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Sum Connectivity Index Under the Cartesian and Strong Products Graph of Monogenic Semigroup

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Abstract. This field's main feature is to implement the sum connectivity index method. This sum connectivity index method can solve the monogenic semigroups under the cartesian and strong products. We will define for an undirected graph as $SCI(\mathcal{G}_{\mathcal{MS}}) = \sum_{\mathfrak{uv}\in E(\mathcal{G}_{\mathcal{MS}})} \left[d_{\mathcal{G}_{\mathcal{MS}}}(\mathfrak{u}) + d_{\mathcal{G}_{\mathcal{MS}}}(\mathfrak{v})\right]^{-\frac{1}{2}}$, where $d_{\mathcal{G}_{\mathcal{MS}}}(\mathfrak{u})$ and $d_{\mathcal{G}_{\mathcal{MS}}}(\mathfrak{v})$ are degree of $\mathfrak{u}$ and $\mathfrak{v}$ in $\mathcal{G}_{\mathcal{MS}}$ respectively. Further, we investigate two different algorithms concerning topological index for computing cartesian and strong products of a monogenic semigroup with a detailed example.

1. Introduction

Here we consider the monogenic semigroup $\mathcal{G}_{\mathcal{MS}}$ having vertices and edges respectively, for graph theoretic concepts two products namely cartesian and strong products are two important operations used to combine two graphs into a new one. A monogenic semigroup is a mathematical structure that combines elements of both algebraic and graph theoretical concepts. The sum connectivity index (SCI) is a graph theoretical parameter that measures the overall connectedness of a graph. In recent years, researchers have investigated the SCI of various types of graphs, as it reflects the robustness and resilience of a network. In particular, the SCI of the cartesian and strong product graphs of monogenic semigroups have gained attention due to their potential applications in computer science, communication networks, and social networks.

Chen [4] devoted the graph and its line graph obtained through the general sum connectivity index. In [1], the authors studied the various type of connectivity indexes. In [8], the authors used the method

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to solve generalized sum connectivity index n-vertex trees and the corresponding extremal trees. Wang *et al.* [3], the authors investigated a triangle-free graph used to sum connectivity index. In [12] have recently focused on the bicyclic graphs to prove the minimum and maximum sum–connectivity indices. For instance, the solution of various product graphs through the monogenic semigroup was obtained by Aydm *et al.* [5] Wiener index over the dot product and Seda [6] Sombor index over the Cartesian and Tensor products. Akgunes [7] demonstrated some importance graph parameter used strong product to solve the monogenic semigroups.

There are many published paper on sum-connectivity index to obtain, unicyclic graphs [11] trees and uncyclic graphs [10] tree [8] and molecular trees [9] through various approach.

Das *et al.* (2013) presented a study on the finite monogenic semigroup S_M with zero, which has elements $\nu_1,\ \nu_2, \nu_3,\ \dots,\ \nu_n$. They determined various properties of S_M, such as the diameter, girth, domination number, chromatic number, clique number, degree sequence, irregularity index, maximum and minimum degrees. In addition, they demonstrated a spectral property for the cartesian product $\Gamma(S_M^1 \times \Gamma S_M^2)$.

Several authors like Zhou and Trinajstić [1, 16, 17], introduced and extended the work on sum-connectivity index which established a new research study of interest.

Motivated and influenced by the above results, The theme of current work is to explore the SCI of the cartesian and strong product graph of the monogenic semigroups. In addition, the entire study of the current work is categorized as follows: In the second segment on some basic notations typically defines key concepts and terminology related to the topic under study. On the other hand, we employed used the sum connectivity index to solve the monogenic semigroups under the Cartesian and Strong product graphs in third and fourth segment. The present results are validated by using some pertinent examples and which are provided in the fifth segment. The significant findings of the present work are illustrated in the last segment.

2. Preliminary Results

In this study, the following monogenic semigroups were utilized: S_M^1 and S_M^2, which defined as follows, respectively, $S_M^1 = \{\nu_1, \nu_1^2, \nu_1^3, \dots, \nu_1^p\} \cup \{0\}$ and $S_M^2 = \{\nu_2, \nu_2^2, \nu_2^3, \dots, \nu_2^q\} \cup \{0\}$.

The vertex set of the cartesian and strong product of S_M^1 and S_M^1 is given as:
$\{(\nu_1, \nu_2), (\nu_1^2, \nu_2), \dots, (\nu_1^p, \nu_2), (\nu_1, \nu_2^2), (\nu_1^2, \nu_2^2), \dots, (\nu_1^p, \nu_2^2), \dots, (\nu_1, \nu_2^{q-1}), (\nu_1^2, \nu_2^{q-1}), \dots,$
$(\nu_1^p, \nu_2^{q-1}), (\nu_1, \nu_2^q), (\nu_1^2, \nu_2^q), \dots, (\nu_1^p, \nu_2^q)\}$.

Definition 2.1. *Let $\mathcal{S}$ be a monogenic semigroup generated by a single element ν and $\mathcal{G}$ be the undirected graph whose vertices are the elements of the monogenic semigroup $\mathcal{G}_{\mathcal{MS}}$.*

Let us consider sum connectivity index of undirected graph and which is expressed as: [1, 16]

$$SCI(\mathcal{G}_{\mathcal{MS}}) = \sum_{\mathfrak{u}\mathfrak{v} \in E(\mathcal{G}_{\mathcal{MS}})} \Big[d(\mathfrak{u}) + d(\mathfrak{v})\Big]^{-\frac{1}{2}}. \tag{2.1}$$

Furthermore, for a real number $\mathfrak{r}$, we identify the most significant integer lessthan or equal $\mathfrak{r}$ by $\lceil \mathfrak{r} \rceil$, and the little integer greater than or equal $\mathfrak{r}$ by $\lfloor \mathfrak{r} \rfloor$. It is evident that $\mathfrak{r} - 1 \leq \lfloor \mathfrak{r} \rfloor \leq \mathfrak{r}$ and $\mathfrak{r} \leq \lceil \mathfrak{r} \rceil \leq \mathfrak{r} + 1$. Moreover, for any positive integer $\mathfrak{v}$, we obtain,

$$\lceil \frac{\mathfrak{v}}{2} \rceil = \begin{cases} \frac{1}{2}, \text{ if } \mathfrak{v} \text{ is even,} \\ \frac{\mathfrak{v}+1}{2}, \text{ if } \mathfrak{v} \text{ is odd.} \end{cases} \tag{2.2}$$

In this case, any pair of vertices $(\nu_1^{\imath}, \nu_2^{\mathfrak{b}})$ and (ν_1^{a}, ν_2^{b}) have a relationship if only if

$$\nu_1^{\imath}, \nu_1^{a} \in E(\Gamma(S_M^1)) \Longleftrightarrow \nu_1^{\imath}, \nu_1^{a} = 0 \Longleftrightarrow \imath + a \geq \mathfrak{v} + 1, \tag{2.3}$$

and

$$\nu_2^{\mathfrak{b}}, \nu_2^{b} \in E(\Gamma(S_M^2)) \Longleftrightarrow \nu_2^{\mathfrak{b}}, \nu_2^{b} = 0 \Longleftrightarrow \jmath + b \geq \mathfrak{u} + 1. \tag{2.4}$$

