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NEW INTEGRAL INEQUALITIES THROUGH INVEXITY WITH APPLICATIONS

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Abstract. In this paper, we obtain some inequalities of Simpson's inequality type for functions whose derivatives absolute values are quasi-preinvex function. Applications to some special means are considered.

1. Introduction

The Simpson's inequality is very important and the well-known in the literature: This inequality is stated that: If $f : [a, b] \to R$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$.then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4$$

Recently, many others [5-7], [1] developed and discussed error estimates of the Simpson's type inequality interms of refinement, counterparts, generalizations and new Simpson's type inequalities.

In [1],Dragomir et.al. proved the following recent developments on Simpson's inequality for which the reminder is expressed in terms of lower derivatives than the fourth.

Theorem 1. Suppose $f : [a, b] \to R$ is a differentiable mapping whose derivative is continuous on (a, b) and $f' \in L[a, b]$.then the following inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{3}\left\|f'\right\|_1 \tag{1.1}$$

holds, where $\|f'\|_1 = \int_a^b |f'(x)|dx$.

The bound of (1.1) for L-Lipschitian mapping was given in [1] by $\frac{5}{36}L(b-a)$.

Theorem 2. Suppose $f : [a, b] \to R$ is an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$.then the following inequality holds,

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p\end{aligned} \tag{1.2}$$

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Where $\frac{1}{p}+\frac{1}{q}=1$.

In [2], Kirmaci established the following Hermite-Hadamard inequalities for different convex functions as:

Theorem 3. Let $f: I \subset R \rightarrow R$ be a differentiable function on I^0 interior of $I^0 a, b \in I$ with $a<b$.if the mapping $|f'|$ is convex on $[a,b]$, and the following inequality holds

(1.3) $\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left[|f'(a)|+|f'(b)|\right].$

Let K be a closed set $R^n$and let $f: K \rightarrow R$and $\eta: K \times K \rightarrow R$be continuous functions. Let $x \in K$,then the set K is said to be invex at $x$with respect to $\eta(.,.)$,

If $x+t\eta(y,x) \in K, \forall x,y \in K, t \in [0,1]$.

$K$is said to be invex set with respect to η if $K$is invex at each $x \in K$.The invex set $K$is also called a η-connected set.

Definition 5[3]. The function f on the invex set $K$is said to be preinvex with respect to η,if

$$f(u+t\eta(v,u)) \leq +(1-t)f(u)+tf(v), \forall u,v \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y)=x-y$ but the converse is not true.

Definition 6[4]. The function f on the invex set $K$is said to be preinvex with respect to η,if

$$f(u+t\eta(v,u)) \leq \max\{f(u), f(v)\}, \forall u,v \in K, t \in [0,1].$$

Also every quasi-convex function is a prequasiinvex with respect to the map $\eta(u,v)$ but the converse does not hold, see for example [8].

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute values are quasi-preinvex . .

2. Main results

Before proceeding towards our main theorem regarding generalization of the Simpson's type inequality using prequasiinvex . We begin with the following Lemma.

Lemma 2.1. Let $K \subseteq R$ be an open invex subset with respect to $\eta: K \times K \rightarrow R$and $a,b \in K$with $a<a+\eta(b,a)$suppose $f: K \rightarrow R$is a differentiable mapping on Ksuch that $f' \in L([a,a+\eta(b,a)])$.Then for every $a,b \in K$ with $\eta(b,a) \neq 0$the following inequality holds:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$=\frac{\eta(b,a)}{2}\left[\int_0^1\left(\frac{\lambda}{2}-\frac{1}{3}\right)f'\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)d\lambda+\int_0^1\left(\frac{1}{3}-\frac{\lambda}{2}\right)f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)d\lambda\right].$$

Proof. Integrating by parts, we have

$$I_1 = \int_0^1 \left(\tfrac{\lambda}{2} - \tfrac{1}{3}\right) f'\left(a + \left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right) d\lambda$$
$$= \frac{2\left(\frac{\lambda}{2} - \frac{1}{3}\right) f\left(a + \left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)}{\eta(b,a)}\Bigg|_0^1 - \frac{1}{\eta(b,a)} \int_0^1 f\left(a + \left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right)$$
$$= \frac{2}{6\eta(b,a)} f(a + \eta(b,a)) + \frac{2}{3\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_0^1 f\left(a + \left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right) d\lambda$$

Setting $x = a + \left(\frac{1+\lambda}{2}\right)\eta(b,a)$ and $dx = \frac{\eta(b,a)}{2} d\lambda$ which gives $I_1 = \frac{2}{6\eta(b,a)} f(a + \eta(b,a)) + \frac{2}{3\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{2}{(\eta(b,a))^2} \int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,dx$

