Int. J. Anal. Appl. (2023), 21:91

International Journal of Analysis and Applications

Transmission Problem Between Two Herschel-Bulkley Fluids in a Three Dimensional Thin Layer

Salim Saf[1,*], Farid Messelmi[2]

[1] *Laboratory of Pure and Applied Mathematics, Amar Telidji University of Laghouat, Laghouat 03000, Algeria*

[2] *Department of Mathematics and LDMM Laboratory Ziane Achour University, Djelfa 17000, Algeria*

[*] *Corresponding author:* s.saf@lagh-univ.dz

Abstract. The paper is devoted to the study of steady-state transmission problem between two Herschel-Bulkley fluids in a three dimensional thin layer.

1. Introduction

The rigid viscoplastic and incompressible fluid of Herschel-Bulkley has been studied and used by many mathematicians, physicists and engineers, to model the flow of metals, plastic solids and a variety of polymers. Due to the existence of the yield limit, the model can capture phenomena connected with the development of discontinuous stresses. A particularity of Herschel-Bulkley fluid lies in the presence of rigid zones located in the interior of the flow and as yield limit increases, the rigid zones become larger and may completely block the flow, this phenomenon is known as the blockage property. The literature concerning this topic is extensive; see e.g. [4, 11, 12, 14, 15]. The purpose of this paper is to study the asymptotic behavior of the steady flow of Herschel-Bulkley fluid in a three-dimensional thin layer. The paper is organized as follows. In section 2 we present the mechanical problem of the steady flow of Herschel-Bulkley fluid in a three-dimensional thin layer. We introduce some notations and preliminaries. Moreover, we define some function spaces and we recall the variational formulation. In Section 3, we are interested in the asymptotic behavior, to this aim we prove some convergence

Received: Jun. 22, 2023.

2020 *Mathematics Subject Classification.* 76A05, 49J40, 76B15.

Key words and phrases. Herschel-Bulkley fluid; transmission; asymptotic behaviour; thin layer.

https://doi.org/10.28924/2291-8639-21-2023-91
© 2023 the author(s).
ISSN: 2291-8639

results concerning the velocity and pressure when the thickness tends to zero. Besides, the uniqueness of a limit solution has been also established.

2. Problem statement

Denoting by ω the fixed region in plan $x = (x_1, x_2) \in \mathbb{R}^2$. Introducing the function $h : \omega \to \mathbb{R}$ such that $0 < h_0 \leq h(x, y) \leq h_1$ for all $(x, y) \in \mathbb{R}^2$, where h_0 and h_1 are constants.

Considering the following domains

$$\begin{aligned}
\Omega_1 &= \left\{(x, y, z) \in \mathbb{R}^3 / \ (x, y) \in \omega \text{ and } 0 < z < h(x, y)\right\},\\
\Omega_1^\varepsilon &= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 / \ (x_1, x_2) \in \omega \text{ and } 0 < x_3 < \varepsilon h(x_1, x_2)\right\},\\
\Omega_2 &= \left\{(x, y, z) \in \mathbb{R}^3 / \ (x, y) \in \omega \text{ and } -h(x, y) < z < 0\right\}\\
\Omega_2^\varepsilon &= \left\{(x_1, x_2; x_3) \in \mathbb{R}^3 / \ (x_1, x_2) \in \omega \text{ and } -\varepsilon h(x_1, x_2) < x_3 < 0 \ \right\},
\end{aligned}$$

where $\varepsilon > 0$.

Remark that if $(x_1, x_2; x_3) \in \Omega_i^\varepsilon$ then $(x, y, z) = (x_1, x_2, \frac{x_3}{\varepsilon}) \in \Omega_i$. This permits us to define, for every function $\varphi_i^\varepsilon : \Omega_i^\varepsilon \to \mathbb{R}$, the function $\widehat{\varphi_i^\varepsilon} : \Omega_i \to \mathbb{R}$ given by $\widehat{\varphi_i^\varepsilon}(x, y, z) = \varphi_i^\varepsilon(x_1, x_2, x_3)$, $i = 1, 2$.

Let $1 < p \leq 2$, p' the conjugate p, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ and $f_i = (f_{i1}, f_{i2}, f_{i3}) \in L^{p_i'}(\Omega_i)^3$ a given functions. We define the functions $f_i^\varepsilon \in L^{p'}(\Omega_i^\varepsilon)^3$ such that $\widehat{f_i^\varepsilon} = f_i$, $i = 1, 2$. We consider a mathematical problem modeling the steady flow of a rigid viscoplastic and incompressible Herschel-Bulkley fluid. We suppose that the consistency and yield limit of the fluid are respectively $\mu_i \varepsilon^p$, $g_i \varepsilon$ where μ_i, $g_i > 0$, $i = 1, 2$ and p represents the power-law index. The first fluid occupies a bounded domain $\Omega_1^\varepsilon \subset \mathbb{R}^3$ with the boundary $\partial\Omega_1^\varepsilon$ of class C^1. The second one occupies a bounded domain $\Omega_2^\varepsilon \subset \mathbb{R}^3$ with the boundary $\partial\Omega_2^\varepsilon$ of class C^1. We denote by Ω^ε the domain $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ and we suppose that $\partial\Omega_1^\varepsilon = \omega \cup \Gamma_1^\varepsilon$ and $\partial\Omega_2^\varepsilon = \omega \cup \Gamma_2^\varepsilon$ the velocity is known and equal to zero, where $\omega, \Gamma_1^\varepsilon, \Gamma_2^\varepsilon$ are measurable domains and $meas(\Gamma_1^\varepsilon)$, $meas(\Gamma_2^\varepsilon) > 0$. The fluids are acted upon by given volume forces of densities f_1^ε, f_2^ε respectively. We denote by S_3 the space of symmetric tensors on $\mathbb{R}^3$. We define the inner product and the Euclidean norm on $\mathbb{R}^3$ and S_3, respectively, by

$$\begin{aligned}
u.\upsilon &= u_l \upsilon_l \quad \forall u, \upsilon \in \mathbb{R}^3 \quad \text{and} \quad \sigma.\tau = \sigma_{lm}\tau_{lm} \ \forall \sigma, \tau \in S_3.\\
|u| &= (u.u)^{\frac{1}{2}} \quad \forall u \in \mathbb{R}^3 \quad \text{and} \quad |\sigma| = (\sigma.\sigma)^{\frac{1}{2}} \ \forall \sigma \in S_3.
\end{aligned}$$

Here and below, the indices l and m run from 1 to 3 and the summation convention over repeated indices is used. We denote by $\widetilde{\sigma_i^\varepsilon}$ the deviator of σ_i^ε given by

$$\sigma_i^\varepsilon = -p_i^\varepsilon I_3 + \widetilde{\boldsymbol{\sigma}}_i^\varepsilon,$$

where p_i^ε, $i = 1, 2$ represents the hydrostatic pressure and I_3 denotes the identity matrix of size 2.

We consider the rate of deformation operator defined for every $v_i^\varepsilon \in W^{1,\, p_i}(\Omega_i^\varepsilon)^3$ by

$$D(v_i^\varepsilon) = (D_{lm}(v_i^\varepsilon)), \ \ D_{lm}(v_i^\varepsilon) = \frac{1}{2}((\upsilon_i^\varepsilon)_{l,\, m} + (\upsilon_i^\varepsilon)_{m,\, l}), \ \ i = 1, 2.$$

We denote by n the unit outward normal vector on the boundary ω oriented to the exterior of Ω_1^ε and to the interior of Ω_2^ε, see the figure below. For every vector field $v_i^\varepsilon \in W^{1,\,p_i}(\Omega_i^\varepsilon)^3$ we also write v_i^ε for its trace on $\partial\Omega_i^\varepsilon$, $i = 1, 2$.

The steady-state transmission problem for the Herschel-Bulkley fluids in thin layer is given by the following mechanical problem.

Problem P_ε. Find the velocity field $u_i^\varepsilon = (u_{i1}^\varepsilon, u_{i2}^\varepsilon, u_{i3}^\varepsilon) : \Omega_i^\varepsilon \to \mathbb{R}^3$, the stress field $\sigma_i^\varepsilon = (\sigma_{i1}^\varepsilon, \sigma_{i2}^\varepsilon, \sigma_{i3}^\varepsilon) : \Omega_i^\varepsilon \to S_3$ and the pressure $p_i^\varepsilon : \Omega_i^\varepsilon \to \mathbb{R}^2$, $i = 1, 2$ such that

$$\operatorname{div}\sigma_1^\varepsilon + f_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon. \tag{2.1}$$

$$\operatorname{div}\sigma_2^\varepsilon + f_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon. \tag{2.2}$$

$$\left.\begin{array}{ll} \widetilde{\sigma_1^\varepsilon} = \mu_1\varepsilon^p \left|D(u_1^\varepsilon)\right|^{p-2} D(u_1^\varepsilon) + g_1\varepsilon\frac{D(u_1^\varepsilon)}{|D(u_1^\varepsilon)|} & \text{if } \left|D(u_1^\varepsilon)\right| \neq 0 \\ \left|\widetilde{\sigma_1^\varepsilon}\right| \leq g_1\varepsilon & \text{if } \left|D(u_1^\varepsilon)\right| = 0 \end{array}\right\} \text{ in } \Omega_1^\varepsilon, \tag{2.3}$$

$$\left.\begin{array}{ll} \widetilde{\sigma_2^\varepsilon} = \mu_2\varepsilon^p \left|D(u_2^\varepsilon)\right|^{p-2} D(u_2^\varepsilon) + g_2\varepsilon\frac{D(u_2^\varepsilon)}{|D(u_2^\varepsilon)|} & \text{if } \left|D(u_2^\varepsilon)\right| \neq 0 \\ \left|\widetilde{\sigma_2^\varepsilon}\right| \leq g_2\varepsilon & \text{if } \left|D(u_2^\varepsilon)\right| = 0 \end{array}\right\} \text{ in } \Omega_2^\varepsilon, \tag{2.4}$$

$$\operatorname{div} u_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon, \tag{2.5}$$

$$\operatorname{div} u_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon\ , \tag{2.6}$$

$$u_1^\varepsilon = 0 \text{ on } \Gamma_1^\varepsilon\ , \tag{2.7}$$

$$u_2^\varepsilon = 0 \text{ on } \Gamma_2^\varepsilon, \tag{2.8}$$

$$u_1^\varepsilon - u_2^\varepsilon = 0 \text{ on } \omega, \tag{2.9}$$

$$\sigma_1^\varepsilon.\mathbf{n} - \sigma_2^\varepsilon.\mathbf{n} = 0 \text{ on } \omega. \tag{2.10}$$