Definition 2.2. *The product of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted as $\mathcal{G}_1 \times \mathcal{G}_2$, is a graph with vertex set $\mathcal{G}_1 \times \mathcal{G}_2$, where (ν_1, ν_2) and (ρ_1, ρ_2) are adjacent in $\mathcal{G}_1 \times \mathcal{G}_2$ if and only if $\nu_1, \rho_1 \in E(\mathcal{G}_1)$ and $\nu_2 = \rho_2$ or $\nu_2, \rho_2 \in E(\mathcal{G}_2)$ and $\nu_1 = \rho_1$.*

Definition 2.3. *The strong product of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is the graph $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ with vertex set $\mathcal{G}_1 \times \mathcal{G}_2$ and $(\nu_1, \nu_2)(\rho_1, \rho_2) \in (\mathcal{G}_1 \boxtimes \mathcal{G}_2)$ whenever $\nu_1, \rho_1 \in E(\mathcal{G}_1)$ and $\nu_2 = \rho_2$ or $\nu_2\rho_2 \in E(\mathcal{G}_2)$ and $\nu_1 = \rho_1$ or $\nu_1\rho_1 \in E(\mathcal{G}_1)$ and $\nu_2\rho_2 \in E(\mathcal{G}_2)$. The Cartesian and strong product are commutative and associative operations, with the trivial graph as the identity element.*

3. Computing the Sum Connectivity Index of the Cartesian Products of Monogenic Semigroup

This segment investigate SCI under the cartesian product graph comprising two monogenic semigroups.

Assume the monogenic semigroup graphs, we introduce an algorithm to adjacent vertices on $\Gamma(S_M^1) \times \Gamma(S_M^2)$.

If $\mathfrak{v}$ is even ($\mathfrak{u}$ is even or odd):

$I_{\mathfrak{v},\mathfrak{u}}$: The vertex $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{u}})$ is related to $(\nu^{\mathfrak{a}}, \nu_2^{\mathfrak{u}})$ and $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{b}})(1 \leq \mathfrak{a} \leq \mathfrak{v} - 1, 1 \leq \mathfrak{b} \leq \mathfrak{u} - 1)$.
$I_{\mathfrak{v},\mathfrak{u}-1}$: The vertex $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{u}-1})$ is related to $(\nu_1^{\mathfrak{a}}, \nu_2^{\mathfrak{u}})$ and $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{b}})(1 \leq \mathfrak{a} \leq \mathfrak{v} - 1, 2 \leq \mathfrak{b} \leq \mathfrak{u} - 2)$.
$I_{\mathfrak{v},\mathfrak{u}-2}$: The vertex $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{u}-2})$ is related to $(\nu_1^{\mathfrak{a}}, \nu_2^{\mathfrak{u}-2})$ and $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{b}})(1 \leq \mathfrak{a} \leq \mathfrak{v} - 1, 3 \leq \mathfrak{b} \leq \mathfrak{u} - 3)$.
$\vdots$
$I_{\mathfrak{v},1}$: The vertex $(\nu_1^{\mathfrak{v}}, \nu_2^{1})$ is related to $(\nu_1^{\mathfrak{a}}, \nu_2^{1})(1 \leq \mathfrak{a} \leq \mathfrak{v} - 1)$ $I_{\mathfrak{v}-1,\mathfrak{u}}$: The vertex $(\nu_1^{\mathfrak{v}}, \nu_2^{\mathfrak{u}})$ is related to $(\nu_1^{\mathfrak{a}}, \nu_2^{\mathfrak{u}})$ and $(\nu_1^{\mathfrak{v}-1}, \nu_2^{\mathfrak{b}})(2 \leq \mathfrak{a} \leq \mathfrak{v} - 2, 1 \leq \mathfrak{b} \leq \mathfrak{u} - 1)$.
$\vdots$
$I_{\mathfrak{v}/2+1,1}$: The vertex $(\nu_1^{\mathfrak{v}/2+1}, \nu_2^{1})$ is related to $(\nu_1^{\mathfrak{v}/2}, \nu_2^{1})$.

By pursing out such circumstances, if $\mathfrak{v}$ is odd, ($\mathfrak{u}$ is even or odd) will determine whether the following circumstance takes place. If $\mathfrak{v}$ is odd ($\mathfrak{u}$ is even or odd): $I_{\frac{\mathfrak{v}}{2}+1,1}$: The vertex $(\nu_1^{\frac{\mathfrak{v}}{2}+1}, \nu_2^{1})$ is related to $(\nu_1^{\frac{\mathfrak{v}}{2}}, \nu_2^1)$. The following lemma provides the degrees of the vertices as

$$\begin{aligned}&(\nu_1,\nu_2),(\nu_1^2,\nu_2),...,(\nu_1^{\mathfrak{v}},\nu_2),(\nu_1,\nu_2^2),(\nu_1^2,\nu_2^2),...,(\nu_1^{\mathfrak{v}},\nu_2^2),...,(\nu_1,\nu_2^{\mathfrak{u}-1}),\\&(\nu_1^2,\nu_2^{\mathfrak{u}-1}),...,(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}-1}),(\nu_1,\nu_2^{\mathfrak{u}}),(\nu_1^2,\nu_2^{\mathfrak{u}}),...,(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}})\in\Gamma(S_M^1)\times\Gamma(S_M^2).\end{aligned} \tag{3.1}$$

These vertex degrees are denote by

$$\begin{aligned}&(\mathcal{M}_{\mathcal{S}1},\mathcal{M}_{\mathcal{S}^{'}1}),(\mathcal{M}_{\mathcal{S}2},\mathcal{M}_{\mathcal{S}^{'}1}),...,(\mathcal{M}_{\mathcal{S}\mathfrak{v}},\mathcal{M}_{\mathcal{S}^{'}1}),(\mathcal{M}_{\mathcal{S}1},\mathcal{M}_{\mathcal{S}^{'}2}),(\mathcal{M}_{\mathcal{S}2},\mathcal{M}_{\mathcal{S}^{'}2}),...,\\&(\mathcal{M}_{\mathcal{S}\mathfrak{v}},\mathcal{M}_{\mathcal{S}^{'}2}),...,(\mathcal{M}_{\mathcal{S}1},\mathcal{M}_{\mathcal{S}^{'}\mathfrak{u}}),(\mathcal{M}_{\mathcal{S}2},\mathcal{M}_{\mathcal{S}^{'}\mathfrak{u}}),...,(\mathcal{M}_{\mathcal{S}\mathfrak{v}},\mathcal{M}_{\mathcal{S}^{'}\mathfrak{u}}).\end{aligned} \tag{3.2}$$

Previous studies have examined the degree series in relation to this series, which are discussed in [13, 14].

