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A Symbolic Algorithm for Solving Doubly Bordered k-Tridiagonal Interval Linear Systems

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Abstract. Doubly bordered k-tridiagonal interval linear systems play a crucial role in various mathematical and engineering applications where uncertainty is inherent in the system's parameters. In this paper, we propose a novel symbolic algorithm for solving such systems efficiently. Our approach combines symbolic computation techniques with interval arithmetic to provide rigorous solutions in the form of tight interval enclosures. By exploiting the tridiagonal structure and employing a divide-and-conquer strategy, our algorithm achieves significantly reduced computational complexity compared to existing numerical methods. We also present theoretical analysis and provide numerical experiments to demonstrate the effectiveness and accuracy of our algorithm. The proposed symbolic algorithm offers a valuable tool for handling doubly bordered k-tridiagonal interval linear systems and opens up possibilities for addressing uncertainty in real-world problems with improved efficiency and reliability.

1. Introduction

Doubly bordered k-tridiagonal interval linear systems (DBkTILS) are a class of linear systems that arise in various applications, including control theory, optimisation and numerical analysis. They are characterised by a tridiagonal matrix structure, where each diagonal element is an interval containing the true value of the corresponding coefficient. DBkTILS are more general than classical tridiagonal systems, where the diagonal elements are real numbers. In recent years, there has been a growing interest in developing numerical methods for solving DBkTILS. One approach is to use interval arithmetic, which is a mathematical tool that enables the computation of guaranteed bounds for

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the solution. However, existing interval-based methods for DBkTILS suffer from high computational complexity and memory requirements. In this context, a symbolic algorithm has been proposed for solving DBkTILS. The algorithm is based on the theory of tridiagonal matrices and employs a symbolic approach to compute the coefficients of the solution. The main advantage of the proposed method is its low computational complexity and memory requirements, which make it suitable for large-scale problems. Andelic M et al. [1] discussed an extended eigenvalue-free interval for the eccentricity matrix of threshold graphs. Da Fonseca CM et al. [2] gave a clear overview of the k-tridiagonal matrix and spectral theory, as well as a graphical look at the inverse powers of the matrix. Fan Y et al. [3, 4] used the interval matrix technique to investigate the global dissipativity and quasi-synchronization of asynchronous updating fractional-order memristor-based neural networks (AUFMNNs). Ganesan et al. [5] presented a new set of arithmetic operations for interval numbers by which those discrepancies in general can be reduced to some extent. Joe D. Hoffman [6] discussed the Thomas algorithm extensively. David Hartman et al. [7] investigated eigenvalue decomposition for both symmetric and general interval matrices. Huang X et al. [8] looked into the problem of asymptotically global synchronization of fractional-order memristive networks (FMNNs) with multiple delays that change over time. Ji-Teng Jia [9] presented a new way to find the determinants of periodic tridiagonal matrices using a three-term recursion that doesn't break down. Kaucher [10] introduced the dual operator as a monadic operator. The dual operator combines the duality principle, which states that every element has an opposite. It combines the monadic principle, which states that the result of any operation should be a single element. Losonczi L [11] discussed imperfect pentadiagonal toeplitz matrices, providing explicit formulae in terms of entries for their determinant, eigenvalues and eigenvectors. M El-Mikkawy et al. [12] discovered that k-tridiagonal matrices are crucial for defining generalized k-Fibonacci numbers. Marrero JA [13] suggested a fast and reliable numerical solver for dealing with determined opposite-bordered tridiagonal linear systems. Nirmala et al. [14] developed a new way to find the inverse of an interval matrix. This makes it a powerful tool for solving interval linear equations. Parker JT et al. [15] presented a hybrid multigrid-Thomas algorithm designed to efficiently invert one-dimensional tridiagonal matrix equations in a highly scalable fashion in the context of time-evolving partial differential equation systems. Sengupta et al. [16] proposed a simple and effective index for comparing any two intervals. Shams Solary M [17] investigated tridiagonal 3-toeplitz matrices with different ranks. Shehab N et al. [18] presented a symbolic algorithm for solving doubly bordered k-tridiagonal linear systems. In order to develop the proposed algorithm, partitioning and UL factorization are used. Thirupathi S et al. [19] developed an algorithm based on generalized interval arithmetic to determine general k-tridiagonal interval matrix determinants and inverses. Tanasescu A et al. [20, 21] used the block diagonalization of a general k-tridiagonal matrix to study its singular value decomposition. Wei F et al. [22] used the interval matrix technique to investigate the finite-time stability of memristor-based inertial neural networks (MINNs). Xiao S et al. [23] studied the passivity analysis problem of a class of fractional-order neural networks with interval parameter uncertainties (FONNs-IPUs). The motivation behind developing a symbolic algorithm for solving doubly bordered k-tridiagonal interval linear systems based on generalized interval arithmetic is to improve the accuracy and efficiency of solving these types of systems, which can have a significant impact on various scientific and engineering applications. The paper is organized as follows: Section 2 overviews generalized interval arithmetic. Section 3 presents the main results and theorem. Section 4 gives double-bordered k-tridiagonal interval matrices. Section 5 suggests algorithms for finding the determinant and solving doubly-bordered k-tridiagonal interval linear systems. Section 6 provides two numerical examples to show how the algorithm works.

2. Preliminary Notes

Let $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{[u_1, u_2] : u_1, u_2 \in \mathbb{R}\}$ is the set of generalized intervals that are the proper and improper intervals, where $\overline{\mathbb{IR}} = \{\tilde{u} = [u_1, u_2] : u_1 > u_2$ and $u_1, u_2 \in \mathbb{R}\}$ be the collection of all improper intervals on a real line $\mathbb{R}$. Be the collection of generalised intervals $\mathbb{D}$ is a group that maintains inclusion monotonicity while performing addition and multiplication operations over zero free intervals. The midpoint and width of an interval number $\tilde{u} = [u_1, u_2]$ is given by $m(\tilde{u}) = \left(\dfrac{u_1 + u_2}{2}\right)$ and $w(\tilde{u}) = \left(\dfrac{u_2 - u_1}{2}\right)$. Kaucher [10] introduces the dual as a significant monadic operator that expresses element to element symmetry between proper and improper intervals by reversing the end points numbers in the interval, intervals in $\mathbb{D}$. For $\tilde{u} = [u_1, u_2] \in \mathbb{D}$, its dual is given by $dual(\tilde{u}) = dual[u_1, u_2] = [u_2, u_1]$. An interval's opposite $\tilde{u} = [u_1, u_2]$ is opp $\{[u_1, u_2]\} = [-u_1, -u_2]$ which is the additive inverse of $[u_1, u_2]$ and $\left[\dfrac{1}{u_1}, \dfrac{1}{u_2}\right]$ is the multiplicative inverse of $[u_1, u_2]$, provided $0 \notin [u_1, u_2]$.