Here, the flow is given by the equations (2.1) and (2.2). Equations (2.3) and (2.4) represents, respectively, the constitutive laws of Herschel-Bulkley fluids where and are the consistencies and yield limits of the two fluids, respectively, $1 < p \leq 2$ are the power law index of the two fluids, respectively. Equations (2.5) and (2.6) represents the incompressibility condition. (2.7), (2.8) give the velocities on the boundaries Γ_1^ε and Γ_2^ε, respectively. Finally, on the boundary part ω, (2.9) and (2.10) represents the transmission condition for liquid-liquid interface.

Let us define now the following Banach spaces

$$W_{\Gamma i}^{1,\,p}(\Omega_i^\varepsilon) = \left\{v_i \in W^{1,\,p}(\Omega_i^\varepsilon) : v_i = 0 \text{ on } \Gamma_i^\varepsilon\right\}, \tag{2.11}$$

$$W_{\operatorname{div}}^{p,\,\varepsilon}(\Omega_i^\varepsilon) = \left\{v_i \in W^{1,\,p}(\Omega_i^\varepsilon)^3 : \operatorname{div}(v_i) = 0 \text{ in } \Omega_i^\varepsilon\right\}, \tag{2.12}$$

$$W_{\operatorname{div}}^{p}(\Omega_i) = \left\{v_i \in W^{1,\,p}(\Omega_i)^3 : \operatorname{div}(v_i) = 0 \text{ in } \Omega_i\right\}, \tag{2.13}$$

$$L_0^p(\Omega_i^\varepsilon) = \left\{\varphi_i^\varepsilon \in L^p(\Omega_i^\varepsilon) : \int_{\Omega_i^\varepsilon} \varphi_i^\varepsilon(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3 = 0\right\}, \tag{2.14}$$

$$L_0^p(\Omega_i) = \left\{\varphi_i \in L^p(\Omega_i) \ : \int_{\Omega_i} \varphi_i(x,y,z)dxdydz = 0\right\}, \quad (2.15)$$

$$W_z(\Omega_i) = \left\{\varphi_i \in L^p(\Omega_i) : \frac{\partial \varphi_i}{\partial z} \in L^p(\Omega_i)\right\}, \ i = 1, 2. \quad (2.16)$$

$$W_z = W_z(\Omega_1)^2 \times W_z(\Omega_2)^2, \quad (2.17)$$

$$W_{\text{div}}^{\varepsilon} = \left\{ \begin{array}{c} (v_1, v_2) \in W_{\text{div}}^{p,\,\varepsilon}(\Omega_1^\varepsilon) \times W_{\text{div}}^{p,\,\varepsilon}(\Omega_2^\varepsilon) : v_1 = v_2 \\ \text{on } \omega,\ v_1 = 0 \text{ on } \Gamma_1^\varepsilon,\ v_2 = 0 \text{ on } \Gamma_2^\varepsilon \end{array} \right\}, \quad (2.18)$$

$$W^\varepsilon = \left\{(v_1, v_2) \in W_{\Gamma_1^\varepsilon}^{1,\,p}(\Omega_1^\varepsilon)^3 \times W_{\Gamma_2^\varepsilon}^{1,\,p}(\Omega_2^\varepsilon)^3 : v_1 = v_2 \text{ on } \omega\right\}. \quad (2.19)$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem.

The use of Green's formula permits us to derive the following variational formulation of the mechanical problem (P_ε), see [4, 13, 15].

Problem PV$_\varepsilon$. For prescribed data $(f_1^\varepsilon, f_2^\varepsilon) \in L^{p'}(\Omega_1^\varepsilon)^3 \times L^{p'}(\Omega_2^\varepsilon)^3$. Find $(u_1^\varepsilon, u_2^\varepsilon) \in W_{\text{div}}^\varepsilon$ and $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$ satisfying the variational inequality

$$\begin{aligned}
&\mu_1 \varepsilon^p \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon).D(v_1 - u_1^\varepsilon)dx_1dx_2dx_3 + g_1\varepsilon \int_{\Omega_1^\varepsilon} |D(v_1)|\, dx_1dx_2dx_3 \\
&-g_1\varepsilon \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|\, dx_1dx_2dx_3 + \mu_2\varepsilon^p \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon).D(v_2 - u_2^\varepsilon)dx_1dx_2dx_3 \\
&+g_2\varepsilon \int_{\Omega_2^\varepsilon} |D(v_2)|\, dx_1dx_2dx_3 - g_2\varepsilon \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|\, dx_1dx_2dx_3 \\
&\geq \int_{\Omega_1^\varepsilon} f_1^\varepsilon.(v_1 - u_1^\varepsilon)dx_1dx_2dx_3 + \int_{\Omega_1^\varepsilon} p_1^\varepsilon \operatorname{div}(v_1 - u_1^\varepsilon)dx_1dx_2dx_3 \\
&+\int_{\Omega_2^\varepsilon} f_2^\varepsilon.(v_2 - u_2^\varepsilon)dx_1dx_2dx_3 + \int_{\Omega_2^\varepsilon} p_2^\varepsilon \operatorname{div}(v_2 - u_2^\varepsilon)dx_1dx_2dx_3,\ \forall (v_1, v_2) \in W^\varepsilon. \quad (2.20)
\end{aligned}$$

It is known that this variational problem has a unique solution $(u_1^\varepsilon, u_2^\varepsilon) \in W_{\text{div}}^\varepsilon$ and $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$, see for more details [11, 13, 15].

3. Asymptotic behavior

In this section, we establish some results concerning the asymptotic behavior of the solution when ε tends to zero. We begin by recalling the following lemmas see [1, 3, 4, 8, 16].

Lemma 3.1. (1) *Poincaré's inequality. For every $v_i \in W_{\Gamma_i}^{1,p}(\Omega_i^\varepsilon)^3$ we have*

$$\|v_i^\varepsilon\|_{L^p(\Omega_i^\varepsilon)^3} \leq \varepsilon \left\| \frac{\partial v_i^\varepsilon}{\partial x_2} \right\|_{L^p(\Omega_i^\varepsilon)^3},\ i = 1, 2. \quad (3.1)$$

(2) *Korn's inequality. For every $v_i \in W_{\Gamma_i}^{1,p}(\Omega_i^\varepsilon)^3$ there exists a positive constant C_0 independent on ε, such that*

$$\|\nabla v_i^\varepsilon\|_{L^p(\Omega_i^\varepsilon)^9} \leq C_0 \,\|D(v_i^\varepsilon)\,\|_{L^p(\Omega_i^\varepsilon)^9},\ i = 1, 2. \quad (3.2)$$

Lemma 3.2 (Minty)**.** *Let E be a banach spaces, $A : E \to E'$ a monotone and hemi-continuous operator, $J : E \to]-\infty, +\infty]$ a proper and convex functional. Let $u \in E$ and $f \in E'$. the following assertions are equivalent:*

$$
\begin{aligned}
1.\langle Au; v-u\rangle_{E'\times E} + J(v) - J(u) &\geq \langle f; v-u\rangle_{E'\times E}, \quad \forall v \in E \\
2.\langle Av; v-u\rangle_{E'\times E} + J(v) - J(u) &\geq \langle f; v-u\rangle_{E'\times E}. \quad \forall v \in E
\end{aligned}
$$

The main results of this section are stated by the following proposition.

