V))$ *for every $\star$-open set V of Y;*
(5) $\star\alpha Cl(f^{-1}(Int^{\star}(\star_{\theta}Cl(B)))) \subseteq f^{-1}(\star_{\theta}Cl(B))$ *for every subset B of Y;*
(6) $Cl^{\star}(Int(Cl^{\star}(f^{-1}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ *for every $\star$-open set V of Y;*
(7) $f^{-1}(V) \subseteq Int^{\star}(Cl(Int^{\star}(f^{-1}(Cl^{\star}(V)))))$ *for every $\star$-open set V of Y;*
(8) $f(Cl^{\star}(Int(Cl^{\star}(A)))) \subseteq \star_{\theta}Cl(f(A))$ *for every subset A of X;*
(9) $Cl^{\star}(Int(Cl^{\star}(f^{-1}(B)))) \subseteq f^{-1}(\star_{\theta}Cl(B))$ *for every subset B of Y.*

Lemma 3.2. [8] *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. If A is a $\star$-regular $\star$-paracompact set of X and each $\star$-open set U containing A, then there exists a $\star$-open set V such that $A \subseteq V \subseteq Cl(V) \subseteq U$.*

Theorem 3.5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $F(x)$ is a $\star$-regular $\star$-paracompact set for each $x \in X$, the following properties are equivalent:*

(1) *F is upper α-$\star$-continuous;*
(2) *F is upper almost α-$\star$-continuous;*
(3) *F is upper weakly α-$\star$-continuous.*

Proof. We prove only the implication (3) $\Rightarrow$ (1). Suppose that F is upper weakly α-$\star$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. Since $F(x)$ is $\star$-regular $\star$-paracompact, by Lemma 3.2, there exists a $\star$-open set G such that $F(x) \subseteq G \subseteq \mathrm{Cl}(G) \subseteq V$. Since F is upper weakly α-$\star$-continuous, there exists an α-$\star$-open set U of X containing x such that $F(U) \subseteq \mathrm{Cl}^{\star}(G)$ and hence $F(U) \subseteq \mathrm{Cl}^{\star}(G) \subseteq \mathrm{Cl}(G) \subseteq V$. This shows that F is upper α-$\star$-continuous. $\square$

Theorem 3.6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $F(x)$ is $\star$-open in Y for each $x \in X$, the following properties are equivalent:*

(1) *F is lower α-$\star$-continuous;*
(2) *F is lower almost α-$\star$-continuous;*
(3) *F is lower weakly α-$\star$-continuous.*

Proof. We prove the only implication (3) $\Rightarrow$ (1). Suppose that F is lower weakly α-$\star$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \cap V \neq \emptyset$. Since F is lower weakly α-$\star$-continuous, there exists an α-$\star$-open set U of X containing x such that $F(z) \cap \mathrm{Cl}^{\star}(V) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\star$-open, we have $F(z) \cap V \neq \emptyset$ for each $z \in U$ and hence F is lower α-$\star$-continuous. $\square$

Acknowledgements: This research project was financially supported by Mahasarakham University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.E. Abd El-Monsef, E.F. Lashien, A.A. Nasef, On $\mathscr{I}$-Open Sets and $\mathscr{I}$- Continuous Functions, Kyungpook Math. J. 32 (1992), 21–30.

[2] A. Açikgöz, T. Noiri, Ş. Yüksel, On α-Operfect Sets and α-$\star$-Closed Sets, Acta Math. Hungarica. 105 (2010), 146–153.

[3] A. Açikgöz, Ş. Yüksel, E. Gursel, On a New Concept of Functions in Ideal Topological Spaces, J. Fac. Sci. Ege Univ. 29 (2006), 30–35.

[4] A. Açikgöz, T. Noiri, Ş. Yüksel, On α-$\mathscr{I}$-Continuous and α-$\mathscr{I}$-Open Functions, Acta Math. Hungarica. 105 (2004), 27–37.

[5] C. Berge, Espaces Topologiques Fonctions Multivoques, Dunod, Paris, 1959.

[6] C. Boonpok, J. Khampakdee, Upper and Lower α-$\star$-Continuity, (accepted).

[7] C. Boonpok, N. Srisarakham, Almost α-$\star$-Continuity for Multifunctions, (submitted).

[8] C. Boonpok, On Some Types of Continuity for Multifunctions in Ideal Topological Spaces, Adv. Math., Sci. J. 9 (2020), 859–886. https://doi.org/10.37418/amsj.9.3.13.

[9] C. Boonpok, Weak Quasi Continuity for Multifunctions in Ideal Topological Spaces, Adv. Math., Sci. J. 9 (2020), 339–355. https://doi.org/10.37418/amsj.9.1.28.

[10] C. Boonpok, On Continuous Multifunctions in Ideal Topological Spaces, Lobachevskii J. Math. 40 (2019), 24–35. https://doi.org/10.1134/s1995080219010049.

[11] J. Cao, J. Dontchev, On Some Weaker Forms of Continuity for Multifunctions, Real Anal. Exchange, 22 (1996/97), 842–852.

[12] E. Ekici, T. Noiri, $\star$-Hyperconnected Ideal Topological Spaces, Ann. Alexandru Ioan Cuza Univ. - Math. 58 (2012), 121–129. https://doi.org/10.2478/v10157-011-0045-9.

[13] E. Ekici, T. Noiri, $\star$-Extremally Disconnected Ideal Topological Spaces, Acta Math. Hungarica. 122 (2009), 81–90.

[14] Hatir and T. Noiri, On Decompositions of Continuity via Idealization, Acta Math. Hungarica. 96 (2002), 341–349.

[15] D. Janković and T. R. Hamlett, New Topologies from Old via Ideals, Amer. Math. Mon. 97 (1990), 295–310. https://doi.org/10.1080/00029890.1990.11995593.

[16] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.

[17] A.S. Mashhour, I.A. Hasanein, S.N. El-Deeb, α-Continuous and α-Open Mapping, Acta Math. Hungarica. 41 (1983), 213–218.

[18] O. Njåstad, On Some Classes of Nearly Open Sets, Pac. J. Math. 15 (1965), 961–970. https://doi.org/10.2140/pjm.1965.15.961.

[19] T. Neubrunn, Strongly Quasi-Continuous Multivalued Mapping, in: General Topology and its Relations to Modern Analysis and Algebra VI (Prague 1986), Heldermann, Berlin, 1988, pp. 351–359.

[20] T. Noiri, On α- Continuous Functions, Časopis Pěst. Mat. 109 (1984), 118–126. http://dml.cz/dmlcz/108508.

[21] T. Noiri, Almost α-Continuous Functions, Kyungpook Math. J. 28 (1988), 71–77.

[22] V. Popa, T. Noiri, On Upper and Lower Weakly α-Continuous Multifunctions, Novi Sad J. Math. 32 (2002), 7–24.

[23] V. Popa, T. Noiri, On Upper and Lower Almost α-Continuous Multifunctions, Demonstr. Math. 24 (1996), 381–396.

[24] V. Popa, T. Noiri, On Upper and Lower α-Continuous Multifunctions, Math. Slovaca, 43 (1993), 477–491.

[25] V. Vaidyanathswamy, The localization theory in set topology, Proc. Indian Acad. Sci. 20 (1945), 51–61.