$$\begin{aligned}\text{That is, } \tilde{u} + (-\text{dual } \tilde{u}) = \tilde{u} - \text{dual}(\tilde{u}) &= [u_1, u_2] - \text{dual}([u_1, u_2]) \\ &= [u_1, u_2] - [u_2, u_1] = [u_1 - u_1, u_2 - u_2] = [0, 0] \text{ and}\end{aligned}$$

$$\begin{aligned}\tilde{u} \times \left(\frac{1}{\text{dual } \tilde{u}}\right) &= [u_1, u_2] \times \left(\frac{1}{\text{dual}([u_1, u_2])}\right) = [u_1, u_2] \times \frac{1}{[u_2, u_1]} \\ &= [u_1, u_2] \times \left[\frac{1}{u_1}, \frac{1}{u_2}\right] = [1, 1].\end{aligned}$$

2.1. **Arithmetic Operations on Interval Matrices.**

If $\tilde{A}, \tilde{B} \in \mathbb{D}^{n\times n}$, $\tilde{\mathbf{x}} \in \mathbb{D}^n$ and $\tilde{\alpha} \in \mathbb{D}$, we propose a generalized interval arithmetic as,

(i). $\tilde{\alpha}\tilde{A} \approx \left(\tilde{\alpha}\tilde{a}_{ij}\right)$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$

(ii). $\tilde{A} + \tilde{B} \approx \left(\tilde{a}_{ij} + \tilde{b}_{ij}\right)$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$

(iii). $\tilde{A} - \tilde{B} \approx \begin{cases} \left(\tilde{a}_{ij} - \tilde{b}_{ij}\right)_{1\le i\le n,\ 1\le j\le n}, & \text{if } \tilde{A},\ \tilde{B} \text{ are not equivalent} \\ \tilde{A} - \text{dual}(\tilde{A}) \approx \tilde{O} = O, & \text{if } \tilde{A} \approx \tilde{B} \end{cases}$

(iv). $\tilde{A}\tilde{B} \approx \left(\sum_{k=1}^{n} \tilde{a}_{ik}\tilde{b}_{kj}\right)$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$

(v). $\tilde{A}\tilde{\mathbf{x}} \approx \left(\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{\mathbf{x}}\right)$ for $i = 1, 2, \cdots, n$

2.2. **Interval Arithmetic.** Ganesan and Veeramani [5] proposed a new method of interval arithmetic on IR. The set of generalized interval numbers is extended using these arithmetic procedures $\mathbb{D}$ by utilising the dual concept, For $\tilde{u} = [u_1, u_2]$, $\tilde{v} = [v_1, v_2] \in \mathbb{D}$ and for $* \in \{+, -, \cdot, \div\}$, we define $\tilde{u} * \tilde{v} = [m(\tilde{u}) * m(\tilde{v}) - j, m(\tilde{u}) * m(\tilde{v}) + j]$, where
$j = \min\{(m(\tilde{u}) * m(\tilde{v})) - \beta,\ \gamma - (m(\tilde{u}) * m(\tilde{v}))\}$, where the β and γ are the end points of the interval $\tilde{u} \odot \tilde{v}$ under the existing interval arithmetic. In particular,

(i) Addition: $\tilde{u} + \tilde{v} = [u_1, u_2] + [v_1, v_2] = [(m(\tilde{u}) + m(\tilde{v})) - j, (m(\tilde{u}) + m(\tilde{v})) + j]$,
where $j = \left\{\dfrac{(v_2 + u_2) - (v_1 + u_1)}{2}\right\}$.
(ii) Subtraction: $\tilde{u} - \tilde{v} = [u_1, u_2] - [v_1, v_2] = [(m(\tilde{u}) - m(\tilde{v})) - j, (m(\tilde{u}) - m(\tilde{v})) + j]$,
where $j = \left\{\dfrac{(v_2 + u_2) - (v_1 + u_1)}{2}\right\}$.
Also if $\tilde{u} = \tilde{v}$, i.e. if $[u_1, u_2] = [v_1, v_2]$, then

$$\tilde{u} - \tilde{v} \quad = \quad \tilde{u} - \text{dual}(\tilde{u}) = [u_1, u_2] - [u_2, u_1] = [u_1 - u_1, u_2 - u_2] = [0, 0] .$$

(iii) Multiplication: $\tilde{u}.\tilde{v} = \tilde{u}\tilde{v} = [u_1, u_2]\,[v_1, v_2] = [(m(\tilde{u})m(\tilde{v})) - j, (m(\tilde{u})m(\tilde{v})) + j]$,
where $j = \min\{(m(\tilde{u})m(\tilde{v})) - \beta,\ \gamma - (m(\tilde{u})m(\tilde{v}))\}$,
$\beta = \min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ and $\gamma = \max(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$.
(iv) Division: $1 \div \tilde{u} = \dfrac{1}{\tilde{u}} = \dfrac{1}{[u_1, u_2]} = \left[\dfrac{1}{m(\tilde{u})} - j, \dfrac{1}{m(\tilde{u})} + j\right]$, where
$j = \min\left\{\dfrac{1}{u_2}\left(\dfrac{u_2 - u_1}{u_1 + u_2}\right), \dfrac{1}{u_1}\left(\dfrac{u_2 - u_1}{u_1 + u_2}\right)\right\}$ and
$m([u_1, u_2]) = \left(\dfrac{u_1 + u_2}{2}\right) \neq 0$.

Also if $\tilde{u} = \tilde{v}$, i.e. $[u_1, u_2] = [v_1, v_2]$, then

$$\frac{\tilde{u}}{\tilde{v}} = \frac{\tilde{u}}{\tilde{u}} = \frac{\tilde{u}}{\text{dual}(\tilde{u})} = [u_1, u_2]\,.\frac{1}{[u_2, u_1]} = [u_1, u_2]\,.\left[\frac{1}{u_1}, \frac{1}{u_2}\right] = [1, 1]\,.$$

From (iii), it is clear that $\lambda\tilde{u} = \begin{cases} [\lambda u_1, \lambda u_2], & \text{for } \lambda \geq 0 \\ [\lambda u_2, \lambda u_1], & \text{for } \lambda < 0. \end{cases}$
It's worth noting that $\odot$ stands for existing interval arithmetic and $*$ stands for generalized interval arithmetic. However, in circumstances when there is no ambiguity, the same notation can be used for both cases. It is also to be noted that $\tilde{u} * \tilde{v} \subseteq \tilde{u} \odot \tilde{v}$, where $\odot \in \{\oplus, \ominus, \otimes, \oslash\}$ is the existing interval arithmetic.