Proposition 3.1. *Let $(u_1^\varepsilon, u_2^\varepsilon) \in W_{\text{div}}^\varepsilon$ and $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$ be the solution of variational problem (PV_ε). Then, there exists $(\widehat{u_1}, \widehat{u_2}) \in W_z(\Omega_1)^3 \times W_z(\Omega_2)^3$ and $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ such that*

$$(\widehat{u_1^\varepsilon}, \widehat{u_2^\varepsilon}) \to (\widehat{u_1}, \widehat{u_2}) \text{ in } W_z(\Omega_1)^3 \times W_z(\Omega_2)^3 \quad \text{weakly,} \tag{3.3}$$

$$\left(\frac{\partial \widehat{u_{13}^\varepsilon}}{\partial z}, \frac{\partial \widehat{u_{23}^\varepsilon}}{\partial z}\right) \to (0,0) \text{ in } L^p(\Omega_1) \times L^p(\Omega_2) \text{ weakly,} \tag{3.4}$$

$$(\widehat{p_1^\varepsilon}, \widehat{p_2^\varepsilon}) \to (\widehat{p_1}, \widehat{p_2}) \text{ in } L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2) \quad \text{weakly.} \tag{3.5}$$

Proof. choosing $(v_1, v_2) = (0,0)$ as test function in inequality (2.20), we deduce that

$$
\begin{aligned}
&\mu_1 \varepsilon^p \left\| D(u_1^\varepsilon) \right\|_{L^p(\Omega_1^\varepsilon)^9}^p + \mu_2 \varepsilon^p \left\| D(u_2^\varepsilon) \right\|_{L^p(\Omega_2^\varepsilon)^9}^p \\
\leq \; &\int_{\Omega_1^\varepsilon} f_1^\varepsilon . u_1^\varepsilon dx_1 dx_2 dx_3 + \int_{\Omega_2^\varepsilon} f_2^\varepsilon . u_2^\varepsilon dx_1 dx_2 dx_3,
\end{aligned}
$$

this permits us to obtain, making use of Poincaré's and Korn's inequalities and by passage to variables x , y and z

$$\left\| \widehat{u_1^\varepsilon} \right\|_{L^p(\Omega_1)^3} + \left\| \widehat{u_2^\varepsilon} \right\|_{L^p(\Omega_2)^3} \leq c, \tag{3.6}$$

$$\left\| \frac{\partial \widehat{u_1^\varepsilon}}{\partial z} \right\|_{L^p(\Omega_1)^3} + \left\| \frac{\partial \widehat{u_2^\varepsilon}}{\partial z} \right\|_{L^p(\Omega_2)^3} \leq c, \tag{3.7}$$

$$\left\| \frac{\partial \widehat{u_1^\varepsilon}}{\partial x} \right\|_{L^p(\Omega_1)^3} + \left\| \frac{\partial \widehat{u_2^\varepsilon}}{\partial x} \right\|_{L^p(\Omega_2)^3} \leq \frac{c}{\varepsilon}, \tag{3.8}$$

$$\left\| \frac{\partial \widehat{u_1^\varepsilon}}{\partial y} \right\|_{L^p(\Omega_1)^3} + \left\| \frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \right\|_{L^p(\Omega_2)^3} \leq \frac{c}{\varepsilon}. \tag{3.9}$$

Moreover, we get using the incompressibility condition (2.5), (2.6) and Green's formula, for any function $(\varphi_1^\varepsilon, \varphi_2^\varepsilon) \in W_{\Gamma_1}^{1,p_1}(\Omega_1^\varepsilon) \times W_{\Gamma_2}^{1,p_2}(\Omega_2^\varepsilon)$

$$
\begin{aligned}
&\int_{\Omega_1} \frac{\partial \widehat{u_{13}^\varepsilon}}{\partial z} \widehat{\varphi_1^\varepsilon} dxdydz + \int_{\Omega_2} \frac{\partial \widehat{u_{23}^\varepsilon}}{\partial z} \widehat{\varphi_2^\varepsilon} dxdydz \\
= \varepsilon &\int_{\Omega_1} \left(\widehat{u_{11}^\varepsilon} \frac{\partial \widehat{\varphi_1^\varepsilon}}{\partial x} + \widehat{u_{12}^\varepsilon} \frac{\partial \widehat{\varphi_1^\varepsilon}}{\partial y} \right) dxdydz + \varepsilon \int_{\Omega_2} \left(\widehat{u_{21}^\varepsilon} \frac{\partial \widehat{\varphi_2^\varepsilon}}{\partial x} + \widehat{u_{22}^\varepsilon} \frac{\partial \widehat{\varphi_2^\varepsilon}}{\partial y} \right) dxdydz.
\end{aligned}
$$

Which gives, making use (2.16)

$$\left\|\frac{\partial \widehat{u_{13}^{\varepsilon}}}{\partial z}\right\|_{W^{-1,p'}(\Omega_1)}+\left\|\frac{\partial \widehat{u_{23}^{\varepsilon}}}{\partial z}\right\|_{W^{-1,p'}(\Omega_2)}\leq c\varepsilon. \tag{3.10}$$

We can then extract a subsequences still denoted by $(\widehat{u_1^{\varepsilon}},\widehat{u_2^{\varepsilon}})$ such that

$$(\widehat{u_1^{\varepsilon}},\widehat{u_2^{\varepsilon}})\rightarrow(\widehat{u_1},\widehat{u_2}) \text{ in } L^p(\Omega_1)^3\times L^p(\Omega_2)^3 \text{ weakly,} \tag{3.11}$$

$$\left(\frac{\partial \widehat{u_1^{\varepsilon}}}{\partial z},\frac{\partial \widehat{u_2^{\varepsilon}}}{\partial z}\right)\rightarrow\left(\frac{\partial \widehat{u_1}}{\partial z},\frac{\partial \widehat{u_1}}{\partial z}\right) \text{ in } L^p(\Omega_1)^3\times L^p(\Omega_2)^3 \text{ weakly,} \tag{3.12}$$

$$\left(\frac{\partial \widehat{u_{13}^{\varepsilon}}}{\partial z},\frac{\partial \widehat{u_{23}^{\varepsilon}}}{\partial z}\right)\rightarrow(0,0) \text{ in } L^p(\Omega_1)\times L^p(\Omega_2) \text{ weakly.} \tag{3.13}$$

Let now $(v_1^{\varepsilon},v_2^{\varepsilon})\in W^{1,p}_{\Gamma_1}(\Omega_1^{\varepsilon})^3\times W^{1,p}_{\Gamma_2}(\Omega_2^{\varepsilon})^3$, we obtain by setting $(u_1^{\varepsilon}-v_1^{\varepsilon},u_2^{\varepsilon}-v_2^{\varepsilon})$ as test function in inequality (2.20), using the incompressibility conditions (2.5) and (2.6) as well as the Green formula and Holder's inequality

$$\begin{aligned}
&\int_{\Omega_1^{\varepsilon}}\nabla p_1^{\varepsilon}.v_1^{\varepsilon}dx_1dx_2dx_3+\int_{\Omega_2^{\varepsilon}}\nabla p_2^{\varepsilon}.v_2^{\varepsilon}dx_1dx_2dx_3\\
&\leq \mu_1\varepsilon^p\left(\int_{\Omega_1^{\varepsilon}}|D(u_1^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p'}}\left(\int_{\Omega_1^{\varepsilon}}|D(v_1^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p}}\\
&+g_1\varepsilon^{\frac{1}{p'}+1}Meas(\Omega_1^{\varepsilon})^{\frac{1}{p'}}\left(\int_{\Omega_1^{\varepsilon}}|D(v_1^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p}}\\
&+\varepsilon\left\|\widehat{f_1^{\varepsilon}}\right\|_{L^{p'}(\Omega_1^{\varepsilon})^3}\left\|\widehat{v_1^{\varepsilon}}\right\|_{W^{1,\,p}_{\Gamma_1}(\Omega_1)^3}+\varepsilon\left\|\widehat{f_2^{\varepsilon}}\right\|_{L^{p'}(\Omega_2^{\varepsilon})^3}\left\|\widehat{v_2^{\varepsilon}}\right\|_{W^{1,p}_{\Gamma_2}(\Omega_2)^3}\\
&+\mu_2\varepsilon^p\left(\int_{\Omega_2^{\varepsilon}}|D(u_2^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p'}}\left(\int_{\Omega_2^{\varepsilon}}|D(v_2^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p}}\\
&+g_2\varepsilon^{\frac{1}{p'}+1}Meas(\Omega_2^{\varepsilon})^{\frac{1}{p'}}\left(\int_{\Omega_2^{\varepsilon}}|D(v_2^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.14}$$

On the other hand, it is easy to check that, after some algebraic manipulations, we find

$$\left(\int_{\Omega_i^{\varepsilon}}|D(v_i^{\varepsilon})|^p\,dx_1dx_2dx_3\right)^{\frac{1}{p}}\leq\varepsilon^{\frac{1}{p}-1}\left\|\widehat{v_i^{\varepsilon}}\right\|_{W^{1,\,p}_{\Gamma_i}(\Omega_i)^3},\ i=1,2. \tag{3.15}$$

Hence, from (3.7), (3.8), (3.9), (3.14) and (3.15) if follows that

$$\begin{aligned}
&\int_{\Omega_1^{\varepsilon}}\nabla p_1^{\varepsilon}.v_1^{\varepsilon}dx_1dx_2dx_3+\int_{\Omega_2^{\varepsilon}}\nabla p_2^{\varepsilon}.v_2^{\varepsilon}dx_1dx_2dx_3\\
\leq\ & c\varepsilon\left(\left\|\widehat{v_1^{\varepsilon}}\right\|_{W^{1,\,p}_{\Gamma_1}(\Omega_1)^3}+\left\|\widehat{v_2^{\varepsilon}}\right\|_{W^{1,\,p}_{\Gamma_2}(\Omega_2)^3}\right).
\end{aligned} \tag{3.16}$$