Similarly we can show that

$$I_2 = \int_0^1 \left(\tfrac{1}{3} - \tfrac{\lambda}{2}\right) f'\left(a + \left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right) d\lambda$$
$$= \frac{2}{6\eta(b,a)} f(a) + \frac{2}{3\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{2}{(\eta(b,a))^2} \int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx$$

Therefore

$$\frac{\eta(b,a)}{2}[I_1 + I_2]$$
$$= \left[\frac{f(a) + f(a+\eta(b,a))}{6} + \frac{2}{3} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right]$$

Which completes the proof.

In the following theorem, we shall propose some new upper bound for the right-hand side of Simpson's type inequality for functions whose derivatives absolute values are prequasiinvex.

Theorem 2.2. Let $K \subseteq [0, \infty)$ be an open invex subset with respect to $\eta : K \times K \to R$ and $a, b \in K$with $a < a + \eta(b,a)$ suppose $f : K \to R$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b,a)])$. If $|f'|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds: (2.1)

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a + \eta(b,a))\right] - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{5\eta(b,a)}{72} \left[\begin{array}{l} \sup\left\{|f'(a)|, \left|f'\left(a + \frac{1}{2}\eta(b,a)\right)\right|\right\} \\ + \sup\left\{\left|f'\left(a + \frac{1}{2}\eta(b,a)\right)\right|, |f'(a + \eta(b,a))|\right\} \end{array}\right].$$

Proof. From Lemma 2.1, and since $|f'|$ is prequasiinvex, then we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq \frac{\eta(b,a)}{2}\left[\int_0^1\left|\frac{\lambda}{2}-\frac{1}{3}\right|\left|f'\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda+\int_0^1\left|\frac{1}{3}-\frac{\lambda}{2}\right|\left|f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda\right]$$
$$\leq \frac{\eta(b,a)}{2}\int_0^1\left|\frac{\lambda}{2}-\frac{1}{3}\right|\sup\left\{|f'(a)|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}d\lambda$$
$$+\frac{\eta(b,a)}{2}\int_0^1\left|\frac{\lambda}{2}-\frac{1}{3}\right|\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|,|f'(a+\eta(b,a))|\right\}d\lambda$$
$$\leq \frac{\eta(b,a)}{2}\sup\left\{|f'(a)|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\int_0^1\left|\frac{\lambda}{2}-\frac{1}{3}\right|d\lambda$$
$$+\frac{\eta(b,a)}{2}\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|,|f'(a+\eta(b,a))|\right\}\int_0^1\left|\frac{\lambda}{2}-\frac{1}{3}\right|d\lambda$$
$$=\frac{5\eta(b,a)}{72}\sup\left\{|f''(a)|,\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}$$
$$+\frac{5\eta(b,a)}{72}\sup\left\{\left|f''\left(a+\frac{1}{2}\eta(b,a)\right)\right|,|f''(a+\eta(b,a))|\right\}.$$

Which completes the proof.

The upper bound for the midpoint inequality for the first derivative is presented as

Corollary 2.3. Let f as in Theorem 2.2, if in addition

(1) $|f'|$ is increasing, then we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\&\leq \frac{5\eta(b,a)}{72}\left[|f'(a+\eta(b,a))|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned}\tag{2.2}$$

(2) $|f'|$ is decreasing, then we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\&\leq \frac{5\eta(b,a)}{72}\left[|f'(a)|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].\end{aligned}\tag{2.3}$$

Proof. It follows directly by Theorem 2.2.

Remark 2.4. We note that the inequalities (2.2) and (2.3), are two new refinements of the trapezoid inequality for prequasiinvex functions, and thus for convex functions.

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 2.5. Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to R$ and $a,b\in K$ with $a<a+\eta(b,a)$ suppose $f:K\to R$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $|f'|^p$ is preinvex on K,from some $p>1$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$ the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\&\leq \frac{\eta(b,a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{1/p}\left(\sup\left\{|f'(a)|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\\&+\frac{\eta(b,a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{1/p}\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},|f'(a+\eta(b,a))|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\end{aligned}\tag{2.4}$$

Where $q = p/(p-1)$.