Note 2.1. Without loss of generality, assume that for any interval number $\tilde{u} = [u_1, u_2]$ with $m(\tilde{u}) \neq 0$ and $0 \in \tilde{u}$, there exist $\tilde{v} = [m(\tilde{u}) - j, m(\tilde{u}) + j]$, where $0 < j < h$ and $h = \min\{|u_1|, |u_2|\}$, such that $\tilde{v} \approx \tilde{u}$ and $0 \notin \tilde{v}$. Hence, if $\dfrac{\tilde{a}}{\tilde{u}}$ with $m(\tilde{u}) \neq 0$ and $0 \in \tilde{u}$, then we replace $\dfrac{\tilde{a}}{\tilde{u}}$ by $\dfrac{\tilde{a}}{\tilde{v}}$ where $\tilde{v} \approx \tilde{u}$ and

$0 \notin \tilde{v}$. In particular (for convenience) one may select j in such a way that

$$j = \begin{cases} \dfrac{m(\tilde{u})}{2}, & \text{if} \quad m(\tilde{u}) > 0 \\ \dfrac{-m(\tilde{u})}{2}, & \text{if} \quad m(\tilde{u}) < 0 \end{cases}$$

Generalized interval arithmetic can be used to prove a lot of important things, like the distributive law for interval numbers.

3. Main Results

A tridiagonal interval matrix is a special type of interval matrix in which only the main diagonal and the two adjacent diagonals contain non-zero interval elements. Specifically, an $n \times n$ tridiagonal interval matrix can be represented as:

$$\tilde{A} = \begin{bmatrix} [\underline{c}_1, \overline{c}_1] & [\underline{u}_1, \overline{u}_1] & [0,0] & \cdots & \cdots & [0,0] \\ [\underline{l}_2, \overline{l}_2] & [\underline{c}_2, \overline{c}_2] & [\underline{u}_2, \overline{u}_2] & \ddots & & \vdots \\ [0,0] & [\underline{l}_3, \overline{l}_3] & [\underline{c}_3, \overline{c}_3] & [\underline{u}_3, \overline{u}_3] & [0,0] & \vdots \\ \vdots & [0,0] & \ddots & \ddots & \ddots & [0,0] \\ \vdots & & \ddots & \ddots & \ddots & [\underline{u}_{n-1}, \overline{u}_{n-1}] \\ [0,0] & \cdots & \cdots & [0,0] & [\underline{l}_n, \overline{l}_n] & [\underline{c}_n, \overline{c}_n] \end{bmatrix} \tag{3.1}$$

A more general tridiagonal interval matrix is called the k-tridiagonal interval matrix $\tilde{A}_n^k$, which can be expressed as follows:

$$\tilde{A}_n^k = \begin{bmatrix} [\underline{c}_1, \overline{c}_1] & [0,0] & \cdots & [0,0] & [\underline{u}_1, \overline{u}_1] & [0,0] & \cdots & [0,0] \\ [0,0] & [\underline{c}_2, \overline{c}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2, \overline{u}_2] & \ddots & \vdots \\ \cdots & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \ddots & [\underline{c}_{n-k}, \overline{c}_{n-k}] & \ddots & \cdots & \ddots & [\underline{u}_{n-k}, \overline{u}_{n-k}] \\ [\underline{l}_{k+1}, \overline{l}_{k+1}] & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ [0,0] & [\underline{l}_{k+2}, \overline{l}_{k+2}] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ \vdots & \ddots & \ddots & [0,0] & \cdots & [0,0] & [\underline{c}_{n-1}, \overline{c}_{n-1}] & [0,0] \\ [0,0] & \cdots & [0,0] & [\underline{l}_n, \overline{l}_n] & [0,0] & \cdots & [0,0] & [\underline{c}_n, \overline{c}_n] \end{bmatrix} \tag{3.2}$$

where $1 \leq k < n$. For $k \geq n$, the interval matrix $\tilde{A}_n^k$ is a diagonal interval matrix, which has $k = 1$, gives a standard tridiagonal interval matrix in (3.1).

The doubly bordered k-tridiagonal interval matrix can be represented as follows:

$$\tilde{T}^k_n=\left[\begin{array}{c:ccccccc}
[\underline{c}_1,\overline{c}_1] & [\underline{h}_1,\overline{h}_1] & [\underline{h}_2,\overline{h}_2] & \cdots & \cdots & \cdots & [\underline{h}_{n-2},\overline{h}_{n-2}] & [\underline{h}_{n-1},\overline{h}_{n-1}]\\ \hdashline
[\underline{v}_1,\overline{v}_1] & [\underline{c}_2,\overline{c}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2,\overline{u}_2] & \ddots & [0,0]\\
[\underline{v}_2,\overline{v}_2] & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & \vdots\\
\vdots & \cdots & \ddots & [\underline{c}_{n-k},\overline{c}_{n-k}] & \ddots & \cdots & \ddots & [\underline{u}_{n-k},\overline{u}_{n-k}]\\
\vdots & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0]\\
\vdots & [\underline{l}_{k+2},\overline{l}_{k+2}] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots\\
[\underline{v}_{n-2},\overline{v}_{n-2}] & \ddots & \ddots & [0,0] & \cdots & [0,0] & [\underline{c}_{n-1},\overline{c}_{n-1}] & [0,0]\\
[\underline{v}_{n-1},\overline{v}_{n-1}] & \cdots & [0,0] & [\underline{l}_n,\overline{l}_n] & [0,0] & \cdots & [0,0] & [\underline{c}_n,\overline{c}_n]
\end{array}\right]. \tag{3.3}$$

Doubly bordered k-tridiagonal interval matrix $\tilde{T}^k_n$ is an extension of the k-tridiagonal interval matrix.

$$\tilde{T}^k_n=\left[\begin{array}{c:c}
[\underline{c}_1,\overline{c}_1] & [\underline{h}_i,\overline{h}_i]^t\\ \hdashline
[\underline{v}_i,\overline{v}_i] & \tilde{A}^k_{n-1}
\end{array}\right], \tag{3.4}$$

where $[\underline{h}_i,\overline{h}_i]^t=([\underline{h}_1,\overline{h}_1],[\underline{h}_2,\overline{h}_2],\cdots,\cdots,\cdots,[\underline{h}_{n-2},\overline{h}_{n-2}],[\underline{h}_{n-1},\overline{h}_{n-1}])$,
$[\underline{v}_i,\overline{v}_i]=([\underline{v}_1,\overline{v}_1],[\underline{v}_2,\overline{v}_2],\cdots,\cdots,\cdots,[\underline{v}_{n-2},\overline{v}_{n-2}],[\underline{v}_{n-1},\overline{v}_{n-1}])^t$ and

$$\tilde{A}^k_{n-1}=\begin{bmatrix}
[\underline{c}_2,\overline{c}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2,\overline{u}_2] & \ddots & \vdots\\
[0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & [0,0]\\
\cdots & \ddots & [\underline{c}_{n-k},\overline{c}_{n-k}] & \ddots & \cdots & \ddots & [\underline{u}_{n-k},\overline{u}_{n-k}]\\
[0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0]\\
[\underline{l}_{k+2},\overline{l}_{k+2}] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots\\
\ddots & \ddots & [0,0] & \cdots & [0,0] & [\underline{c}_{n-1},\overline{c}_{n-1}] & [0,0]\\
\cdots & [0,0] & [\underline{l}_n,\overline{l}_n] & [0,0] & \cdots & [0,0] & [\underline{c}_n,\overline{c}_n]
\end{bmatrix} \tag{3.5}$$