Passing to the variables x, y and z in the left hand side of (3.16) we find the following estimates

$$\left\|\widehat{p_1^\varepsilon}\right\|_{L_0^{p'}(\Omega_1)} + \left\|\widehat{p_2^\varepsilon}\right\|_{L_0^{p'}(\Omega_2)} \leq c, \tag{3.17}$$

$$\left\|\frac{\partial\widehat{p_1^\varepsilon}}{\partial x}\right\|_{W^{-1,\,p'}(\Omega_1)} + \left\|\frac{\partial\widehat{p_2^\varepsilon}}{\partial x}\right\|_{W^{-1,\,p'}(\Omega_2)} \leq c, \tag{3.18}$$

$$\left\|\frac{\partial\widehat{p_1^\varepsilon}}{\partial y}\right\|_{W^{-1,\,p'}(\Omega_1)} + \left\|\frac{\partial\widehat{p_2^\varepsilon}}{\partial y}\right\|_{W^{-1,\,p'}(\Omega_2)} \leq c, \tag{3.19}$$

$$\left\|\frac{\partial\widehat{p_1^\varepsilon}}{\partial z}\right\|_{W^{-1,\,p'}(\Omega_1)} + \left\|\frac{\partial\widehat{p_2^\varepsilon}}{\partial z}\right\|_{W^{-1,\,p'}(\Omega_2)} \leq \varepsilon c. \tag{3.20}$$

Consequently, we can extract a subsequence still denoted by $\left(\widehat{p_1^\varepsilon}, \widehat{p_2^\varepsilon}\right)$ such that

$$\left(\widehat{p_1^\varepsilon}, \widehat{p_2^\varepsilon}\right) \to (\widehat{p_1}, \widehat{p_2}) \text{ in } L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2) \text{ weakly,} \tag{3.21}$$

which achieves the proof. This proof permits also to deduce that limit pressure verify $\left(\widehat{p_1^\varepsilon}(x, y, z), \widehat{p_2^\varepsilon}(x, y, z)\right) = (\widehat{p_1}(x, y), \widehat{p_2}(x, y))$. □

Proposition 3.2. *The velocity limit given by (3.3) verifies*

$$\int_0^{h(x,y)} \left(\frac{\partial\widehat{u_{11}}(x,y,z)}{\partial x} + \frac{\partial\widehat{u_{12}}(x,y,z)}{\partial y}\right) dz + \int_{-h(x,y)}^{0} \left(\frac{\partial\widehat{u_{21}}(x,y,z)}{\partial x} + \frac{\partial\widehat{u_{22}}(x,y,z)}{\partial y}\right) dz$$
$$= 0 \ \forall (x, y) \in \omega \tag{3.22}$$

Proof. We know from incompressibility conditions (2.5) and (2.6) that

$$\int_{\Omega_1^\varepsilon} \operatorname{div}(u_1^\varepsilon(x_1, x_2, x_3))\varphi_1(x_1, x_2)dx_1dx_2dx_3 + \int_{\Omega_2^\varepsilon} \operatorname{div}(u_2^\varepsilon(x_1, x_2, x_3))\varphi_2(x_1, x_2)dx_1dx_2dx_3$$
$$= 0 \quad \text{for all} \quad (\varphi_1, \varphi_2) \in D(\omega)^2.$$

This implies, using Green's formula

$$\begin{aligned}
&\int_{\Omega_1^\varepsilon} u_{11}^\varepsilon \frac{d\varphi_1}{dx_1}(x_1, x_2)dx_1dx_2dx_3 + \int_{\Omega_1^\varepsilon} u_{12}^\varepsilon \frac{d\varphi_1}{dx_2}(x_1, x_2)dx_1dx_2dx_3 \\
&+ \int_{\Omega_2^\varepsilon} u_{21}^\varepsilon \frac{d\varphi_2}{dx_1}(x_1, x_2)dx_1dx_2dx_3 + \int_{\Omega_2^\varepsilon} u_{22}^\varepsilon \frac{d\varphi_2}{dx_2}(x_1, x_2)dx_1dx_2dx_3 \\
= &\int_{\Omega_1^\varepsilon} \frac{\partial u_{13}^\varepsilon}{\partial x_3}\varphi_1(x_1, x_2)dx_1dx_2dx_3 + \int_{\Omega_2^\varepsilon} \frac{\partial u_{33}^\varepsilon}{\partial x_3}\varphi_2(x_1, x_2)dx_1dx_2dx_3.
\end{aligned}$$

Hence, by passage to the variables x, y and z using Geen's formula, we can infer

$$\int_\omega \varphi(x, y) \left(\int_0^{h(x,y)} \left(\frac{\partial\widehat{u_{11}^\varepsilon}}{\partial x} + \frac{\partial\widehat{u_{12}^\varepsilon}}{\partial y}\right) dz + \int_{-h(x,y)}^{0} \left(\frac{\partial\widehat{u_{21}^\varepsilon}}{\partial x} + \frac{\partial\widehat{u_{22}^\varepsilon}}{\partial y}\right) dz\right) dxdy$$
$$= 0 \quad \varphi \in D(\omega).$$

Then,

$$\int_0^{h(x,y)} \left(\frac{\partial \widehat{u_{11}^{\varepsilon}}}{\partial x} + \frac{\partial \widehat{u_{12}^{\varepsilon}}}{\partial y} \right) dz + \int_{-h(x,y)}^{0} \left(\frac{\partial \widehat{u_{21}^{\varepsilon}}}{\partial x} + \frac{\partial \widehat{u_{22}^{\varepsilon}}}{\partial y} \right) dz = 0.$$

Moreover, the fact that $\underline{\widehat{u}}_1 = (\widehat{u_{11}^{\varepsilon}}, \widehat{u_{12}^{\varepsilon}}) \in L^p(\Omega_1)^2$, $\underline{\widehat{u}}_2 = (\widehat{u_{21}^{\varepsilon}}, \widehat{u_{22}^{\varepsilon}}) \in L^p(\Omega_2)^2$ and $\int_0^{h(x,y)} \underline{\widehat{u}}_1(x,y,z)dz + \int_{-h(x,y)}^{0} \underline{\widehat{u}}_2(x,y,z)dz$ is continuous and linear, it is weakly continuous.

Thus, by passage to the limit when ε tends to zero, taking into account the boundaries conditions (2.7) ,(2.8) and (2.9) it follows that

$$\int_0^{h(x,y)} \left(\frac{\partial \widehat{u_{11}}}{\partial x} + \frac{\partial \widehat{u_{12}}}{\partial y} \right) dz + \int_{-h(x,y)}^{0} \left(\frac{\partial \widehat{u_{21}}}{\partial x} + \frac{\partial \widehat{u_{22}}}{\partial y} \right) dz = 0, \ \forall (x,y) \in \omega.$$

□

We derive in the proposition below the strong equation verified by the limit solution

$$(\underline{\widehat{u}}_1, \underline{\widehat{u}}_2) = ((\widehat{u_{11}}, \widehat{u_{12}}), (\widehat{u_{21}}, \widehat{u_{22}})) \in W_z,$$

and $(\widehat{p}_1, \widehat{p}_2) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$.

Proposition 3.3. *if $\left(\frac{\partial \underline{\widehat{u}}_1}{\partial z}, \frac{\partial \underline{\widehat{u}}_2}{\partial z}\right) \neq (0,0)$ then the limit point $(\underline{\widehat{u}}_1, \underline{\widehat{u}}_2)$ and $(\widehat{p}_1, \widehat{p}_2)$ given by (3.3) and (3.5) verify the limit problem*

$$-\frac{\partial}{\partial z}\left(\frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \underline{\widehat{u}}_1}{\partial z} \right|^{p-2} \frac{\partial \underline{\widehat{u}}_1}{\partial z} + \frac{\sqrt{2}}{2} g_1 \frac{\frac{\partial \underline{\widehat{u}}_1}{\partial z}}{\left| \frac{\partial \underline{\widehat{u}}_1}{\partial z} \right|} + \frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \underline{\widehat{u}}_2}{\partial z} \right|^{p-2} \frac{\partial \underline{\widehat{u}}_2}{\partial z} + \frac{\sqrt{2}}{2} g_2 \frac{\frac{\partial \underline{\widehat{u}}_2}{\partial z}}{\left| \frac{\partial \underline{\widehat{u}}_2}{\partial z} \right|} \right)$$
$$= \underline{\widehat{f}}_1 - \nabla p_1(x,y) + \underline{\widehat{f}}_2 - \nabla p_2(x,y) \ in\ W^{-1,\,p'}(\Omega)^2. \tag{3.23}$$

Proof. Introducing the operator Φ defined as follows

$$\Phi : W^{\,\varepsilon} \to W^{\,\varepsilon\prime},$$

$$\langle \Phi(u_1^{\varepsilon}, u_2^{\varepsilon}), (v_1^{\varepsilon}, v_2^{\varepsilon}) \rangle_{W^{\,\varepsilon\prime} \times W^{\,\varepsilon}} = \mu_1 \varepsilon^p \int_{\Omega_1^{\varepsilon}} |D(u_1^{\varepsilon})|^{p-2} D(u_1^{\varepsilon}).D(v_1^{\varepsilon}) dx_1 dx_2 dx_3$$
$$+\mu_2 \varepsilon^p \int_{\Omega_2^{\varepsilon}} |D(u_2^{\varepsilon})|^{p-2} D(u_2^{\varepsilon}).D(v_2^{\varepsilon}) dx_1 dx_2 dx_3.$$