Lemma 3.1.

$$\begin{aligned}&(\mathcal{M}_{\mathcal{S}1},\mathcal{M}_{\mathcal{S}^{'}1})=1,(\mathcal{M}_{\mathcal{S}2},\mathcal{M}_{\mathcal{S}^{'}1})=2,...,(\mathcal{M}_{\mathcal{S}\lceil\frac{\mathfrak{v}}{2}\rceil},\mathcal{M}_{\mathcal{S}^{'}1})=\lceil\frac{\mathfrak{v}}{2}\rceil,(\mathcal{M}_{\mathcal{S}\lceil\frac{\mathfrak{v}}{2}\rceil}+1,\mathcal{M}_{\mathcal{S}^{'}2})=\lceil\frac{\mathfrak{v}}{2}\rceil,...,\\&(\mathcal{M}_{\mathcal{S}\mathfrak{v}},\mathcal{M}_{\mathcal{S}^{'}1})=\mathfrak{v}-1,...,(\mathcal{M}_{\mathcal{S}\lceil\frac{\mathfrak{v}}{2}\rceil}+1,\mathcal{M}_{\mathcal{S}^{'}\lceil\frac{\mathfrak{u}}{2}\rceil}+1)=(\lceil\frac{\mathfrak{v}}{2}\rceil)(\lceil\frac{\mathfrak{u}}{2}\rceil).\end{aligned}$$

Remark 3.1. *Upon careful examination of Lemma 3.1, we can observe the recurring phrases used in the following:* $\mathcal{M}_{\mathcal{S}\lceil\frac{\mathfrak{u}}{2}\rceil}=\lceil\frac{\mathfrak{u}}{2}\rceil=\mathcal{M}_{\mathcal{S}\lceil\frac{\mathfrak{u}}{2}\rceil}+1$.

Thus, the degree of $\mathcal{M}_{\mathcal{S}\mathfrak{u}}$ is equal to $\mathfrak{u}-1$, regardless of the number of vertices being $\mathfrak{u}$.

Theorem 3.1. *In this case of any monogenic semigroup $S_{\mathcal{M}}^1\times S_{\mathcal{M}}^2$, the sum connectivity index under catesian product of two monogenic semigroup graphs, $\Gamma(S_{\mathcal{M}}^1)\times\Gamma(S_{\mathcal{M}}^2)$ are given by:*

$$[SCI](\Gamma S_{\mathcal{M}}^1\times\Gamma S_{\mathcal{M}}^2)=\begin{cases}\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[[(\ell-1)+(\mathfrak{s}-1)]+[\beta+(\mathfrak{s}-1)]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=1}^{\mathfrak{u}}\Big[[(\ell-1)+\mathfrak{s}]+[\beta+\mathfrak{s}]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\Big[[(\ell-1)+(\mathfrak{s}-1)]+[(\beta-1)+(\mathfrak{s}-1)]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[[(\ell-1)+h]+[(\ell-1)+(\mathfrak{s}-1)]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[[(\ell-1)+(\mathfrak{s}-1)]+[(\ell-1)+(h-1)]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\Big[[(\ell-1)+\mathfrak{s}]+[(\beta-1)+\mathfrak{s}]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[[\ell+(\mathfrak{s}-1)]+[\ell+h]\Big]^{-\frac{1}{2}}+\\\sum_{\ell=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[[\ell+(\mathfrak{s}-1)]+[\ell+(h-1)]\Big]^{-\frac{1}{2}}.\end{cases}$$

In the method above will take ℓ,β and $\mathfrak{s},\mathfrak{v}$ according to the regulations $\ell+\beta\geq\mathfrak{v}+1$, $\ell>\beta,\mathfrak{s}+h\geq\mathfrak{u}+1$, and $\mathfrak{s}>h$.

Proof. Our main goal is to develop $SCI(\Gamma S_{\mathcal{M}}^1\times\Gamma S_{\mathcal{M}}^2)$ in terms of the overall degrees, we must regard the sum as total quantity of various blocks, which will then be decided separately. Including equations (2.2), (3.1) and Remark (3.1), the method presented in 3 is used to find the structures of the degrees of vertices.

$[SCI](\Gamma S^1_{\mathcal{M}} \times \Gamma S^2_{\mathcal{M}})$

$$
\begin{aligned}
&= \Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}}\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}+1}}-1)\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}}))+(\mathcal{M}_{\mathcal{S}_1}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}}+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}_2})+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_3}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\ldots+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_3}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-2}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_1})+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_1})+(\mathcal{M}_{\mathcal{S}_3}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}+\ldots+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_1})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-2}}-1)+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)\Big]^{-\frac{1}{2}}+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+\mathcal{M}_{\mathcal{S}'_1})+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_1}+\mathcal{M}_{\mathcal{S}'_1})\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_1}+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)\Big]^{-\frac{1}{2}}+\Big[(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_1}+\mathcal{M}_{\mathcal{S}'_2})\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}))\Big]^{-\frac{1}{2}}.
\end{aligned}
$$

Consequently, the SCI of $[SCI](\Gamma(S_M^1)\times(\Gamma(S_M^2))$ is expressed as the sum below

$$\begin{aligned}[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2)) &= [SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u})}+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u}-1)}+\ldots\\ &+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},1)}+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v}-1,\mathfrak{u})}\\ &+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v}-1,\mathfrak{u}-1)}+\ldots+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v}-1,1)}\\ &+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(1,\mathfrak{u})}+\ldots+[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(1,\frac{\mathfrak{u}}{2}+1)}.\end{aligned} \tag{3.3}$$

Because estimating the SCI value, the smallest quantity is obtained after multiple calculation. If $\mathfrak{v}$ is odd, we use the equality $\lceil\frac{\mathfrak{u}}{2}\rceil=\frac{\mathfrak{u}+1}{2}$ given in (2.2). Then we have

$$\begin{aligned}&[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u})}\\ &=\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+(1+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((2-2)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}+\ldots\\ &\quad+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+(\frac{\mathfrak{v}}{2}+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\frac{\mathfrak{v}}{2}+1-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}+\ldots\\ &\quad+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+(((\mathfrak{v}-1)-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\mathfrak{v}-1)+1)\Big]^{-\frac{1}{2}}\\ &\quad+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\mathfrak{v}-1)+\frac{\mathfrak{u}}{2})\Big]^{-\frac{1}{2}}+\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\mathfrak{v}-1)+(\frac{\mathfrak{u}}{2}+1-1))\Big]^{-\frac{1}{2}}.\end{aligned} \tag{3.4}$$