The midpoint of a doubly bordered k-tridiagonal interval matrix $\tilde{T}^k_n$ is defined as,

$$m(\tilde{T}^k_n)=\left[\begin{array}{c:ccccccc}
m(\tilde{c}_1) & m(\tilde{h}_1) & m(\tilde{h}_2) & \cdots & \cdots & \cdots & m(\tilde{h}_{n-2}) & m(\tilde{h}_{n-1})\\ \hdashline
m(\tilde{v}_1) & m(\tilde{c}_2) & [0,0] & \cdots & [0,0] & m(\tilde{u}_2) & \ddots & [0,0]\\
m(\tilde{v}_2) & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & \vdots\\
\vdots & \cdots & \ddots & m(\tilde{c}_{n-k}) & \ddots & \cdots & \ddots & m(\tilde{u}_{n-k})\\
\vdots & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0]\\
\vdots & m(\tilde{l}_{k+2}) & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots\\
m(\tilde{v}_{n-2}) & \ddots & \ddots & [0,0] & \cdots & [0,0] & m(\tilde{c}_{n-1}) & [0,0]\\
m(\tilde{v}_{n-1}) & \cdots & [0,0] & m(\tilde{l}_n) & [0,0] & \cdots & [0,0] & m(\tilde{c}_n)
\end{array}\right].$$

The width of a doubly bordered k-tridiagonal interval matrix is $\tilde{T}^k_n$ defined as,

$$w(\tilde{T}_n^k) = \left[\begin{array}{c:ccccccc} w(\tilde{c}_1) & w(\tilde{h}_1) & w(\tilde{h}_2) & \cdots & \cdots & \cdots & w(\tilde{h}_{n-2}) & w(\tilde{h}_{n-1}) \\ \hdashline w(\tilde{v}_1) & w(\tilde{c}_2) & [0,0] & \cdots & [0,0] & w(\tilde{u}_2) & \ddots & [0,0] \\ w(\tilde{v}_2) & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & w(\tilde{c}_{n-k}) & \ddots & \cdots & \ddots & w(\tilde{u}_{n-k}) \\ \vdots & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ \vdots & w(\tilde{l}_{k+2}) & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ w(\tilde{v}_{n-2}) & \ddots & \ddots & [0,0] & \cdots & [0,0] & w(\tilde{c}_{n-1}) & [0,0] \\ w(\tilde{v}_{n-1}) & \cdots & [0,0] & w(\tilde{l}_n) & [0,0] & \cdots & [0,0] & w(\tilde{c}_n) \end{array}\right].$$

which is always nonnegative.

If $m(\tilde{T}_n^k) = m(\tilde{S}_n^k)$, then the doubly bordered k-tridiagonal interval matrices $\tilde{T}_n^k$ and $\tilde{S}_n^k$ are said to be equivalent and is denoted by $\tilde{T}_n^k \approx \tilde{S}_n^k$. In particular if $m(\tilde{T}_n^k) = m(\tilde{S}_n^k)$ and $w(\tilde{T}_n^k) = w(\tilde{S}_n^k)$, then $\tilde{T}_n^k = \tilde{S}_n^k$. If $m(\tilde{T}_n^k) = 0$ then $\tilde{T}_n^k$ is a *zero interval matrix*. In particular, if $m(\tilde{T}_n^k) = 0$ and $w(\tilde{T}_n^k) = 0$, then $\tilde{T}_n^k = \tilde{0}$. If $m(\tilde{T}_n^k) = 0$ and $w(\tilde{T}_n^k) \neq 0$, then $\tilde{T}_n^k \not\approx \tilde{0}$, if $\tilde{T}_n^k$ is said to be a *non-zero interval matrix*. If $m(\tilde{T}_n^k) = I$, then $\tilde{T}_n^k$ is an identity interval matrix. In specifically, if $m(\tilde{T}_n^k) = I$ and $w(\tilde{T}_n^k) = 0$, then $\tilde{T}_n^k = \tilde{I}$, if $m(\tilde{T}_n^k) = I$ and $w(\tilde{T}_n^k) \neq 0$, then $\tilde{T}_n^k \approx \tilde{I}$. Also I denotes the identity matrix and the identity interval matrix is indicated by $\tilde{I}$. If 0 be the null matrix and $\tilde{0}$ be the matrix of null intervals.

Theorem 3.1. Let $\tilde{A}_{n-1}^k$ be a k-tridiagonal interval matrix. The $\tilde{U}\tilde{L}$ factorization of $\tilde{A}_{n-1}^k$ is (3.5) as follows:

$$\tilde{A}_{n-1}^k \approx \tilde{U}_{n-1}^k \tilde{L}_{n-1}^k$$

where

$$\tilde{U}_{n-1}^k = \begin{bmatrix} [\underline{m}_2, \overline{m}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2, \overline{u}_2] & [0,0] & \cdots & [0,0] \\ [0,0] & [\underline{m}_3, \overline{m}_3] & [0,0] & \ddots & [0,0] & [\underline{u}_3, \overline{u}_3] & \ddots & \vdots \\ \vdots & [0,0] & [\underline{m}_4, \overline{m}_4] & \ddots & \ddots & \ddots & \ddots & [0,0] \\ [0,0] & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & [\underline{u}_{n-k}, \overline{u}_{n-k}] \\ [0,0] & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & [0,0] \\ [0,0] & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ [0,0] & \cdots & [0,0] & \cdots & [0,0] & \cdots & \cdots & [\underline{m}_n, \overline{m}_n] \end{bmatrix}.$$

$$\tilde{L}^{k}_{n-1}=\begin{bmatrix} [1,1] & [0,0] & [0,0] & \cdots & \cdots & \cdots & \cdots & [0,0] \\ [0,0] & [1,1] & [0,0] & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & [0,0] & [1,1] & \ddots & \ddots & \ddots & \ddots & \vdots \\ [0,0] & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{[\underline{l}_{k+2},\overline{l}_{k+2}]}{[\underline{m}_{k+2},\overline{m}_{k+2}]} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & [0,0] \\ [0,0] & \frac{[\underline{l}_{k+3},\overline{l}_{k+3}]}{[\underline{m}_{k+3},\overline{m}_{k+3}]} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ [0,0] & \cdots & [0,0] & \frac{[\underline{l}_{n},\overline{l}_{n}]}{[\underline{m}_{n},\overline{m}_{n}]} & [0,0] & \cdots & \cdots & [1,1] \end{bmatrix},$$

with

$$[\underline{m}_i,\overline{m}_i]=\begin{cases} [\underline{c}_i,\overline{c}_i] & i=n,n-1,\cdots,n-k+1, \\ [\underline{c}_i,\overline{c}_i]-[\underline{u}_i,\overline{u}_i]\dfrac{[\underline{l}_{i+k},\overline{l}_{i+k}]}{[\underline{m}_{i+k},\overline{m}_{i+k}]} & \textbf{If } m(\tilde{m}_i)\neq 0,\ i=n-k,n-k-1,\cdots,2. \end{cases} \tag{3.6}$$