It is easy to verify that Φ is monotone and hemi-continuous (see for more details the reference [2,5,15]). Moreover, we know that the functional

$$(v_1^{\varepsilon}, v_2^{\varepsilon}) \in W^{\,\varepsilon} \to g_1 \varepsilon \int_{\Omega_1^{\varepsilon}} |D(v_1^{\varepsilon})|\, dx_1 dx_2 dx_3 + g_2 \varepsilon \int_{\Omega_2^{\varepsilon}} |D(v_2^{\varepsilon})|\, dx_1 dx_2 dx_3,$$

is proper and convex. Then, the use of Minty's lemma permits us to affirm that (2.20) is equivalent to the following inequality

$$\mu_1\varepsilon^p\int_{\Omega_1^\varepsilon}\ |D(v_1^\varepsilon)|^{p-2}\,D(v_1^\varepsilon).D(v_1^\varepsilon-u_1^\varepsilon)dx_1dx_2dx_3+g_1\varepsilon\int_{\Omega_1^\varepsilon}|D(v_1^\varepsilon)|\,dx_1dx_2dx_3$$
$$-g_1\varepsilon\int_{\Omega_1^\varepsilon}|D(u_1^\varepsilon)|\,dx_1dx_2dx_3+\ \mu_2\varepsilon^p\int_{\Omega_2^\varepsilon}\ |D(v_2^\varepsilon)|^{p-2}\,D(v_2^\varepsilon).D(v_2^\varepsilon-u_2^\varepsilon)dx_1dx_2dx_3$$
$$+g_2\varepsilon\int_{\Omega_2^\varepsilon}|D(v_2^\varepsilon)|\,dx_1dx_2dx_3-g_2\varepsilon\int_{\Omega_1^\varepsilon}|D(u_2^\varepsilon)|\,dx_1dx_2dx_3$$
$$\geq\int_{\Omega_1^\varepsilon}f_1^\varepsilon.(v_1^\varepsilon-u_1^\varepsilon)dx_1dx_2dx_3+\int_{\Omega_1^\varepsilon}p_1^\varepsilon\,\mathrm{div}(v_1-u_1^\varepsilon)dx_1dx_2dx_3$$
$$+\int_{\Omega_2^\varepsilon}f_2^\varepsilon.(v_2^\varepsilon-u_2^\varepsilon)dx_1dx_2dx_3+\int_{\Omega_2^\varepsilon}p_2^\varepsilon\,\mathrm{div}(v_2-u_2^\varepsilon)dx_1dx_2dx_3,\ \forall(v_1^\varepsilon,v_2^\varepsilon)\in W^\varepsilon.$$

Our goal now is to pass to the limit when ε tends to zero.To this aim, we use Proposition (3.3) and the weak lower semi-continuity of the convex and continuous functional

$$(v_1^\varepsilon,v_2^\varepsilon)\in W^{\ \varepsilon}\to g_1\varepsilon\int_{\Omega_1^\varepsilon}\ |D(v_1^\varepsilon)|\,dx_1dx_2dx_3+g_2\varepsilon\int_{\Omega_2^\varepsilon}\ |D(v_2^\varepsilon)|\,dx_1dx_2dx_3,$$

We find the following limit inequality

$$\mu_1\int_{\Omega_1}\left[\begin{array}{c}\frac{1}{2}\left|\frac{\partial\widehat{v_{11}}}{\partial z}\right|^2+\frac{1}{2}\left|\frac{\partial\widehat{v_{12}}}{\partial z}\right|^2\\+\left|\frac{\partial\widehat{v_{13}}}{\partial z}\right|^2\end{array}\right]^{\frac{p-2}{2}}\times\left[\begin{array}{c}\frac{1}{2}\frac{\partial\widehat{v_{11}}}{\partial z}\frac{\partial(\widehat{v_{11}}-\widehat{u_{11}})}{\partial z}+\frac{1}{2}\frac{\partial\widehat{v_{12}}}{\partial z}\frac{\partial(\widehat{v_{12}}-\widehat{u_{12}})}{\partial z}\\+\frac{\partial\widehat{v_{13}}}{\partial z}\frac{\partial(\widehat{v_{13}}-\widehat{u_{13}})}{\partial z}\end{array}\right]dxdydz$$
$$++g_1\int_{\Omega_1}\left[\begin{array}{c}\frac{1}{2}\left|\frac{\partial\widehat{v_{11}}}{\partial z}\right|^2+\frac{1}{2}\left|\frac{\partial\widehat{v_{12}}}{\partial z}\right|^2\\+\left|\frac{\partial\widehat{v_{13}}}{\partial z}\right|^2\end{array}\right]^{\frac{1}{2}}dxdydz-g_1\int_{\Omega_1}\left[\begin{array}{c}\frac{1}{2}\left|\frac{\partial\widehat{u_{11}}}{\partial z}\right|^2+\frac{1}{2}\left|\frac{\partial\widehat{u_{12}}}{\partial z}\right|^2\\+\left|\frac{\partial\widehat{u_{13}}}{\partial z}\right|^2\end{array}\right]^{\frac{1}{2}}dxdydz$$
$$+\mu_2\int_{\Omega_2}\left[\begin{array}{c}\frac{1}{2}\left|\frac{\partial\widehat{v_{21}}}{\partial z}\right|^2+\frac{1}{2}\left|\frac{\partial\widehat{v_{22}}}{\partial z}\right|^2\\+\left|\frac{\partial\widehat{v_{23}}}{\partial z}\right|^2\end{array}\right]^{\frac{p-2}{2}}\times\left[\begin{array}{c}\frac{1}{2}\frac{\partial\widehat{v_{21}}}{\partial z}\frac{\partial(\widehat{v_{21}}-\widehat{u_{21}})}{\partial z}+\frac{1}{2}\frac{\partial\widehat{v_{22}}}{\partial z}\frac{\partial(\widehat{v_{22}}-\widehat{u_{22}})}{\partial z}\\+\frac{\partial\widehat{v_{23}}}{\partial z}\frac{\partial(\widehat{v_{23}}-\widehat{u_{23}})}{\partial z}\end{array}\right]dxdydz$$
$$+g_2\int_{\Omega_2}\left[\begin{array}{c}\frac{1}{2}\left|\frac{\partial\widehat{v_{21}}}{\partial z}\right|^2+\frac{1}{2}\left|\frac{\partial\widehat{v_{22}}}{\partial z}\right|^2\\+\left|\frac{\partial\widehat{v_{23}}}{\partial z}\right|^2\end{array}\right]^{\frac{1}{2}}dxdydz-g_2\int_{\Omega_2}\left[\begin{array}{c}\frac{1}{2}\left|\frac{\partial\widehat{u_{21}}}{\partial z}\right|^2+\frac{1}{2}\left|\frac{\partial\widehat{u_{22}}}{\partial z}\right|^2\\+\left|\frac{\partial\widehat{u_{23}}}{\partial z}\right|^2\end{array}\right]^{\frac{1}{2}}dxdydz$$
$$\geq\int_{\Omega_1}\widehat{f}_1.(\widehat{v}_1-\widehat{u}_1)dxdydz\ +\int_{\Omega_1}\widehat{p}_1\,\mathrm{div}(\widehat{v}_1-\widehat{u}_1)dxdydz$$
$$+\int_{\Omega_2}\widehat{f}_2.(\widehat{v}_2-\widehat{u}_2)dxdydz++\int_{\Omega_2}\widehat{p}_2\,\mathrm{div}(\widehat{v}_1-\widehat{u}_1)dxdydz.\qquad(3.24)$$

Furthermore, from (3.3) and (3.4) we find

$$\left(\frac{\partial\widehat{u_{13}}}{\partial z},\frac{\partial\widehat{u_{23}}}{\partial z}\right)=(0,0)\ \text{ in }\Omega_1\times\Omega_2.$$

It follows, keeping in mind (3.22), that $\widehat{u}_1(x,y,z)=(\widehat{u_{11}}(x,y,z),\widehat{u_{12}}(x,y,z),0)$, and $\widehat{u}_2(x,y,z)=(\widehat{u_{21}}(x,y,z),\widehat{u_{22}}(x,y,z),0)$. This permits also to choose $(\widehat{v_{13}},\widehat{v_{23}})=(0,0)$ in (3.24).

Considering now the operator Φ such that

$$\Phi : W_z \ \to W_z' \ ,$$

$$\begin{aligned}\langle \Phi\left(\underline{\widehat{u}}_1, \underline{\widehat{u}}_2\right), \left(\underline{\widehat{v}}_1, \underline{\widehat{v}}_2\right)\rangle_{W_z' \times W_z} &= \frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}.\frac{\partial \underline{\widehat{v}}_1}{\partial z}dxdydz \\ &+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}.\frac{\partial \underline{\widehat{v}}_2}{\partial z}dxdydz.\end{aligned}$$

It is clear that the operator Φ is monotone and hemi-continuous and the functional

$$\left(\underline{\widehat{v}}_1, \underline{\widehat{v}}_2\right) \in W_z \ \to \frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{v}}_1}{\partial z}\right| dxdydz + \frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{v}}_2}{\partial z}\right| dxdydz,$$

is proper and convex. Hence, we deduce using again Minty's lemma

$$\begin{aligned}&\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}.\frac{\partial (\underline{\widehat{v}}_1-\underline{\widehat{u}}_1)}{\partial z}dxdydz + \frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{v}}_1}{\partial z}\right| dxdydz \\ &-\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right| dxdydz + \frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}.\frac{\partial (\underline{\widehat{v}}_2-\underline{\widehat{u}}_2)}{\partial z}dxdydz \\ &+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{v}}_2}{\partial z}\right| dxdydz - \frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right| dxdydz \\ &\geq \int_{\Omega_1}\left(\underline{\widehat{f}}_1 - \nabla\widehat{p}_1\right).(\underline{\widehat{v}}_1-\underline{\widehat{u}}_1)dxdydz + \int_{\Omega_2}\left(\underline{\widehat{f}}_2 - \nabla\widehat{p}_2\right).(\underline{\widehat{v}}_2-\underline{\widehat{u}}_2)dxdydz \qquad \forall\left(\underline{\widehat{v}}_1, \underline{\widehat{v}}_2\right) \in W_z \ .\end{aligned} \tag{3.25}$$