Now the equation (3.4) is expressed as sum below

$$\begin{aligned}&[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u})}\\ &=\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+(\beta+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}+\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\beta-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}\\ &\quad+\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\mathfrak{v}-1)+h)\Big]^{-\frac{1}{2}}+\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}-1}\Big[((\mathfrak{v}-1)+(\mathfrak{u}-1))+((\mathfrak{v}-1)+(h-1))\Big]^{-\frac{1}{2}}.\end{aligned} \tag{3.5}$$

If identical procedures were out in $[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u})}$ are applied to $[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u})}$, we obtain

$$\begin{aligned}[SCI](\Gamma(S_{\mathcal{M}}^1)\times(\Gamma(S_{\mathcal{M}}^2))_{(\mathfrak{v},\mathfrak{u}-1)}&=\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\Big[((\mathfrak{v}-1)+((\mathfrak{u}-1)-1))+(\beta+((\mathfrak{u}-1)-1))\Big]^{-\frac{1}{2}}\\ &+\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\Big[((\mathfrak{v}-1)+((\mathfrak{u}-1)-2))+((\beta-1)+((\mathfrak{u}-1)-1))\Big]^{-\frac{1}{2}}\\ &+\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\sum_{h=2}^{\frac{\mathfrak{u}}{2}}\Big[((\mathfrak{v}-1)+((\mathfrak{u}-1)))+((\beta-1)+h)\Big]^{-\frac{1}{2}}\\ &+\sum_{h=2}^{\frac{\mathfrak{u}}{2}}\Big[((\mathfrak{v}-1)+((\mathfrak{u}-1)-1))+((\mathfrak{v}-1)+h)\Big]^{-\frac{1}{2}}\end{aligned} \tag{3.6}$$

$$+\sum_{\beta=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}-2}\Big[((\mathfrak{v}-1)+((\mathfrak{u}-1)-1))+((\mathfrak{v}-1)+((h-1))\Big]^{-\frac{1}{2}}.$$

If it is continued, the following equalities were obtained for $[SCI](\Gamma(S^1_{\mathcal{M}})\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\frac{\mathfrak{u}}{2}+1)}$, $[SCI](\Gamma(S^1_{\mathcal{M}})\times(\Gamma(S^1_{\mathcal{M}}))_{(\mathfrak{v},\frac{\mathfrak{u}}{2})}$ and $[SCI](\Gamma(S^1_{\mathcal{M}})\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},1)}$, respectively,

$$[SCI](\Gamma(S^1_{\mathcal{M}})\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\frac{\mathfrak{u}}{2})}=\Big[((\mathfrak{v}-1)+(\frac{\mathfrak{u}}{2}-\frac{\mathfrak{u}}{2}))+((b-b)+(\frac{\mathfrak{u}}{2}-\frac{\mathfrak{u}}{2}))\Big]^{-\frac{1}{2}}, \tag{3.7}$$

and

$$\begin{aligned}&[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},1)}\\&=\sum_{b=1}^{\frac{\mathfrak{v}}{2}}\Big[((\mathfrak{v}-1)+1)+(b+1)\Big]^{-\frac{1}{2}}+\sum_{b=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\Big[((\mathfrak{v}-1)+1)+((b-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}.\end{aligned} \tag{3.8}$$

In this approach, $[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,\mathfrak{u})}$, $[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,\mathfrak{u}-1)}$, ..., $[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,1)}$, ..., $[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{v}}{2}+1,\mathfrak{u})}$, $[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{v}}{2}+1,\mathfrak{u}-1)}$, ..., $[SCI](\Gamma(S^1_{\mathcal{M}}))\times(\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{v}}{2}+1,1)}$ are generated one by one to get the general sum methods shown below,

$$[SCI](\Gamma S^1_{\mathcal{M}}\times\Gamma S^2_{\mathcal{M}})=\begin{cases}\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\ell-1)+(\mathfrak{s}-1))+(\beta+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=1}^{\mathfrak{u}}\Big[((\ell-1)+\mathfrak{s})+(\beta+\mathfrak{s})\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\Big[((\ell-1)+(\mathfrak{s}-1))+((\beta-1)+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[((\ell-1)+h)+((\ell-1)+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+(h-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\Big[((\ell-1)+\mathfrak{s})+((\beta-1)+\mathfrak{s})\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[(\ell+(\mathfrak{s}-1))+(\ell+h)\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=1}^{\frac{\mathfrak{v}}{2}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[(\ell+(\mathfrak{s}-1))+(\ell+(h-1))\Big]^{-\frac{1}{2}}.\end{cases} \tag{3.9}$$

□

4. Computing the Sum Connectivity Index of the Strong Products of Monogenic Semigroup

Hereinafter, We calculate the SCI for the strong product graph that consists of two monogenic semigroups. Our attention is directed towards the relationship between the SCI of the strong product graph and that of its constituent graph. Furthermore, we scrutinize the impact of modifications in the analysis of the monogenic semigroups on the SCI of the strong product graph.

Employing the algorithm mentioned above, the data we collect in this region will enable us to obtain a precise technique for computing the sum connectivity index across the powerful products of monogenic semigroup graphs.

Assuming, the constrains of the monogenic semigroup graphs, we implement an algorithm to the adjacent vertices on $\Gamma(S_M^1)\boxed{\times}\Gamma(S_M^2)$.

If $\mathfrak{v}$ is even ($\mathfrak{u}$ is even or odd):

$I_{\mathfrak{v},\mathfrak{u}}$: The vertex $(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}})$ is related to $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{u}})$ and $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{b}})(1\leq\mathfrak{a}\leq\mathfrak{v}-1, 1\leq\mathfrak{b}\leq\mathfrak{u}-1)$.

$I_{\mathfrak{v},\mathfrak{u}-1}$: The vertex $(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}-1})$ is related to $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{u}})$ and $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{b}})(1\leq\mathfrak{a}\leq\mathfrak{v}-1, 2\leq\mathfrak{b}\leq\mathfrak{u}-2)$.

$I_{\mathfrak{v},\mathfrak{u}-2}$: The vertex $(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}-2})$ is related to $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{u}-2})$ and $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{b}})(1\leq\mathfrak{a}\leq\mathfrak{v}-1, 3\leq\mathfrak{b}\leq\mathfrak{u}-3)$.

$\vdots$

$I_{\mathfrak{v},1}$: The vertex $(\nu_1^{\mathfrak{v}},\nu_2^{1})$ is related to $(\nu_1^{\mathfrak{a}},\nu_2^{1})(1\leq\mathfrak{a}\leq\mathfrak{v}-1)$

$I_{\mathfrak{v}-1,\mathfrak{u}}$: The vertex $(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}})$ is related to $(\nu_1^{\mathfrak{a}},\nu_2^{\mathfrak{u}})$ and $(\nu_1^{\mathfrak{a}-1},\nu_2^{\mathfrak{b}})(2\leq\mathfrak{a}\leq\mathfrak{v}-2, 1\leq\mathfrak{b}\leq\mathfrak{u}-1)$.