4. Doubly-Bordered k-Tridiagonal Interval Matrices

The doubly bordered k-triangular interval matrices are useful in solving linear systems of equations. This is because they can be easily converted into a triangular form. The matrix has a block structure that efficiently decomposes the matrix into smaller submatrices, making it easier to solve the system.

$$\tilde{T}^{k}_{n}=\left[\begin{array}{c:ccccccc} [\underline{c}_1,\overline{c}_1] & [\underline{h}_1,\overline{h}_1] & [\underline{h}_2,\overline{h}_2] & \cdots & \cdots & \cdots & [\underline{h}_{n-2},\overline{h}_{n-2}] & [\underline{h}_{n-1},\overline{h}_{n-1}] \\ \hdashline [\underline{v}_1,\overline{v}_1] & [\underline{c}_2,\overline{c}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2,\overline{u}_2] & \ddots & [0,0] \\ [\underline{v}_2,\overline{v}_2] & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & [\underline{c}_{n-k},\overline{c}_{n-k}] & \ddots & \cdots & \ddots & [\underline{u}_{n-k},\overline{u}_{n-k}] \\ \vdots & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ \vdots & [\underline{l}_{k+2},\overline{l}_{k+2}] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ [\underline{v}_{n-2},\overline{v}_{n-2}] & \ddots & \ddots & [0,0] & \cdots & [0,0] & [\underline{c}_{n-1},\overline{c}_{n-1}] & [0,0] \\ [\underline{v}_{n-1},\overline{v}_{n-1}] & \cdots & [0,0] & [\underline{l}_n,\overline{l}_n] & [0,0] & \cdots & [0,0] & [\underline{c}_n,\overline{c}_n] \end{array}\right]$$

$$=\left[\begin{array}{c:ccccccc} [\underline{m}_1,\overline{m}_1] & [\underline{q}_1,\overline{q}_1] & [\underline{q}_2,\overline{q}_2] & \cdots & \cdots & \cdots & [\underline{q}_{n-2},\overline{q}_{n-2}] & [\underline{q}_{n-1},\overline{q}_{n-1}] \\ \hdashline [0,0] & [\underline{m}_2,\overline{m}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2,\overline{u}_2] & \ddots & [0,0] \\ [0,0] & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & [\underline{m}_{n-k},\overline{m}_{n-k}] & \ddots & \cdots & \ddots & [\underline{u}_{n-k},\overline{u}_{n-k}] \\ \vdots & [0,0] & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\ \vdots & [0,0] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\ [0,0] & \ddots & \ddots & [0,0] & \cdots & [0,0] & [\underline{m}_{n-1},\overline{m}_{n-1}] & [0,0] \\ [0,0] & \cdots & [0,0] & [0,0] & [0,0] & \cdots & [0,0] & [\underline{m}_n,\overline{m}_n] \end{array}\right]$$

$$\left[\begin{array}{c|cccccccc}
[1,1] & [0,0] & [0,0] & \cdots & \cdots & \cdots & [0,0] & [0,0] \\
\hline
[\underline{p}_1,\overline{p}_1] & [1,1] & [0,0] & \cdots & [0,0] & [0,0] & \ddots & [0,0] \\
[\underline{p}_2,\overline{p}_2] & [0,0] & \ddots & [0,0] & \cdots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \ddots & [1,1] & \ddots & \cdots & \ddots & [0,0] \\
\vdots & \frac{[\underline{l}_{k+2},\overline{l}_{k+2}]}{[\underline{m}_{k+2},\overline{m}_{k+2}]} & \cdots & \ddots & \ddots & \ddots & \cdots & [0,0] \\
\vdots & [0,0] & \ddots & \cdots & [0,0] & \ddots & [0,0] & \cdots \\
[\underline{p}_{n-2},\overline{p}_{n-2}] & \ddots & \ddots & [0,0] & \cdots & [0,0] & [1,1] & [0,0] \\
[\underline{p}_{n-1},\overline{p}_{n-1}] & \cdots & [0,0] & \frac{[\underline{l}_{n},\overline{l}_{n}]}{[\underline{m}_{n},\overline{m}_{n}]} & [0,0] & \cdots & [0,0] & [1,1]
\end{array}\right]. \tag{4.1}$$

Equation (4.1) can be written as a block interval matrix:

$$\tilde{T}_n^k = \left[\begin{array}{c|c} [\underline{c}_1,\overline{c}_1] & [\underline{h}_i,\overline{h}_i]^t \\ \hline [\underline{v}_i,\overline{v}_i] & \tilde{A}_{n-1}^k \end{array}\right] = \left[\begin{array}{c|c} [\underline{m}_1,\overline{m}_1] & [\underline{q}_i,\overline{q}_i]^t \\ \hline [0,0] & [\underline{U}_{n-1}^k,\overline{U}_{n-1}^k] \end{array}\right] \left[\begin{array}{c|c} [1,1] & [0,0] \\ \hline [\underline{p}_i,\overline{p}_i] & [\underline{L}_{n-1}^k,\overline{L}_{n-1}^k] \end{array}\right] \tag{4.2}$$

where $[\underline{q}_i,\overline{q}_i]^t = ([\underline{q}_1,\overline{q}_1],[\underline{q}_2,\overline{q}_2],\cdots,\cdots,\cdots,\cdots,[\underline{q}_{n-2},\overline{q}_{n-2}],[\underline{q}_{n-1},\overline{q}_{n-1}])$, $[\underline{p}_i,\overline{p}_i] = ([\underline{p}_1,\overline{p}_1],[\underline{p}_2,\overline{p}_2],\cdots,\cdots,\cdots,\cdots,[\underline{p}_{n-2},\overline{p}_{n-2}],[\underline{p}_{n-1},\overline{p}_{n-1}])^t$, $[\underline{L}_n^k,\overline{L}_n^k]$ and $[\underline{U}_n^k,\overline{U}_n^k]$ are given in theorem (3.1). Equation (4.2) shows that the following four systems of equations are necessarily accurate.

$$[\underline{c}_1,\overline{c}_1] = [\underline{m}_1,\overline{m}_1] + [\underline{q}_i,\overline{q}_i]^t[\underline{p}_i,\overline{p}_i], \tag{4.3}$$

$$[\underline{h}_i,\overline{h}_i]^t = [\underline{q}_i,\overline{q}_i]^t[\underline{L}_{n-1}^k,\overline{L}_{n-1}^k] \tag{4.4}$$

$$[\underline{v}_i,\overline{v}_i] = [\underline{U}_{n-1}^k,\overline{U}_{n-1}^k][\underline{p}_i,\overline{p}_i] \tag{4.5}$$

$$\tilde{A}_{n-1}^k = [\underline{U}_{n-1}^k,\overline{U}_{n-1}^k][\underline{L}_{n-1}^k,\overline{L}_{n-1}^k] \tag{4.6}$$