Proof . From Lemma 2.1, and using the well known Holder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(x)dx\right| \\
&\leq \frac{\eta(b,a)}{2}\int\limits_0^1\left|\frac{\lambda}{2}-\frac{1}{3}\right|\left|f'\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda \\
&+\frac{\eta(b,a)}{2}\int\limits_0^1\left|\frac{1}{3}-\frac{\lambda}{2}\right|\left|f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|d\lambda \\
&\leq \frac{\eta(b,a)}{2}\left(\int\limits_0^1\left(\frac{\lambda}{2}-\frac{1}{3}\right)^p\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|f'\left(a+\left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^{\frac{p}{p-1}}d\lambda\right)^{\frac{p-1}{p}} \\
&+\frac{\eta(b,a)}{2}\left(\int\limits_0^1\left(\frac{1}{3}-\frac{\lambda}{2}\right)^p\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|f'\left(a+\left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^{\frac{p}{p-1}}d\lambda\right)^{\frac{p-1}{p}} \\
&\leq \frac{\eta(b,a)}{2}\left(\int\limits_0^1\left(\frac{\lambda}{2}-\frac{1}{3}\right)^p\right)^{\frac{1}{p}}\left(\int\limits_0^1\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}} \\
&+\frac{\eta(b,a)}{2}\left(\int\limits_0^1\left(\frac{1}{3}-\frac{\lambda}{2}\right)^p\right)^{\frac{1}{p}}\left(\int\limits_0^1\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},\left|f'(a+\eta(b,a))\right|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}} \\
&= \frac{\eta(b,a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{1/p}\left(\int\limits_0^1\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}} \\
&+\frac{\eta(b,a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{1/p}\left(\int\limits_0^1\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},\left|f'(a+\eta(b,a))\right|^{\frac{p}{p-1}}\right\}d\lambda\right)^{\frac{p-1}{p}}
\end{aligned}$$

Which completes the proof.

Corollary 2.6. Let f as in Theorem 2.5, if in addition

(1) $|f'|^{p/(p-1)}$ is increasing, then we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(x)dx\right| \\
&\leq \frac{\eta(b,a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{1/p}\left[\left|f'(a+\eta(b,a))\right|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].
\end{aligned} \tag{2.5}$$

(2) $|f'|^{p/(p-1)}$ is decreasing, then we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)}f(x)dx\right| \\
&\leq \frac{\eta(b,a)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{1/p}\left[\left|f'(a)\right|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].
\end{aligned} \tag{2.6}$$

An improvement of constants in Theorem 2.5 and a consolidation of this result with Theorem 2.2. are given in the following theorem.

Theorem 2.7. Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to R$ and $a, b \in K$ with $a < a+\eta(b,a)$ suppose $f : K \to R$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $|f'|^q$ is preinvex on K, $q \geq 1$, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ the following inequality holds:

$$\begin{aligned} &\left| \tfrac{1}{6}\left[f(a) + 4f\left(\tfrac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \tfrac{5\eta(b,a)}{72} \left(\sup\left\{ |f'(a)|^q, \left| f'\left(a + \tfrac{1}{2}\eta(b,a)\right)\right|^q \right\} \right)^{\frac{1}{q}} \\ &+ \tfrac{5\eta(b,a)}{72} \left(\sup\left\{ \left| f'\left(a + \tfrac{1}{2}\eta(b,a)\right)\right|^q, |f'(a+\eta(b,a))|^q \right\} \right)^{\frac{1}{q}}. \end{aligned} \tag{2.7}$$

Proof . Suppose that$q \geq 1$. From Lemma 2.1 and using the well known power mean inequality, we have

$$\begin{aligned} &\left| \tfrac{1}{6}\left[f(a) + 4f\left(\tfrac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \tfrac{\eta(b,a)}{2} \int_0^1 \left| \tfrac{\lambda}{2} - \tfrac{1}{3} \right| \left| f'\left(a + \left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)\right| d\lambda \\ &+ \tfrac{\eta(b,a)}{2} \int_0^1 \left| \tfrac{1}{3} - \tfrac{\lambda}{2} \right| \left| f'\left(a + \left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right)\right| d\lambda \\ &\leq \tfrac{\eta(b,a)}{2} \left(\int_0^1 \left(\tfrac{\lambda}{2} - \tfrac{1}{3}\right) d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| f'\left(a + \left(\tfrac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^q d\lambda \right)^{\frac{1}{q}} \\ &+ \tfrac{\eta(b,a)}{2} \left(\int_0^1 \left(\tfrac{1}{3} - \tfrac{\lambda}{2}\right) d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| f'\left(a + \left(\tfrac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^q d\lambda \right)^{\frac{1}{q}} \end{aligned}$$

Since $|f'|^q$ is quasi-preinvexity , we have

$$\left| f'\left(a + \left(\frac{1+\lambda}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left(|f'(a)|^q, \left| f'\left(a + \frac{1}{2}\eta(b,a)\right)\right|^q \right)$$