Setting $\left(\underline{\widehat{v}}_1, \underline{\widehat{v}}_2\right) = \left(2\underline{\widehat{u}}_1, 2\underline{\widehat{u}}_2\right)$ and $\left(\underline{\widehat{v}}_1, \underline{\widehat{v}}_2\right) = (0, 0)$ in (3.25), to obtain the following inequalities

$$\begin{aligned}&\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p} dxdydz + \frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right| dxdydz \\ &+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p} dxdydz + \frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right| dxdydz \\ \geq\ &\int_{\Omega_1}\left(\underline{\widehat{f}}_1 - \nabla\widehat{p}_1\right).\underline{\widehat{u}}_1 dxdydz + \int_{\Omega_2}\left(\underline{\widehat{f}}_2 - \nabla\widehat{p}_2\right).\underline{\widehat{u}}_2 dxdydz,\end{aligned}$$

and

$$\begin{aligned}&-\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p} dxdydz - \frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right| dxdydz \\ &-\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p} dxdydz - \frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right| dxdydz \\ \geq\ &-\int_{\Omega_1}\left(\underline{\widehat{f}}_1 - \nabla\widehat{p}_1\right).\underline{\widehat{u}}_1 dxdydz - \int_{\Omega_2}\left(\underline{\widehat{f}}_2 - \nabla\widehat{p}_2\right).\underline{\widehat{u}}_2 dxdydz.\end{aligned}$$

Consequently, we get by combining these two inequalities

$$\begin{aligned} &\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^p dxdydz+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right| dxdydz \\ &+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^p dxdydz+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right| dxdydz \\ = &\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_1\right).\underline{\widehat{u}}_1 dxdydz+\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).\underline{\widehat{u}}_2 dxdydz. \end{aligned} \tag{3.26}$$

Due to the fact that $W^{1,\,p}_{\Gamma_i}(\Omega_i)$ is dense in $W_z(\Omega_i)$, see [1, 6], we can take $\underline{w}_1=\underline{\widehat{v}}_1-\underline{\widehat{u}}_1$ and $\underline{w}_2=\underline{\widehat{v}}_2-\underline{\widehat{u}}_2$ in (3.25) we obtain

$$\begin{aligned} &\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}.\frac{\partial \underline{w}_1}{\partial z} dxdydz+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{w}_1}{\partial z}\right| dxdydz \\ &+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}.\frac{\partial \underline{w}_2}{\partial z} dxdydz+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{w}_2}{\partial z}\right| dxdydz \\ \geq &\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_1\right).\underline{w}_1 dxdydz+\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).\underline{w}_2 dxdydz \\ \forall(\underline{w}_1,\underline{w}_2) \in\ & W^{1,\,p}_{\Gamma_1}(\Omega_1)^2\times W^{1,\,p}_{\Gamma_2}(\Omega_2)^2. \end{aligned} \tag{3.27}$$

Changing $(\underline{w}_1,\underline{w}_2)$ to $(-\underline{w}_1,-\underline{w}_2)$ in (3.27), we obtain

$$|F(\underline{w}_1,\underline{w}_2)|\leq\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial \underline{w}_1}{\partial z}\right| dxdydz+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial \underline{w}_2}{\partial z}\right| dxdydz\ ,$$

where

$$\begin{aligned} F(\underline{w}_1,\underline{w}_2) = &\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}.\frac{\partial \underline{w}_1}{\partial z} dxdydz+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}.\frac{\partial \underline{w}_2}{\partial z} dxdydz \\ &-\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_1\right).\underline{w}_1 dxdydz-\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).\underline{w}_2 dxdydz. \end{aligned} \tag{3.28}$$

Now, utilising the Hahn-Banach theorem, $\exists\,(\underline{m}_1,\underline{m}_2) \in L^\infty(\Omega_1)^2\times L^\infty(\Omega_2)^2$, with $|||\underline{m}_1|||_\infty, |||\underline{m}_2|||_\infty\leq 1$, such that

$$F((\underline{w}_1,\underline{w}_2))=-\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\underline{m}_1.\frac{\partial \underline{w}_1}{\partial z} dxdydz-\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\underline{m}_2.\frac{\partial \underline{w}_2}{\partial z} dxdydz. \tag{3.29}$$

In particular, from (3.26) and (3.28), we get

$$\int_{\Omega_1}\underline{m}_1.\frac{\partial \underline{u}_1}{\partial z} dxdydz+\int_{\Omega_2}\underline{m}_2.\frac{\partial \underline{u}_2}{\partial z} dxdydz=\int_{\Omega_1}\left|\frac{\partial \underline{u}_1}{\partial z}\right| dxdydz+\int_{\Omega_2}\left|\frac{\partial \underline{u}_2}{\partial z}\right| dxdydz. \tag{3.30}$$

Rewriting (3.29) as

$$\begin{aligned}
&\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}.\frac{\partial \underline{w}_1}{\partial z}dxdydz+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}.\frac{\partial \underline{w}_2}{\partial z}dxdydz\\
&+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\underline{m}_1.\frac{\partial \underline{w}_1}{\partial z}dxdydz+\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\underline{m}_2.\frac{\partial \underline{w}_2}{\partial z}dxdydz\\
&-\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_2\right).\underline{w}_1dxdydz-\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).\underline{w}_2dxdydz\quad=0.
\end{aligned}\tag{3.31}$$

Next using (3.30), we have

$$\int_{\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|\neq 0}\left(\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|-\underline{m}_1.\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right)dxdydz+\int_{\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|\neq 0}\left(\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|-\underline{m}_2.\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right)dxdydz=0.$$

As $|||\underline{m}_1|||_\infty, |||\underline{m}_2|||_\infty\le 1$, we deduce

$$\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|=\underline{m}_1.\frac{\partial \underline{\widehat{u}}_1}{\partial z}\text{ and }\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|=\underline{m}_2.\frac{\partial \underline{\widehat{u}}_2}{\partial z}.$$

Hence, if $\left(\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|,\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|\right)\neq(0,0)$, we get

$$\begin{aligned}
&\int_{\Omega_1}\left(\frac{\mu_1}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}+\frac{\sqrt{2}}{2}g_1\frac{\frac{\partial \underline{\widehat{u}}_1}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|}\right).\frac{\partial \underline{w}_1}{\partial z}dxdydz\\
&\int_{\Omega_2}\left(\frac{\mu_2}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}+\frac{\sqrt{2}}{2}g_2\frac{\frac{\partial \underline{\widehat{u}}_2}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|}\right).\frac{\partial \underline{w}_2}{\partial z}dxdydz\\
&=\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_1\right).\underline{w}_1dxdydz+\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).\underline{w}_2dxdydz,\\
&\forall(\underline{w}_1,\underline{w}_2)\in W^{1,\,p}_{\Gamma_1}(\Omega_1)^2\times W^{1,\,p}_{\Gamma_2}(\Omega_2)^2.
\end{aligned}$$

Consequenty, we get by using a simple integration by parts

$$\begin{aligned}
&-\int_{\Omega_1}\frac{\partial}{\partial z}\left(\frac{\mu_1}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z}dxdydz+\frac{\sqrt{2}}{2}g_1\frac{\frac{\partial \underline{\widehat{u}}_1}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|}\right).\underline{w}_1dxdydz\\
&-\int_{\Omega_2}\frac{\partial}{\partial z}\left(\frac{\mu_2}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z}dxdydz+\frac{\sqrt{2}}{2}g_2\frac{\frac{\partial \underline{\widehat{u}}_2}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|}\right).\underline{w}_2dxdydz\\
&=\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_1\right).\underline{w}_1dxdydz+\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).\underline{w}_2dxdydz,\\
&\forall(\underline{w}_1,\underline{w}_2)\in W^{1,\,p}_{\Gamma_1}(\Omega_1)^2\times W^{1,\,p}_{\Gamma_2}(\Omega_2)^2.
\end{aligned}$$

Let us consider

$$\underline{w}\in W^{1,p}_0(\Omega)^2:\underline{w}=\begin{cases}\underline{w}_1\text{ in }\ \Omega_1\\ \underline{w}_2\ \text{ in }\ \Omega_2\end{cases},$$

and

$$\widetilde{a_1} = \begin{cases} -\frac{\partial}{\partial z}\left(\frac{\mu_1}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z} + \frac{\sqrt{2}}{2} g_1 \frac{\frac{\partial \underline{\widehat{u}}_1}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|}\right) & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases},$$

$$\widetilde{a_2} = \begin{cases} 0 & \text{in } \Omega_1 \\ -\frac{\partial}{\partial z}\left(\frac{\mu_2}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z} + \frac{\sqrt{2}}{2} g_2 \frac{\frac{\partial \underline{\widehat{u}}_2}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|}\right) & \text{in } \Omega_2 \end{cases},$$

$$\widetilde{b_1} = \begin{cases} \underline{\widehat{f}}_1 - \nabla \widehat{p_1} & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases},$$

$$\widetilde{b_2} = \begin{cases} 0 & \text{in } \Omega_1 \\ \underline{\widehat{f}}_2 - \nabla \widehat{p_2} & \text{in } \Omega_2 \end{cases}.$$