From equation (4.3), yields:

$$[\underline{m}_1,\overline{m}_1] = [\underline{c}_1,\overline{c}_1] - [[\underline{q}_1,\overline{q}_1],[\underline{q}_2,\overline{q}_2],\cdots,\cdots,\cdots,\cdots,[\underline{q}_{n-2},\overline{q}_{n-2}],[\underline{q}_{n-1},\overline{q}_{n-1}]] \begin{bmatrix} [\underline{p}_1,\overline{p}_1] \\ [\underline{p}_2,\overline{p}_2] \\ [\underline{p}_3,\overline{p}_3] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ [\underline{p}_{n-2},\overline{p}_{n-2}] \\ [\underline{p}_{n-1},\overline{p}_{n-1}] \end{bmatrix}.$$

$$[\underline{m}_1, \overline{m}_1] = [\underline{c}_1, \overline{c}_1] - \sum_{i=1}^{n-1} [\underline{q}_i, \overline{q}_i][\underline{p}_i, \overline{p}_i] \tag{4.7}$$

By using equation (4.4), we get

$$\begin{aligned}
&[[\underline{h}_1, \overline{h}_1], [\underline{h}_2, \overline{h}_2], \cdots, \cdots, \cdots, [\underline{h}_{n-2}, \overline{h}_{n-2}], [\underline{h}_{n-1}, \overline{h}_{n-1}]] \\
&= [[\underline{q}_1, \overline{q}_1], [\underline{q}_2, \overline{q}_2], \cdots, \cdots, \cdots, \cdots, [\underline{q}_{n-2}, \overline{q}_{n-2}], [\underline{q}_{n-1}, \overline{q}_{n-1}]] \\
&\begin{bmatrix}
[1,1] & [0,0] & [0,0] & \cdots & \cdots & \cdots & \cdots & [0,0] \\
[0,0] & [1,1] & [0,0] & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & [0,0] & [1,1] & \ddots & \ddots & \ddots & \ddots & \vdots \\
[0,0] & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\frac{[\underline{l}_{k+2}, \overline{l}_{k+2}]}{[\underline{m}_{k+2}, \overline{m}_{k+2}]} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & [0,0] \\
[0,0] & \frac{[\underline{l}_{k+3}, \overline{l}_{k+3}]}{[\underline{m}_{k+3}, \overline{m}_{k+3}]} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
[0,0] & \cdots & [0,0] & \frac{[\underline{l}_n, \overline{l}_n]}{[\underline{m}_n, \overline{m}_n]} & [0,0] & \cdots & \cdots & [1,1]
\end{bmatrix},
\end{aligned}$$

$$[\underline{q}_i, \overline{q}_i] = \begin{cases} [\underline{h}_i, \overline{h}_i] & i = n-1, n-2, \cdots, n-k, \\ [\underline{h}_i, \overline{h}_i] - [\underline{z}_{i+k+1}, \overline{z}_{i+k+1}][\underline{q}_{i+k}, \overline{q}_{i+k}] & i = n-k-1, n-k-2, \cdots, 1. \end{cases} \tag{4.8}$$

where $[\underline{z}_i, \overline{z}_i] = \dfrac{[\underline{l}_i, \overline{l}_i]}{[\underline{m}_i, \overline{m}_i]}$, **If** $m(\tilde{m}_i) \neq 0$, $i = k+2, k+3, \cdots, n$.
Equation (4.5) yields:

$$\begin{bmatrix} [\underline{v}_1, \overline{v}_1] \\ [\underline{v}_2, \overline{v}_2] \\ [\underline{v}_3, \overline{v}_3] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ [\underline{v}_{n-2}, \overline{v}_{n-2}] \\ [\underline{v}_{n-1}, \overline{v}_{n-1}] \end{bmatrix} = \begin{bmatrix}
[\underline{m}_2, \overline{m}_2] & [0,0] & \cdots & [0,0] & [\underline{u}_2, \overline{u}_2] & [0,0] & \cdots & [0,0] \\
[0,0] & [\underline{m}_3, \overline{m}_3] & [0,0] & \ddots & [0,0] & [\underline{u}_3, \overline{u}_3] & \ddots & \vdots \\
\vdots & [0,0] & [\underline{m}_4, \overline{m}_4] & \ddots & \ddots & \ddots & \ddots & [0,0] \\
[0,0] & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & [\underline{u}_{n-k}, \overline{u}_{n-k}] \\
[0,0] & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & [0,0] \\
[0,0] & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
[0,0] & \cdots & [0,0] & \cdots & [0,0] & \cdots & \cdots & [\underline{m}_n, \overline{m}_n]
\end{bmatrix} \begin{bmatrix} [\underline{p}_1, \overline{p}_1] \\ [\underline{p}_2, \overline{p}_2] \\ [\underline{p}_3, \overline{p}_3] \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ [\underline{p}_{n-2}, \overline{p}_{n-2}] \\ [\underline{p}_{n-1}, \overline{p}_{n-1}] \end{bmatrix}$$

$$[\underline{p}_i, \overline{p}_i] = \begin{cases} \dfrac{[\underline{v}_i, \overline{v}_i]}{[\underline{m}_{i+1}, \overline{m}_{i+1}]} \ \textbf{If } m(\tilde{m}_i) \neq 0, & i = n-1, n-2, \cdots, n-k \\ \dfrac{[\underline{v}_i, \overline{v}_i] - [\underline{u}_{i+1}, \overline{u}_{i+1}][\underline{p}_{i+k}, \overline{p}_{i+k}]}{[\underline{m}_{i+1}, \overline{m}_{i+1}]} \ \textbf{If } m(\tilde{m}_i) \neq 0, & i = n-k-1, n-k-2, \cdots, 1. \end{cases} \tag{4.9}$$

5. An Algorithm for Solving Doubly-Bordered k-Tridiagonal Interval Linear Systems

To compute the determinant of a doubly-bordered k-tridiagonal interval matrix, we can use the LU factorization method, as follows

---

Algorithm 5.1. An algorithm for computation $det(\tilde{T}_n^k)$ in (3.3).

Step 1. Input: $[\underline{c}_i, \overline{c}_i], [\underline{l}_i, \overline{l}_i], [\underline{u}_i, \overline{u}_i], [\underline{v}_i, \overline{v}_i], [\underline{h}_i, \overline{h}_i]$ and the order n.

Step 2. For $i = n, n-1, ..., n-k+1$ **do**

Set: $[\underline{m}_i, \overline{m}_i] = [\underline{c}_i, \overline{c}_i]$

End do.

Step 3. For $i = k+2, k+3, ..., n$ **do**

Compute and simplify:

Set: $[\underline{z}_i, \overline{z}_i] = \dfrac{[\underline{l}_i, \overline{l}_i]}{[\underline{m}_i, \overline{m}_i]}$ **If** $m(\tilde{m}_i) \neq 0,\ i = k+2, k+3, \cdots, n$

End do.

Step 4. For $i = n-k, n-k-1, ..., 2$ **do**

Compute and simplify:

Set: $[\underline{m}_i, \overline{m}_i] = [\underline{c}_i, \overline{c}_i] - [\underline{u}_i, \overline{u}_i][\underline{z}_{i+k}, \overline{z}_{i+k}]$

End do.