And

$$\left| f'\left(a + \left(\frac{1-\lambda}{2}\right)\eta(b,a)\right)\right|^q \leq \sup\left(\left| f'\left(a + \frac{1}{2}\eta(b,a)\right)\right|^q, |f'(a+\eta(b,a))|^q \right)$$

$$\begin{aligned} &\left| \tfrac{1}{6}\left[f(a) + 4f\left(\tfrac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \tfrac{5\eta(b,a)}{72} \left(\sup\left\{ |f'(a)|^q, \left| f'\left(a + \tfrac{1}{2}\eta(b,a)\right)\right|^q \right\} \right)^{\frac{1}{q}} \\ &+ \tfrac{5\eta(b,a)}{72} \left(\sup\left\{ \left| f'\left(a + \tfrac{1}{2}\eta(b,a)\right)\right|^q, |f'(a+\eta(b,a))|^q \right\} \right)^{\frac{1}{q}}. \end{aligned}$$

Which completes the proof.

3. Application to some Special Means

In what follows we give certain generalization of some notions for a positive valued function of a positive variable.

Definition 3[9]. A function $M : R \to R$,is called a mean function if it has the following properties:

(1) Homogeneity: $M(ax, ay) = aM(x,y)$,for all $a > 0$,
(2) Symmetry: $M(x,y) = M(x,y)$,
(3) Reflexivity: $M(x,x) = x$,
(4) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) = M(x', y')$,
(5) Internality: $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.

We consider some means for arbitrary positive real numbers a, b (see for instance [9]).

We now consider the applications of our theorem to the special means.

The arithmetic mean;

$$A := A(a,b) = \frac{a+b}{2}$$

The geometic mean;

$$G := G(a,b) = \sqrt{ab}$$

The power mean;

$$P_r := P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \quad , \quad r \geq 1,$$

The indentric mean:

$$I = I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right) \quad , & ifa \neq b \\ a \;\;, & ifa = b \end{cases}$$

The Harmonic mean:

$$H := H(a,b) = \frac{2ab}{a+b} \quad ,$$

The logarithmic mean:

$$L = L(a,b) = \frac{a-b}{\ln|a| - \ln|b|}, \quad |a| \neq |b|$$

The *p*- logarithmic mean:

$$L_p \equiv L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right], \quad a \neq b$$

$p \in \Re \setminus \{-1, 0\}$: $a, b > 0$.

It is well known that L_P is monotonic nondecreasing over $p \in \mathrm{R}$ with $L_{-1} :=$ $L$and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

Now let a and $b$be positive real numbers such that $a < b$.consider the function $a < b$. $M : M(b,a) : [a, a + \eta(b,a)] \times [a, a + \eta(b,a)] \to R$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

$\eta(b,a) = M(b,a)$in (2.1), (2.4) and (2.7), one can obtain the following interesting inequalities involving means:

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{2a+M(b,a)}{2}\right) + f(a+M(b,a))\right]? - \frac{1}{M(b,a)}\int\limits_a^{a+M(b,a)} f(x)dx\right|$$
$$\leq \frac{5M(b,a)}{72}\left[\begin{array}{c} \sup\left\{|f'(a)|, \left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|\right\} \\ +\sup\left\{\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|, |f'(a+M(b,a))|\right\} \end{array}\right].$$

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+M(b,a)}{2}\right)+f(a+M(b,a))\right]-\frac{1}{M(b,a)}\int_{a}^{a+M(b,a)}f(x)dx\right|$$
$$\leq \frac{M(b,a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{1/p}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} \tag{3.1}$$
$$+\frac{M(b,a)}{2}\left(\frac{2\left(1+2^{p+1}\right)}{6^{p+1}(p+1)}\right)^{1/p}\left(\sup\left\{\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}},\left|f'(a+M(b,a))\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}$$

(3.2)

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2a+M(b,a)}{2}\right)+f(a+M(b,a))\right]-\frac{1}{\eta(b,a)}\int_{a}^{a+M(b,a)}f(x)dx\right|$$
$$\leq \frac{5M(b,a)}{72}\left(\sup\left\{\left|f'(a)\right|^{q},\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}$$
$$+\frac{5M(b,a)}{72}\left(\sup\left\{\left|f'\left(a+\frac{1}{2}M(b,a)\right)\right|^{q},\left|f'(a+M(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

For $q \geq 1$.Letting $M = A, G, P_r, I, H, L, L_p$ in (3.1), (3.2) and (3.3), we can get the required inequalities, and the details are left to the interested reader.

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