$\vdots$

$I_{\mathfrak{v}/2+1,1}$: The vertex $(\nu_1^{\mathfrak{v}/2+1},\nu_2^{1})$ is related to $(\nu_1^{\mathfrak{v}/2},\nu_2^{1})$.

By pursing out such circumstances, if $\mathfrak{v}$ is odd, whether $\mathfrak{u}$ is even or odd will determine whether the following circumstance takes place. If $\mathfrak{v}$ is odd ($\mathfrak{u}$ is even or odd): $I_{\frac{\mathfrak{v}}{2}+1,1}$: The vertex $(\nu_1^{\frac{\mathfrak{v}}{2}+1},\nu_2^{1})$ is related to $(\nu_1^{\frac{\mathfrak{v}}{2}},\nu_2^{1})$. In the following Lemma 3.1, the vertex degrees are given as

$$\begin{aligned}&(\nu_1,\nu_2),(\nu_1^2,\nu_2),...,(\nu_1^{\mathfrak{v}},\nu_2),(\nu_1,\nu_2^2),(\nu_1^2,\nu_2^2),...,(\nu_1^{\mathfrak{v}},\nu_2^2),...,(\nu_1,\nu_2^{\mathfrak{u}-1}),\\&(\nu_1^2,\nu_2^{\mathfrak{u}-1}),...,(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}-1}),(\nu_1,\nu_2^{\mathfrak{u}}),(\nu_1^2,\nu_2^{\mathfrak{u}}),...,(\nu_1^{\mathfrak{v}},\nu_2^{\mathfrak{u}})\in\Gamma(S_M^1)\times\Gamma(S_M^2)\end{aligned}\tag{4.1}$$

Theorem 4.1. *In this case of any monogenic semigroup $S_{\mathcal{M}}^1\boxed{\times}S_{\mathcal{M}}^2$, the sum connectivity index of two monogenic semigroup graphs are strong products $\Gamma(S_{\mathcal{M}}^1)\boxed{\times}\Gamma(S_{\mathcal{M}}^2)$ by*

$$[SCI](\Gamma S_{\mathcal{M}}^1\boxed{\times}\Gamma S_{\mathcal{M}}^2)=$$
$$\begin{cases}\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\left[(\ell+(\mathfrak{s}-1))+(\ell+h)(\beta+(\mathfrak{s}-1))+(\ell+h)\right]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\left[((\ell-1)(\mathfrak{s}-1))+(\beta(h-1))((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+(h-1))\right]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\left[((\ell-1)(\mathfrak{s}-1))+((\beta-1)+h)((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+h)\right]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\left[((\ell-1)(\mathfrak{s}-1))+((\beta-1)(h-1))((\ell-1)+(\mathfrak{s}-1))+((\beta-1)+(\mathfrak{s}-1))\right]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\left[((\ell-1)+\mathfrak{s})+(\beta+\mathfrak{s}))((\ell-1)+\mathfrak{s})+(\beta+(h-1))\right]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\left[((\ell-1)+\mathfrak{s})+((\beta-1)(h-1))((\ell-1)+(\mathfrak{s}-1))+(\beta+(\mathfrak{s}-1))\right]^{-\frac{1}{2}}\end{cases}\tag{4.2}$$

In the method above will take ℓ,β and $\mathfrak{s},\mathfrak{v}$ according to the regulations $\ell+\beta\geq\mathfrak{v}+1$, $\ell>\beta,\mathfrak{s}>h$ and $\mathfrak{s}+h\geq\mathfrak{u}+1$.

Proof. Our main goal is to develop $SCI(\Gamma S_{\mathcal{M}}^1\boxed{\times}\Gamma S_{\mathcal{M}}^2)$ in terms of the overall number of degrees, we must regard the sum as the total quantity of various blocks, which will then be decided separately. Including equation (2.2), (3.1) and Remark (3.1), the method presented in 4 is used to find the structures of the degrees of vertices.

$[SCI](\Gamma S^1_{\mathcal{M}} \boxed{\times} \Gamma S^2_{\mathcal{M}})$

$$= \left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+\mathcal{M}_{\mathcal{S}_1})\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+\mathcal{M}_{\mathcal{S}_2})\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}})+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+((\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_1}))+((\mathcal{M}_{\mathcal{S}'_1})+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}+1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}+1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1))+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_1}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_1}))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_2}))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_2}))+((\mathcal{M}_{\mathcal{S}'_2})+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}}))+(\mathcal{M}_{\mathcal{S}_1}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_1}))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_2}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_2}))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_1})+((\mathcal{M}_{\mathcal{S}'_1})+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_2})+((\mathcal{M}_{\mathcal{S}'_\mathfrak{u}}-1)+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+\mathcal{M}_{\mathcal{S}'_\mathfrak{u}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}_2})+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_2}))\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_3}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+\mathcal{M}_{\mathcal{S}_3})\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+1))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1)+(\mathcal{M}_{\mathcal{S}'_2})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}'_2})+(\mathcal{M}_{\mathcal{S}_\mathfrak{v}}-1))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))\right]^{-\frac{1}{2}}+\ldots$$

$$+\left[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_2}+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}}))+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+\mathcal{M}_{\mathcal{S}_2})\right]^{-\frac{1}{2}}$$

$$+\left[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_3}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_3}))\right]^{-\frac{1}{2}}+\ldots$$

$$
\begin{aligned}
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-2}}-1)+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-2}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{2}}))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{3}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{3}}))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1)+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-2}}-1)+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{\mathfrak{v}-1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)+((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}}))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1)+\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{1}})+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}+1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}})\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}})\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{\frac{\mathfrak{v}}{2}}})\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{2}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{2}}))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{2}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{2}}))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{2}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{2}}))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{2}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{2}}))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{2}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{\frac{\mathfrak{u}}{2}}})+((\mathcal{M}_{\mathcal{S}_{2}})+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{2}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{2}}+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{2}}))\Big]^{-\frac{1}{2}}\\
&+\Big[((\mathcal{M}_{\mathcal{S}_{1}}-1)+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{1}}+\mathcal{M}_{\mathcal{S}'_{1}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{1}}))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{1}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{1}}+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{1}}))\Big]^{-\frac{1}{2}}+\ldots\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{1}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1))+(\mathcal{M}_{\mathcal{S}_{1}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}}}-1)+(\mathcal{M}_{\mathcal{S}_{1}}))\Big]^{-\frac{1}{2}}\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{1}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{1}}+\mathcal{M}_{\mathcal{S}'_{2}})+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{1}}))\Big]^{-\frac{1}{2}}+\ldots \qquad (4.3)\\
&+\Big[(\mathcal{M}_{\mathcal{S}_{1}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1))+(\mathcal{M}_{\mathcal{S}_{1}}+(\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-2}}))+((\mathcal{M}_{\mathcal{S}'_{\mathfrak{u}-1}}-1)+(\mathcal{M}_{\mathcal{S}_{1}}))\Big]^{-\frac{1}{2}}.
\end{aligned}
$$