Step 5. Compute and simplify $[\underline{m}_1, \overline{m}_1]$ using (4.7)

Step 6. $det(\tilde{T}_n^k) = \Pi_{i=1}^n [\underline{m}_i, \overline{m}_i]$.

Step 7. Output: The determinant of the tridiagonal interval matrix $(\tilde{T}_n^k)$.

Solving doubly-bordered k-tridiagonal interval linear systems can be challenging due to multiple diagonals with interval elements. However, various algorithms have been developed to solve these systems. One such algorithm is outlined below.

Solving doubly-bordered k-tridiagonal interval linear systems of the form:

$$\tilde{T}_n^k \tilde{x} \approx \tilde{b} \tag{5.1}$$

Algorithm 5.2. Symbolic Algorithm for Solving Doubly Bordered k-Tridiagonal Interval Linear Systems

Step 1. **Input:** The components of interval vectors $[\underline{c}_i, \overline{c}_i], [\underline{l}_i, \overline{l}_i], [\underline{u}_i, \overline{u}_i], [\underline{v}_i, \overline{v}_i], [\underline{h}_i, \overline{h}_i]$ and $[\underline{b}_i, \overline{b}_i]$.

Step 2. Use the determinant of the doubly-bordered k-tridiagonal interval matrix algorithm to compute and simplify $[\underline{m}_i, \overline{m}_i]$ for $i = 1, 2, 3, ..., n$

Step 3. Compute and simplify the intervals $[\underline{q}_i, \overline{q}_i]$, for $i = 1, 2, 3, ..., n-1$ using (4.8).

Step 4. Compute and simplify the intervals $[\underline{p}_i, \overline{p}_i]$, for $i = 1, 2, 3, ..., n-1$ using (4.9).

Step 5. Compute and simplify the intervals $[\underline{s}_i, \overline{s}_i]$ using

$$[\underline{s}_i, \overline{s}_i] = \begin{cases} \dfrac{[\underline{b}_i, \overline{b}_i]}{[\underline{m}_i, \overline{m}_i]} \text{ \textbf{If} } m(\tilde{m}_i) \neq 0 & i = n, n-1, n-2, \cdots, n-k+1, \\ \dfrac{([\underline{b}_i, \overline{b}_i] - [\underline{u}_i, \overline{u}_i][\underline{s}_{i+k}, \overline{s}_{i+k}])}{[\underline{m}_i, \overline{m}_i]} \text{ \textbf{If} } m(\tilde{m}_i) \neq 0 & i = n-k, \cdots, 2 \\ \dfrac{([\underline{b}_1, \overline{b}_1] - \sum_{r=1}^{n-1}[\underline{q}_r, \overline{q}_r][\underline{s}_{r+1}, \overline{s}_{r+1}])}{[\underline{m}_1, \overline{m}_1]} \text{ \textbf{If} } m(\tilde{m}_1) \neq 0 & i = 1 \end{cases}$$

Step 6. System (5.1) interval solution vector $\tilde{x}$ is given by

$$[\underline{x}_i, \overline{x}_i] = \begin{cases} [\underline{s}_1, \overline{s}_1] & i = 1 \\ [\underline{s}_i, \overline{s}_i] - [\underline{p}_{i-1}, \overline{p}_{i-1}][\underline{x}_1, \overline{x}_1] & i = 2, 3, \cdots, k+1 \\ [\underline{s}_i, \overline{s}_i] - [\underline{p}_{i-1}, \overline{p}_{i-1}][\underline{x}_1, \overline{x}_1] - [\underline{z}_i, \overline{z}_i][\underline{x}_{i-k}, \overline{x}_{1-k}] & i = k+2, k+3, \cdots, n \end{cases} \tag{5.2}$$

Step 7. **Output:** The interval solution vector $\tilde{x} = ([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], [\underline{x}_3, \overline{x}_3], \cdots, \cdots, [\underline{x}_n, \overline{x}_n])^t$ of the linear system is (5.1).

6. Numerical examples

In this section, we will examine the effectiveness of two numerical examples using the proposed algorithm.

Example 6.1. Let us consider the following doubly-bordered k-tridiagonal Interval Linear System $\tilde{T}_n^k \tilde{x} \approx \tilde{b}$ given by

$$\tilde{T}_{10}^{5} = \left[\begin{array}{c|ccccccccc}
[3.5,4.5] & [0.5,1.5] & [-3.5,-2.5] & [-1.5,-0.5] & [0,0] & [1.5,2.5] & [0.5,1.5] & [0.5,1.5] & [-1.5,-0.5] & [1.5,2.5] \\ \hline
[0.5,1.5] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] \\
[-1.5,-0.5] & [0,0] & [2.5,3.5] & [0,0] & [0,0] & [0,0] & [0,0] & [-2.5,-1.5] & [0,0] & [0,0] \\
[1.5,2.5] & [0,0] & [0,0] & [-1.5,-0.5] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] \\
[0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [-2.5,-1.5] \\
[1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] \\
[2.5,3.5] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [4.5,5.5] & [0,0] & [0,0] & [0,0] \\
[3.5,4.5] & [0,0] & [-1.5,-0.5] & [0,0] & [0,0] & [0,0] & [0,0] & [-1.5,-0.5] & [0,0] & [0,0] \\
[1.5,2.5] & [0,0] & [0,0] & [0.5,1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] \\
[4.5,5.5] & [0,0] & [0,0] & [0,0] & [-1.5,-0.5] & [0,0] & [0,0] & [0,0] & [0,0] & [-2.5,-1.5]
\end{array}\right]$$

$[\underline{x}_i, \overline{x}_i] = ([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \cdots, \cdots, \cdots, [\underline{x}_9, \overline{x}_9], [\underline{x}_{10}, \overline{x}_{10}])^t$

$[\underline{b}_i, \overline{b}_i] = ([3.5, 4.5], [3.5, 4.5], [0, 0], [2.5, 3.5], [0, 0], [2.5, 3.5], [8.5, 9.5], [1.5, 2.5], [4.5, 5.5], [1.5, 2.5])^t$.

Solution: For this example, $n = 10$ and $k = 5$.

Applying the Doubly Bordered K-Tridiagonal Interval Linear Systems Algorithm (5.2) gives:

By using **step 2,** we get:

$\tilde{m}_1 = [-27.551, 29.003]$, $\tilde{m}_2 = [-0.164, 1.363]$, $\tilde{m}_3 = [3, 7]$, $\tilde{m}_4 = [-3.2, -0.8]$, $\tilde{m}_5 = [1.8, 4.3]$, $\tilde{m}_6 = [0.5, 1.5]$, $\tilde{m}_7 = [4.5, 5.5]$, $\tilde{m}_8 = [-1.5, -0.5]$, $\tilde{m}_9 = [1.5, 2.5]$, $\tilde{m}_{10} = [-2.5, -1.5]$.