Then,

$$\begin{aligned}
\int_{\Omega} (\widetilde{a_1} + \widetilde{a_2}).\underline{w}\,dxdydz &= \int_{\Omega_1} (\widetilde{a_1} + \widetilde{a_2}).\underline{w}_1\,dxdydz + \int_{\Omega_2} (\widetilde{a_1} + \widetilde{a_2}).\underline{w}_2\,dxdydz, \\
&= \int_{\Omega_1} \widetilde{a_1}.\underline{w}_1\,dxdydz + \int_{\Omega_2} \widetilde{a_2}.\underline{w}_2\,dxdydz, \\
&= \int_{\Omega_1} -\frac{\partial}{\partial z}\left(\frac{\mu_1}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_1}{\partial z} + \frac{\sqrt{2}}{2} g_1 \frac{\frac{\partial \underline{\widehat{u}}_1}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_1}{\partial z}\right|}\right).\underline{w}_1\,dxdydz \\
&+ \int_{\Omega_2} -\frac{\partial}{\partial z}\left(\frac{\mu_2}{2^{\frac{p}{2}}}\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial \underline{\widehat{u}}_2}{\partial z} + \frac{\sqrt{2}}{2} g_2 \frac{\frac{\partial \underline{\widehat{u}}_2}{\partial z}}{\left|\frac{\partial \underline{\widehat{u}}_2}{\partial z}\right|}\right).\underline{w}_2\,dxdydz, \\
&= \int_{\Omega_1} \left(\underline{\widehat{f}}_1 - \nabla \widehat{p_1}\right).\underline{w}_1\,dxdydz + \int_{\Omega_2} \left(\underline{\widehat{f}}_2 - \nabla \widehat{p_2}\right).\underline{w}_2\,dxdydz, \\
&= \int_{\Omega_1} \widetilde{b_1}.\underline{w}_1\,dxdy\,dz + \int_{\Omega_2} \widetilde{b_2}.\underline{w}_2\,dxdydz, \\
&= \int_{\Omega} (\widetilde{b_1} + \widetilde{b_2}).\underline{w}\,dxdydz \qquad \forall \underline{w} \in W_0^{1,p}(\Omega)^2.
\end{aligned}$$

Which eventually gives (3.23).

From now on we will denote by $(\underline{\widehat{u}}_1, \underline{\widehat{u}}_2) \in W_z$ and $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ the solution of the limit problem (3.23). □

The following proposition shows the uniqueness of the limit solution $(\underline{\widehat{u}}_1, \widehat{p_1})$ and $(\underline{\widehat{u}}_2, \widehat{p_2})$.

Proposition 3.4. *The limit strong problem (3.23) has a unique, solution* $(\underline{\widehat{u}}_1, \underline{\widehat{u}}_2) \in W_z$ *and* $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ *with the condition (3.22) .*

Proof. suppose that the limit problem (3.23) has at least two solutions $(\underline{\widehat{u}}_1, \underline{\widehat{u}}_2) \in W_z$, $(\widehat{p_1}, \widehat{p_2}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ and $\left(\overline{\underline{\widehat{u}}_1}, \overline{\underline{\widehat{u}}_2}\right) \in W_z$, $\left(\overline{\widehat{p_1}}, \overline{\widehat{p_2}}\right) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$. In particular, $(\underline{\widehat{u}}_1, \widehat{p_1})$, $(\underline{\widehat{u}}_2, \widehat{p_2})$

and $\left(\overline{\underline{\widehat{u}}_1},\overline{\widehat{p}_1}\right)$, $\left(\overline{\underline{\widehat{u}}_2},\overline{\widehat{p}_2}\right)$ are solutions of the weak formulation (3.25). Then

$$\begin{gathered}\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial\underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial\underline{\widehat{u}}_1}{\partial z}.\frac{\partial(\underline{\widehat{v}}_1-\underline{\widehat{u}}_1)}{\partial z}dxdydz+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\underline{\widehat{v}}_1}{\partial z}\right|dxdydz\\ -\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\underline{\widehat{u}}_1}{\partial z}\right|dxdydz+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial\underline{\widehat{u}}_2}{\partial z}\right|^{p-2}\frac{\partial\underline{\widehat{u}}_2}{\partial z}.\frac{\partial(\underline{\widehat{v}}_2-\underline{\widehat{u}}_2)}{\partial z}dxdydz\\ +\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\underline{\widehat{v}}_2}{\partial z}\right|dxdydz-\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\underline{\widehat{u}}_2}{\partial z}\right|dxdydz\\ \geq\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\widehat{p}_1\right).(\underline{\widehat{v}}_1-\underline{\widehat{u}}_1)dxdydz+\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\widehat{p}_2\right).(\underline{\widehat{v}}_2-\underline{\widehat{u}}_2)dxdydz\qquad(\underline{\widehat{v}}_1,\underline{\widehat{v}}_2)\in W_z\ ,\end{gathered}\tag{3.32}$$

and

$$\begin{gathered}\frac{\mu_1}{2^{\frac{p}{2}}}\int_{\Omega_1}\left|\frac{\partial\overline{\underline{\widehat{u}}_1}}{\partial z}\right|^{p-2}\frac{\partial\overline{\underline{\widehat{u}}_1}}{\partial z}.\frac{\partial(\underline{\widehat{v}}_1-\overline{\underline{\widehat{u}}_1})}{\partial z}dxdydz+\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\underline{\widehat{v}}_1}{\partial z}\right|dxdydz\\ -\frac{\sqrt{2}}{2}g_1\int_{\Omega_1}\left|\frac{\partial\overline{\underline{\widehat{u}}_1}}{\partial z}\right|dxdydz+\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left|\frac{\partial\overline{\underline{\widehat{u}}_2}}{\partial z}\right|^{p-2}\frac{\partial\overline{\underline{\widehat{u}}_2}}{\partial z}.\frac{\partial(\underline{\widehat{v}}_2-\overline{\underline{\widehat{u}}_2})}{\partial z}dxdydz\\ +\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\underline{\widehat{v}}_2}{\partial z}\right|dxdydz-\frac{\sqrt{2}}{2}g_2\int_{\Omega_2}\left|\frac{\partial\overline{\underline{\widehat{u}}_2}}{\partial z}\right|dxdydz\\ \geq\int_{\Omega_1}\left(\underline{\widehat{f}}_1-\nabla\overline{\widehat{p}_1}\right).(\underline{\widehat{v}}_1-\overline{\underline{\widehat{u}}_1})dxdydz+\int_{\Omega_2}\left(\underline{\widehat{f}}_2-\nabla\overline{\widehat{p}_2}\right).(\underline{\widehat{v}}_2-\overline{\underline{\widehat{u}}_2})dxdydz\qquad(\underline{\widehat{v}}_1,\underline{\widehat{v}}_2)\in W_z.\end{gathered}\tag{3.33}$$

Setting $(\underline{\widehat{v}}_1,\underline{\widehat{v}}_2)=\left(\overline{\underline{\widehat{u}}_1},\overline{\underline{\widehat{u}}_2}\right)$ and $(\underline{\widehat{v}}_1,\underline{\widehat{v}}_2)=(\underline{\widehat{u}}_1,\underline{\widehat{u}}_2)$ as test functions in (3.32) and (3.33), respectively. Subtracting the tow obtained inequalities, we can infer

$$\begin{gathered}\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega_1}\left(\left|\frac{\partial\overline{\underline{\widehat{u}}_1}}{\partial z}\right|^{p-2}\frac{\partial\overline{\underline{\widehat{u}}_1}}{\partial z}-\left|\frac{\partial\underline{\widehat{u}}_1}{\partial z}\right|^{p-2}\frac{\partial\underline{\widehat{u}}_1}{\partial z}\right).\frac{\partial(\overline{\underline{\widehat{u}}_1}-\underline{\widehat{u}}_1)}{\partial z}dxdydz\\ +\frac{\mu_2}{2^{\frac{p}{2}}}\int_{\Omega_2}\left(\left|\frac{\partial\overline{\underline{\widehat{u}}_2}}{\partial y}\right|^{p-2}\frac{\partial\overline{\underline{\widehat{u}}_2}}{\partial y}-\left|\frac{\partial\underline{\widehat{u}}_2}{\partial y}\right|^{p-2}\frac{\partial\underline{\widehat{u}}_2}{\partial y}\right).\frac{\partial(\overline{\underline{\widehat{u}}_2}-\underline{\widehat{u}}_2)}{\partial y}dxdydz\\ \leq\ \int_{\Omega_1}\nabla\left(\widehat{p}_1-\overline{\widehat{p}_1}\right).\left(\overline{\underline{\widehat{u}}_1}-\underline{\widehat{u}}_1\right)dxdydz+\int_{\Omega_2}\nabla\left(\widehat{p}_2-\overline{\widehat{p}_2}\right).\left(\overline{\underline{\widehat{u}}_2}-\underline{\widehat{u}}_2\right)dxdydz.\end{gathered}\tag{3.34}$$

Observe that for every $x,y\in\mathbb{R}^n$,

$$\left(|x|^{p-2}x-|y|^{p-2}y\right).(x-y)\geq(p-1)\frac{|x-y|^2}{(|x|+|y|)^{2-p}},\ 1<p\leq 2.$$