Consequently, the sum connctivity index of $[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))$ is expressed as the sum below

$$\begin{aligned}
&[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))\\
&= [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u})} + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u}-1)} + \dots\\
&\quad + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},1)} + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,\mathfrak{u})}\\
&\quad + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,\mathfrak{u}-1)} + \dots + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,1)}\\
&\quad + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(1,\mathfrak{u})} + \dots + [SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{u}}{2}+1,1)}.
\end{aligned} \tag{4.4}$$

While estimating the sum connectivity index value, the smallest quantity has obtained following several computations. Where $\mathfrak{u}$ is odd, we utilize the equality $\lceil \frac{\mathfrak{u}}{2} \rceil = \frac{\mathfrak{u}+1}{2}$ given in (2.2). then we obtain

$$\begin{aligned}
&[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u})}\\
&= \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + (1+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}} + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((2-2)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}} + \dots\\
&\quad + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + (\frac{\mathfrak{v}}{2}+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}} + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((\frac{\mathfrak{v}}{2}+1-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}} + \dots\\
&\quad + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + (((\mathfrak{v}-1)-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}} + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((\mathfrak{v}-1)+1)\Big]^{-\frac{1}{2}}\\
&\quad + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((\mathfrak{v}-1)+\frac{\mathfrak{u}}{2})\Big]^{-\frac{1}{2}} + \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((\mathfrak{v}-1)+(\frac{\mathfrak{u}}{2}+1-1))\Big]^{-\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

Now the equation (4.5) is expressed as sum below

$$\begin{aligned}
&[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u})}\\
&= \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[(\mathfrak{v}+(\mathfrak{u}-1)) + (\mathfrak{v}+h)(\beta+(\mathfrak{u}-1)) + (\mathfrak{v}+h)\Big]^{-\frac{1}{2}}\\
&+ \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}-1} \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + (\beta(h-1))((\mathfrak{v}-1)+(\mathfrak{u}-1)) + ((\mathfrak{v}-1)+(h-1))\Big]^{-\frac{1}{2}}\\
&+ \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[((\mathfrak{v}-1)+(\mathfrak{u}-1)) + ((\mathfrak{v}-1)+h))((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((\beta-1)+h)\Big]^{-\frac{1}{2}}\\
&+ \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}-1} \Big[((\mathfrak{v}-1)(\mathfrak{u}-1)) + ((\beta-1)(h-1))((\mathfrak{v}-1)+(\mathfrak{u}-1)) + ((\beta-1)+(\mathfrak{u}-1))\Big]^{-\frac{1}{2}}.
\end{aligned} \tag{4.6}$$

The similar operation in $[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u})}$ are applied to $[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u}-1)}$, we obtain

$$\begin{aligned}
&[SCI](\Gamma(S^1_{\mathcal{M}})\ \boxed{\times}\ (\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\mathfrak{u}-1)} = \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[(\mathfrak{v}+(\mathfrak{u}-1)-1)) + (\mathfrak{v}+h)(\mathfrak{v}+((\mathfrak{u}-1)-1)) + (\mathfrak{v}+h)\Big]^{-\frac{1}{2}}\\
&+ \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}-1} \Big[((\mathfrak{v}-1)((\mathfrak{u}-1)-1)) + (\beta(h-1))((\mathfrak{v}-1)+((\mathfrak{u}-1)-1)) + ((\mathfrak{v}-1)+(h-1))\Big]^{-\frac{1}{2}}\\
&+ \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[((\beta-1)+h)((\mathfrak{v}-1)+((\mathfrak{v}-1)((\mathfrak{u}-1)-1)) + ((\mathfrak{u}-1)-1)) + ((\mathfrak{v}-1)+h))\Big]^{-\frac{1}{2}}
\end{aligned}$$

$$+\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}-1}\Big[((\mathfrak{v}-1)((\mathfrak{u}-1)-1))+((\beta-1)(h-1))((\mathfrak{v}-1)+((\mathfrak{u}-1)-1))+((\beta-1)+((\mathfrak{u}-1)-1))\Big]^{-\frac{1}{2}}. \tag{4.7}$$

If this pattern persists, the following equality will be achieved for $[SCI](\Gamma(S^1_{\mathcal{M}})\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\frac{\mathfrak{u}}{2}+1)}$, $[SCI](\Gamma(S^1_{\mathcal{M}})\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\frac{\mathfrak{u}}{2})}$ and $[SCI](\Gamma(S^1_{\mathcal{M}})\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},1)}$, respectively

$$\begin{aligned}&[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},\frac{\mathfrak{u}}{2})}\\&=\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\mathfrak{v}-1)+\frac{\mathfrak{u}}{2})+(\beta+\frac{\mathfrak{u}}{2}))((\mathfrak{v}-1)+\frac{\mathfrak{u}}{2})+(\beta+(h-1))\Big]^{-\frac{1}{2}}\\&+\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\mathfrak{v}-1)+\frac{\mathfrak{u}}{2})+((\beta-1)(h-1))((\mathfrak{v}-1)+(\beta+(\frac{\mathfrak{u}}{2}-1))\Big]^{-\frac{1}{2}},\end{aligned} \tag{4.8}$$

$$\begin{aligned}[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v},1)}&=\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\Big[((\beta-1)+\mathfrak{u})+(h+\mathfrak{u}))((\mathfrak{v}-1)+\mathfrak{u})+(\beta+1)\Big]^{-\frac{1}{2}}\\&+\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\mathfrak{v}-1)+\mathfrak{u})+(h+\mathfrak{u}))((\mathfrak{v}-1)+\mathfrak{u})+(h-1)\Big]^{-\frac{1}{2}}.\end{aligned} \tag{4.9}$$

In this approach, $[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,\mathfrak{u})}$, $[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,\mathfrak{u}-1)}$, ..., $[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\mathfrak{v}-1,1)}$, ..., $[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{v}}{2}+1,\mathfrak{u})}$, $[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{v}}{2}+1,\mathfrak{u}-1)}$, ..., $[SCI](\Gamma(S^1_{\mathcal{M}}))\boxed{\times}(\Gamma(S^2_{\mathcal{M}}))_{(\frac{\mathfrak{v}}{2}+1,1)}$ are generated one by one to get the general sum methods shown below.

$$[SCI](\Gamma S^1_{\mathcal{M}}\boxed{\times}\Gamma S^2_{\mathcal{M}})=$$

$$\begin{cases}\sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[(\ell+(\mathfrak{s}-1))+(\ell+h)(\ell+(\mathfrak{s}-1))+(\ell+h)\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\ell-1)(\mathfrak{s}-1))+(\beta(h-1))((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+(h-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=1}^{\frac{\mathfrak{u}}{2}}\Big[((\ell-1)(\mathfrak{s}-1))+((\beta-1)+h)((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+h))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\ell-1)(\mathfrak{s}-1))+((\beta-1)(h-1))((\ell-1)+(\mathfrak{s}-1))+((\beta-1)+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}}\sum_{\beta=1}^{\frac{\mathfrak{v}}{2}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\ell-1)+\mathfrak{s})+(\beta+\mathfrak{s}))((\ell-1)+\mathfrak{s})+(\beta+(h-1))\Big]^{-\frac{1}{2}}+\\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}}\sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}}\sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}}\Big[((\ell-1)+\mathfrak{s})+((\beta-1)(h-1))((\ell-1)+(\mathfrak{s}-1))+(\beta+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}\end{cases} \tag{4.10}$$

□

5. Examples

In this segment, we examine and utilize the proposed algorithm, the method of the sum connectivity index of monogenic semigroups involving cartesian and strong products.