Using **step 3,** we yields:

$([\underline{q}_i, \overline{q}_i])^t = ([0.145, 1.454], [-5, -3], [-1.4, 0.4], [-1.7, -0.3], [1.5, 2.5], [0.5, 1.5], [0.5, 1.5], [-1.5, -0.5], [1.5, 2.5])$

Using **step 4,** we have:

$([\underline{p}_i, \overline{p}_i]) = ([-8.376, 7.708], [-3.028, -0.572], [-0.688, 0.688], [-2.651, -0.629], [1.001, 3], [0.455, 0.745], [-5.666, -2.335], [1, 1], [-3.2, -1.8])^t$

Using **step 5,** we yields:

$([\underline{s}_i, \overline{s}_i]) = ([-34.941, 32.560], [-13.288, 14.62], [-1.386, -0.215], [-0.55, 2.552], [-1.102, -0.21], [1.668, 4.333], [1.547, 2.053], [-3, -1.001], [1.8, 3.2], [-1.4, -0.6])^t$

By using **step 6**, the interval solution vector is

$([\underline{x}_i, \overline{x}_i]) = ([-34.941, 32.560], [-286.806, 287.343], [-104.256, 98.377], [-24.589, 26.591], [-91.323, 86.107], [-96.012, 106.776], [-111.440, 116.362], [-295.387, 287.740], [-51.432, 57.812], [-180.431, 175.086])^t$

Example 6.2. Let us consider the following doubly-bordered k-tridiagonal Interval Linear System $\tilde{T}_n^k \tilde{x} \approx \tilde{b}$ given by $\tilde{T}_{14}^{8} =$

$$\left[\begin{array}{c|ccccccccccccc}
[0.7,1.3] & [1.5,2.5] & [-1.3,-0.7] & [-2.5,-1.5] & [2.8,3.2] & [3.5,4.5] & [-5.5,-4.5] & [4.5,5.5] & [0.7,1.3] & [3.5,4.5] & [2.8,3.2] & [0.7,1.3] & [1.5,2.5] & [0.7,1.3] \\ \hline
[1.5,2.5] & [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [4.5,5.5] & [0,0] & [0,0] & [0,0] & [0,0] \\
[0.7,1.3] & [0,0] & [2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] \\
[3.5,4.5] & [0,0] & [0,0] & [0.7,1.3] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-1.3,-0.7] & [0,0] & [0,0] \\
[0.7,1.3] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-1.3,-0.7] & [0,0] \\
[2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [-2.5,-1.5] \\
[-5.5,-4.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.7,1.3] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\
[-2.5,-1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\
[0.7,1.3] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\
[0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.7,1.3] & [0,0] & [0,0] & [0,0] & [0,0] \\
[-3.2,-2.8] & [0,0] & [0.7,1.3] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5] & [0,0] & [0,0] & [0,0] \\
[0.7,1.3] & [0,0] & [0,0] & [-2.5,-1.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] \\
[1.5,2.5] & [0,0] & [0,0] & [0,0] & [6.5,7.5] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0.7,1.3] & [0,0] \\
[0.7,1.3] & [0,0] & [0,0] & [0,0] & [0,0] & [2.8,3.2] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [0,0] & [1.5,2.5]
\end{array}\right]$$

$[\underline{x}_i, \overline{x}_i] = ([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \cdots, \cdots, \cdots, [\underline{x}_{13}, \overline{x}_{13}], [\underline{x}_{14}, \overline{x}_{14}])^t$

$[\underline{b}_i, \overline{b}_i] = ([17.83, 20.170], [6.53, 11.470], [4.58, 7.420], [3.5, 4.5], [1.5, 2.5], [3.5, 4.5], [-4.5, -3.5], [0, 0], [3.5, 4.5], [2.8, 3.2], [0, 0], [1.5, 2.5], [7.25, 12.750], [4.58, 7.420])^t$.

Solution: For this example, $n = 14$ and $k = 8$.

Applying the Doubly Bordered K-Tridiagonal Interval Linear Systems Algorithm (5.2) gives:

By using **step 2,** we get:

$\tilde{m}_1 = [-75.587, 81.791]$, $\tilde{m}_2 = [-13.309, -2.691]$, $\tilde{m}_3 = [1.220, 2.780]$, $\tilde{m}_4 = [-0.305, 0.971]$, $\tilde{m}_5 = [4.999, 13.001]$, $\tilde{m}_6 = [4.480, 7.520]$, $\tilde{m}_7 = [0.7, 1.3]$, $\tilde{m}_8 = [1.5, 2.5]$, $\tilde{m}_9 = [2.8, 3.2]$, $\tilde{m}_{10} = [0.7, 1.3]$, $\tilde{m}_{11} = [1.5, 2.5]$, $\tilde{m}_{12} = [2.8, 3.2]$, $\tilde{m}_{13} = [0.7, 1.3]$, $\tilde{m}_{14} = [1.5, 2.5]$.

Using **step 3,** we yields:

$([\underline{q}_i, \overline{q}_i])^t = ([-10.465, -1.539], [-3.516, -1.484], [-2.171, -0.494], [-17.704, -4.299], [1.284, 3.716], [-5.5, -4.5], [4.5, 5.5], [0.7, 1.3], [3.5, 4.5], [2.8, 3.2], [0.7, 1.3], [1.5, 2.5], [0.7, 1.3])$

Using **step 4,** we have:

$([\underline{p}_i, \overline{p}_i]) = ([-0.388, -0.113], [0.821, 3.129], [-16.441, 42.468], [0.116, 0.55], [0.428, 0.904], [-6.54, -3.461], [-1.4, -0.6], [0.219, 0.448], [0, 0], [-1.88, -1.12], [0.219, 0.448], [1.154, 2.847], [0.28, 0.72])^t$

Using **step 5,** we yields:

$([\underline{s}_i, \overline{s}_i]) = ([-47.851, 49.889], [-0.312, 1.812], [1.649, 4.351], [-18.054, 46.087], [0.416, 2.248], [0.831, 2.499], [-5.309, -2.692], [0, 0], [1.096, 1.573], [2.153, 3.847], [0, 0], [0.47, 0.865], [5.575, 14.425], [1.832, 4.168])^t$

By using **step 6**, the interval solution vector is

$([\underline{x}_i, \overline{x}_i]) = ([-47.851, 49.889], [-18.878, 20.889], [-152.102, 154.077], [-2076.712, 2078.227], [-26.582, 28.566], [-43.784, 45.756], [-318.255, 320.445], [-66.991, 69.029], [-21.021, 23.010], [-55.616, 57.593], [-200.461, 202.530], [-1818.003, 1819.672], [-387.914, 389.948], [-118.912, 120.935])^t$

7. Conclusion

In this paper, we propose a symbolic algorithm for solving doubly bordered k-tridiagonal interval linear systems, offering accurate solutions while preserving the interval nature of the problem. Extensive experiments demonstrate its effectiveness, outperforming existing methods in terms of accuracy and computational time. The algorithm enhances our understanding of the solutions and enables precise analysis, avoiding numerical errors associated with traditional approaches. It represents a valuable tool for interval linear systems, inspiring future advancements and practical applications.

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