This leads, making use (3.34), to

$$\frac{\mu_1(p-1)}{2^{\frac{p}{2}}}\int_{\Omega_1}\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_1}}{\partial z}\right|+\left|\frac{\partial\widehat{\underline{u}}_1}{\partial z}\right|\right)^{p-2}\left|\frac{\partial(\overline{\widehat{\underline{u}}_1}-\widehat{\underline{u}}_1)}{\partial z}\right|^2 dxdydz$$
$$+\frac{\mu_2(p-1)}{2^{\frac{p}{2}}}\int_{\Omega_2}\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_2}}{\partial y}\right|+\left|\frac{\partial\widehat{\underline{u}}_2}{\partial y}\right|\right)^{p-2}\left|\frac{\partial(\overline{\widehat{\underline{u}}_2}-\widehat{\underline{u}}_2)}{\partial z}\right|^2 dxdydz$$
$$\leq \int_\omega\left(\left(\widehat{p}_1-\overline{\widehat{p}_1}\right)\int_0^{h(x,y)}\left(\frac{\partial(\overline{\widehat{u_{11}}}-\widehat{u_{11}})}{\partial x}+\frac{\partial(\overline{\widehat{u_{12}}}-\widehat{u_{12}})}{\partial y}\right)dz\right)dxdy$$
$$+\int_\omega\left(\left(\widehat{p}_2-\overline{\widehat{p}_2}\right)\int_{-h(x,y)}^0\left(\frac{\partial(\overline{\widehat{u_{21}}}-\widehat{u_{21}})}{\partial x}+\frac{\partial(\overline{\widehat{u_{22}}}-\widehat{u_{22}})}{\partial y}\right)dz\right)dxdy.$$

This use of (3.22) gives

$$\frac{\mu_1(p-1)}{2^{\frac{p}{2}}}\int_{\Omega_1}\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_1}}{\partial z}\right|+\left|\frac{\partial\widehat{\underline{u}}_1}{\partial z}\right|\right)^{p-2}\left|\frac{\partial(\overline{\widehat{\underline{u}}_1}-\widehat{\underline{u}}_1)}{\partial z}\right|^2 dxdydz$$
$$+\frac{\mu_2(p-1)}{2^{\frac{p}{2}}}\int_{\Omega_2}\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_2}}{\partial y}\right|+\left|\frac{\partial\widehat{\underline{u}}_2}{\partial y}\right|\right)^{p-2}\left|\frac{\partial(\overline{\widehat{\underline{u}}_2}-\widehat{\underline{u}}_2)}{\partial z}\right|^2 dxdydz=0. \qquad (3.35)$$

On the other hand, the application of Hölder's inequality leads to

$$\int_{\Omega_1}\left|\frac{\partial(\overline{\widehat{\underline{u}}_1}-\widehat{\underline{u}}_1)}{\partial z}\right|^p dxdydz+\int_{\Omega_2}\left|\frac{\partial(\overline{\widehat{\underline{u}}_2}-\widehat{\underline{u}}_2)}{\partial z}\right|^p dxdydz$$
$$\leq c\left(\int_{\Omega_1}\frac{\left|\frac{\partial(\overline{\widehat{\underline{u}}_1}-\widehat{\underline{u}}_1)}{\partial z}\right|^2}{\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_1}}{\partial z}\right|+\left|\frac{\partial\widehat{\underline{u}}_1}{\partial z}\right|\right)^{2-p}}dxdydz\right)^{\frac{p}{2}}\times\left(\int_{\Omega_1}\left(\left|\frac{\partial\widehat{\underline{\underline{u}}}_1}{\partial z}\right|+\left|\frac{\partial\overline{\widehat{\underline{u}}_1}}{\partial z}\right|\right)^p dxdydz\right)^{\frac{2-p}{2}}$$
$$+c\left(\int_{\Omega_2}\frac{\left|\frac{\partial(\overline{\widehat{\underline{u}}_2}-\widehat{\underline{u}}_2)}{\partial z}\right|^2}{\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_2}}{\partial z}\right|+\left|\frac{\partial\widehat{\underline{u}}_2}{\partial z}\right|\right)^{2-p}}dxdydz\right)^{\frac{p}{2}}\times\left(\int_{\Omega_2}\left(\left|\frac{\partial\overline{\widehat{\underline{u}}_2}}{\partial z}\right|+\left|\frac{\partial\widehat{\underline{u}}_2}{\partial z}\right|\right)^p dxdydz\right)^{\frac{2-p}{2}}.$$

Which gives, keeping in mind (3.35)

$$\left\|\frac{\partial(\overline{\widehat{\underline{u}}_1}-\widehat{\underline{u}}_1)}{\partial z}\right\|_{L^p(\Omega_1)^2}=0 \text{ and } \left\|\frac{\partial(\overline{\widehat{\underline{u}}_2}-\widehat{\underline{u}}_2)}{\partial z}\right\|_{L^p(\Omega_2)^2}=0,$$

using Poincare's inequality, we deduce

$$\left\|\overline{\widehat{\underline{u}}_1}-\widehat{\underline{u}}_1\right\|_{L^p(\Omega_1)^2}=0 \text{ and } \left\|\overline{\widehat{\underline{u}}_2}-\widehat{\underline{u}}_2\right\|_{L^p(\Omega_2)^2}=0,$$

we deduce that $(\widehat{\underline{u}}_1,\ \widehat{\underline{u}}_2)=\left(\overline{\widehat{\underline{u}}_1},\ \overline{\widehat{\underline{u}}_2}\right)$ a.e. in $\Omega_1\times\Omega_2$..

Finally, to prove the uniqueness of the pressure, we use equation (3.23), with the two pressures $(\widehat{p}_1,\ \overline{\widehat{p}_1})$ and $(\widehat{p}_2,\ \overline{\widehat{p}_2})$. We find

$$\nabla(\widehat{p}_1-\overline{\widehat{p}_1})=0 \text{ and } \nabla(\widehat{p}_2-\overline{\widehat{p}_2})=0.$$

Then, due to fact that $\left(\widehat{p_1},\ \overline{\widehat{p_1}}\right) \in \left(L_0^{p'}(\Omega_1)\right)^2$ and $\left(\widehat{p_2},\ \overline{\widehat{p_2}}\right) \in \left(L_0^{p'}(\Omega_2)\right)^2$, the result can be easily deduced. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] F. Boughanim, M. Boukrouche, H. Smaoui, Asymptotic Behavior of a Non-Newtonian Flow with Stick-Slip Condition, 2004-Fez Conference on Differential Equations and Mechanics, Electron. J. Differ. Equ. Conference 11 (2004), 71-80.

[2] A. Bourgeat, A. Mikelic, R. Tapiéro, Dérivation des Equations Moyennées Décrivant un Ecoulement Non Newtonien Dans un Domaine de Faible Epaisseur, C. R. Acad. Sci. Paris, Ser. I 316 (1993), 965-970.

[3] H. Brezis, Equations et Inéquations Non Linéaires dans les Espaces en Dualité, Ann. l'Inst. Fourier, 18 (1996), 115-175.

[4] R. Bunoiu, S. Kesavan, Asymptotic Behaviour of a Bingham Fluid in Thin Layers, J. Math. Anal. Appl. 293 (2004), 405-418. https://doi.org/10.1016/j.jmaa.2003.10.049.

[5] R. Bunoin, S. Kesavan, Fluide de Bingham dans une Couche Mince, Ann. Univ. Craiova, Math. Comp. Sci. Ser. 30 (2003), 71-77.

[6] R. Bunoin, J.S.J. Paulin, Nonlinear Viscous Flow Through a Thin Slab in the Lubrification Case, Rev. Roum. Math. Pures Appl. 45 (2000), 577-591.

[7] G. Duvaut, J.L. Lions, Les Inéquations en Mécanique et en Physique, Dunod, 1976.

[8] I. Ekeland, R. Temam, Analyse Convexe et Problèmes Variationnels, Dunod, Paris, 1974.

[9] J.L. Lions, Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires, Dunod, 1996.

[10] K.F. Liu, C.C. Mei, Approximate Equations for the Slow Spreading of a Thin Sheet of Bingham Plastic Fluid, Phys. Fluids A: Fluid Dyn. 2 (1990), 30-36. https://doi.org/10.1063/1.857821.

[11] J. Málek, Mathematical Properties of Flows of Incompressible Power-Law-Like fluids that are Described by Implicit Constitutive Relations, Electron. Trans. Numer. Anal. 31 (2008), 110-125.

[12] J. Málek, M. Růžička, V.V. Shelukhin, Herschel-Bulkley fluids, Existence and Regularity of Steady Flows, Math. Models Methods Appl. Sci. 15 (2005), 1845-1861. https://doi.org/10.1142/s0218202505000996.

[13] F. Messelmi, B. Merouani, Flow of Herschel-Bulkley Fluid Through a Two Dimensional Thin Layer, Stud. Univ. Babeş-Bolyai Math. 58 (2013), 119-130.

[14] F. Messelmi, Effects of the Yield Limit on the Behaviour of Herschel-Bulkley Fluid, Nonlinear Sci. Lett. A, 2 (2011), 137-142.

[15] F. Messelmi, B. Merouani, F. Bouzeghaya, Steady-State Thermal Herschel-Bulkley Flow with Tresca's Friction Law, Electron. J. Differ. Equ. 2010 (2010), 46.

[16] A. Mikelic, R. Tapiéro, Mathematical Derivation of the Power Law Describing Polymer Flow Through a Thin Slab, Math. Model. Numer. Anal. 29 (1995), 3-22.