Example 5.1. *We will examine the sum connectivity index by utilize the Theorem 3.1 which involves that the Cartesian product graph of* $\Gamma S^1_{\mathcal{M}} \times \Gamma S^2_{\mathcal{M}}$ *are given as follows:*

$$[SCI](\Gamma S^1_{\mathcal{M}} \times \Gamma S^2_{\mathcal{M}}) = \begin{cases} \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[((\ell-1)+(\mathfrak{s}-1))+(\beta+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{\mathfrak{s}=1}^{\mathfrak{u}} \Big[((\ell-1)+\mathfrak{s})+(\beta+\mathfrak{s})\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \Big[((\ell-1)+(\mathfrak{s}-1))+((\beta-1)+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+h)\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+(h-1))\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}} \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}-1} \Big[((\ell-1)+\mathfrak{s})+((\beta-1)+\mathfrak{s})\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=1}^{\frac{\mathfrak{v}}{2}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[(\ell+(\mathfrak{s}-1))+(\ell+h)\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=1}^{\frac{\mathfrak{v}}{2}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[(\ell+(\mathfrak{s}-1))+(\ell+(h-1))\Big]^{-\frac{1}{2}}, \end{cases}$$

we have

$$\begin{aligned}[SCI](\Gamma S^1_{\mathcal{M}} \times \Gamma S^2_{\mathcal{M}}) &= \big[2+4\big]^{-\frac{1}{2}} + \big[2+6\big]^{-\frac{1}{2}} + \big[3+3\big]^{-\frac{1}{2}} + 2\big[3+4\big]^{-\frac{1}{2}} + 2\big[3+5\big]^{-\frac{1}{2}} \\ &+ 2\big[3+6\big]^{-\frac{1}{2}} + 2\big[3+7\big]^{-\frac{1}{2}} + 2\big[4+4\big]^{-\frac{1}{2}} + 4\big[4+5\big]^{-\frac{1}{2}} + 6\big[4+6\big]^{-\frac{1}{2}} \\ &+ 2\big[4+7\big]^{-\frac{1}{2}} + \big[4+8\big]^{-\frac{1}{2}} + 4\big[5+5\big]^{-\frac{1}{2}} + 9\big[5+6\big]^{-\frac{1}{2}} + 6\big[5+7\big]^{-\frac{1}{2}} \\ &+ \big[5+8\big]^{-\frac{1}{2}} + 6\big[6+7\big]^{-\frac{1}{2}} + 3\big[6+8\big]^{-\frac{1}{2}} + \big[7+7\big]^{-\frac{1}{2}} + 3\big[7+8\big]^{-\frac{1}{2}}.\end{aligned}$$

Example 5.2. *We will examine the sum connectivity index by utilize the Theorem 4.1 which involves that the strong product graph of* $\Gamma S^1_{\mathcal{M}} \boxtimes \Gamma S^2_{\mathcal{M}}$ *are given as follows:*

$$[SCI](\Gamma S^1_{\mathcal{M}} \boxtimes \Gamma S^2_{\mathcal{M}}) =$$

$$\begin{cases} \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[(\ell+(\mathfrak{s}-1))+(\ell+h)(\ell+(\mathfrak{s}-1))+(\ell+h)\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[((\ell-1)(\mathfrak{s}-1))+(\beta(h-1))((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+(h-1))\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{h=1}^{\frac{\mathfrak{u}}{2}} \Big[((\ell-1)(\mathfrak{s}-1))+((\beta-1)+h)((\ell-1)+(\mathfrak{s}-1))+((\ell-1)+h)\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[((\ell-1)(\mathfrak{s}-1))+((\beta-1)(h-1))((\ell-1)+(\mathfrak{s}-1))+((\beta-1)+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}} \sum_{\beta=1}^{\frac{\mathfrak{v}}{2}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[((\ell-1)+\mathfrak{s})+(\beta+\mathfrak{s}))((\ell-1)+\mathfrak{s})+(\beta+(h-1))\Big]^{-\frac{1}{2}}+ \\ \sum_{\ell=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{\mathfrak{s}=1}^{\frac{\mathfrak{u}}{2}} \sum_{\beta=\frac{\mathfrak{v}}{2}+1}^{\mathfrak{v}} \sum_{h=\frac{\mathfrak{u}}{2}+1}^{\mathfrak{u}} \Big[((\ell-1)+\mathfrak{s})+((\beta-1)(h-1))((\ell-1)+(\mathfrak{s}-1))+(\beta+(\mathfrak{s}-1))\Big]^{-\frac{1}{2}} \end{cases}$$

we have

$$\begin{aligned}&[SCI](\Gamma S^1_{\mathcal{M}} \boxtimes \Gamma S^2_{\mathcal{M}}) \\ &= \big[3+7\big]^{-\frac{1}{2}} + \big[3+11\big]^{-\frac{1}{2}} + \big[3+23\big]^{-\frac{1}{2}} + 2\big[5+5\big]^{-\frac{1}{2}} + 2\big[5+7\big]^{-\frac{1}{2}} \\ &+ \big[5+9\big]^{-\frac{1}{2}} + 2\big[5+11\big]^{-\frac{1}{2}} + 5\big[5+17\big]^{-\frac{1}{2}} + \big[5+19\big]^{-\frac{1}{2}} + 2\big[5+23\big]^{-\frac{1}{2}} \\ &+ \big[7+7\big]^{-\frac{1}{2}} + 2\big[7+9\big]^{-\frac{1}{2}} + 3\big[7+11\big]^{-\frac{1}{2}} + 4\big[7+15\big]^{-\frac{1}{2}} + 2\big[7+17\big]^{-\frac{1}{2}} \\ &+ 2\big[7+19\big]^{-\frac{1}{2}} + 3\big[7+23\big]^{-\frac{1}{2}} + \big[8+8\big]^{-\frac{1}{2}} + 2\big[8+11\big]^{-\frac{1}{2}} + 4\big[8+14\big]^{-\frac{1}{2}} \\ &+ 4\big[8+17\big]^{-\frac{1}{2}} + 2\big[8+19\big]^{-\frac{1}{2}} + 2\big[8+23\big]^{-\frac{1}{2}} + 2\big[9+11\big]^{-\frac{1}{2}} + \big[9+15\big]^{-\frac{1}{2}}\end{aligned}$$

$$+ \left[9+19\right]^{-\frac{1}{2}} + 2\left[9+23\right]^{-\frac{1}{2}} + 7\left[11+11\right]^{-\frac{1}{2}} + 10\left[11+14\right]^{-\frac{1}{2}} + 10\left[11+15\right]^{-\frac{1}{2}}$$
$$+ 10\left[11+17\right]^{-\frac{1}{2}} + 6\left[11+19\right]^{-\frac{1}{2}} + 5\left[11+23\right]^{-\frac{1}{2}} + \left[14+14\right]^{-\frac{1}{2}} + 4\left[14+15\right]^{-\frac{1}{2}}$$
$$+ 4\left[14+17\right]^{-\frac{1}{2}} + 2\left[14+19\right]^{-\frac{1}{2}} + \left[15+15\right]^{-\frac{1}{2}} + 4\left[15+17\right]^{-\frac{1}{2}} + 2\left[15+19\right]^{-\frac{1}{2}}$$
$$+ 2\left[15+23\right]^{-\frac{1}{2}} + \left[17+17\right]^{-\frac{1}{2}} + 2\left[17+19\right]^{-\frac{1}{2}} + 4\left[17+23\right]^{-\frac{1}{2}}.$$

Thus we have computed SCI for the cartesian and strong product of two $\mathcal{M}_{\mathcal{S}}$ by using the present formula obtained in the significant theorems.

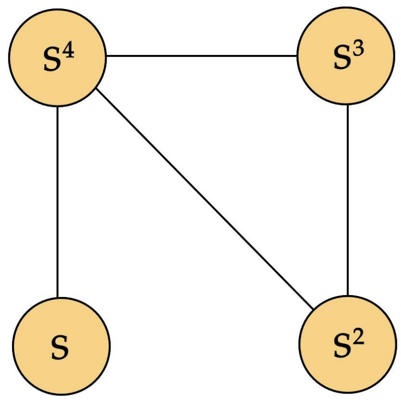

Figure 1. Graphical representation of $S_{\mathcal{M}_4}$ monogenic semigroup

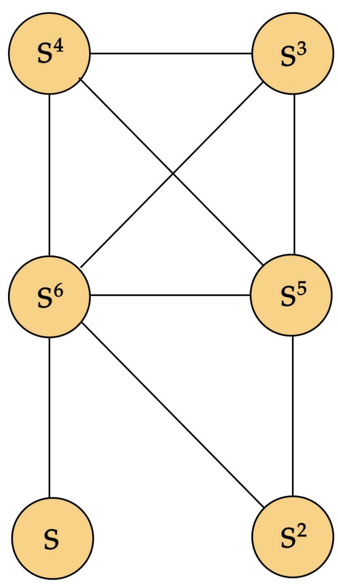

Figure 2. Graphical representation of $S_{\mathcal{M}_6}$ monogenic semigroup

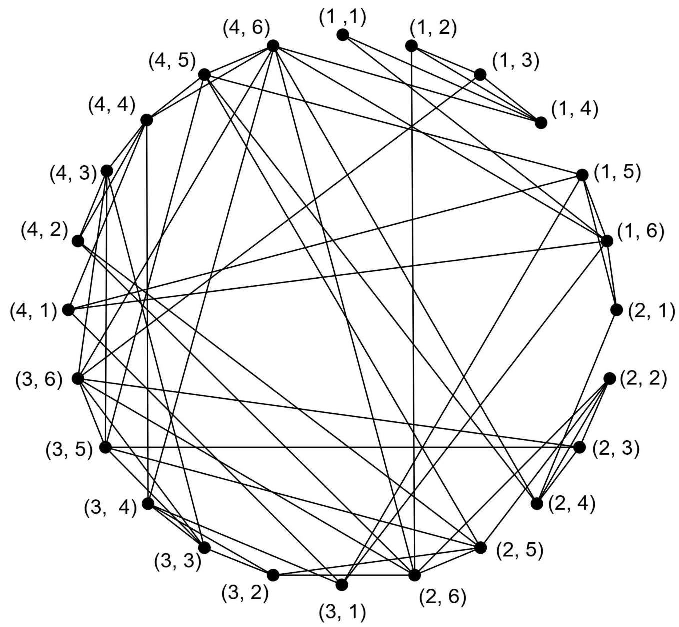

Figure 3. Cartesian products of $S_{\mathcal{M}_4}$ and $S_{\mathcal{M}_6}$ monogenic semigroup graph

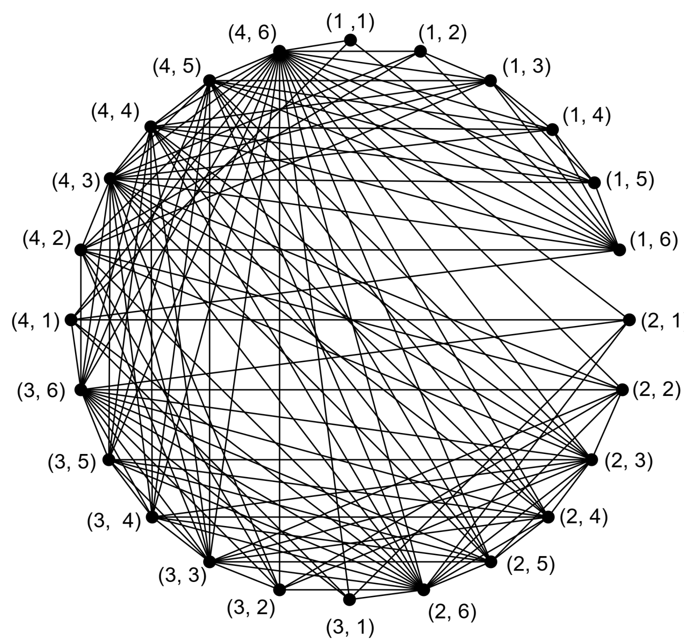

Figure 4. Strong products of $S_{\mathcal{M}_4}$ and $S_{\mathcal{M}_6}$ monogenic semigroup graph

6. Conclusions

In this study, we investigated the sum connectivity index (SCI) of the cartesian and strong product graphs of monogenic semigroups. We found that the SCI of these product graphs is a function of the SCI of their constituent graphs and the degree distribution and sequence of the monogenic semigroups. Our finding provide insights the into the properties and dynamics of these product graphs, which hane potential applications in various fields, including communication networks, computer science, and social networks. Our results show that the SCI of the product graph can be increased by increasing the number of generators and the size of the monogenic semigroups. In summary, the study of the SCI of the cartesian and strong product graphs of monogenic semigroups provids a deeper understanding of the connectivity properties of these graphs and their potential application in various fields. Our finding contribute to the ongoing research on using graph theory and algebraic structures in real-world problems.

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