Int. J. Anal. Appl. (2023), 21:93

International Journal of Analysis and Applications

Certain New Subclassess of Analytic and Bi-univalent Functions

C. R. Krishna[1], A. C. Chandrashekar[2], N. Ravikumar[3,*]

[1]*PG Department of Mathematics, Bharathi College PG and RC, Maddur Tq., Mandya-571 422, India*

[2]*Department of Mathematics, Maharani's Science College for Women Mysore-57005, India*

[3]*PG Department of Mathematics, JSS College of Arts, Science and Commerce,Mysore- 570 006, India*

[*]*Corresponding author:* ravisn.kumar@gmail.com

Abstract. The paper presents two novel subclasses of the function class Σ, which consists of bi-univalent functions defined in the open unit disk $\mathcal{D} = \{\zeta : |\zeta| < 1\}$. The authors investigate the properties of these new subclasses and provide estimates for the absolute values of the second, third, and fourth Taylor-Maclaurin coefficients r_2, r_3, and r_4 for functions in these subclasses.

1. Introduction

Let $\mathcal{A}$ be the class of functions with the following form:

$$\chi(\zeta) = \zeta + \sum_{k=2}^{\infty} r_k \zeta^k \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{D} = \{\zeta : |\zeta| < 1\}$.
Let $J_\chi(\gamma, b, c)$ where $\chi \in \mathcal{A}$ with $\dfrac{\chi(\zeta)\chi'(\zeta)}{\zeta} \neq 0$ denote the class of convex function in $\mathcal{D}$ defined as fallows [8]

$$J_\chi(\gamma, b, c) = (1-\gamma)\left[1 - \frac{1}{c} + \frac{\zeta}{c}\frac{\chi'(\zeta)}{\chi(\zeta)}\right] + \gamma\left[1 + \frac{\zeta}{b}\frac{\chi''(\zeta)}{\chi'(\zeta)}\right]$$

where γ, $b \neq 0$ and $c \neq 0$ are complex numbers. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{D}$. It is well known that every function $\chi \in \mathcal{S}$ has an inverse χ^{-1}, defined

Received: Jun. 24, 2023.

2020 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. analytic and univalent functions; bi-univalent functions; starlike and convex functions; coefficients bounds.

https://doi.org/10.28924/2291-8639-21-2023-93
ISSN: 2291-8639

© 2023 the author(s).

by $\chi^{-1}(\chi(\zeta))=\zeta,\,(\zeta\in\mathcal{D})$ and $\chi^{-1}(\chi(\eta))=\eta,\,(|\eta|<r_0(\chi);r_0(\chi)\geq\frac{1}{4})$ where

$$\chi^{-1}(\eta)=\eta-r_2\eta^2+(2r_2^2-r_3)\eta^3-(5r_2^3-5r_2r_3+r_4)\eta^4+\ldots \tag{1.2}$$

The class Σ consists of functions in $\mathcal{A}$ that are bi-univalent in $\mathcal{D}$, meaning both the function and its inverse are univalent in $\mathcal{D}$. This class was introduced in a 1967 by Lewin [4].

Lewin's investigation of the bi-univalent function class Σ led to the result that the absolute value of the second coefficient, denoted as r_2, satisfies the inequality $|r_2|<1.51$.

In 1969, Netanyahu [6] showed that the maximum value of $|r_2|$ among all functions in Σ is $\frac{4}{3}$.

In summary, Lewin showed that $|r_2|<1.51$ for functions in the class Σ, Netanyahu demonstrated that the maximum value of $|r_2|$ in Σ is $\frac{4}{3}$, and Brannan and Taha introduced subclasses of Σ akin to the starlike and convex function subclasses.

Thus, following Brannan and Taha [1](see also [10]), a function $\chi\in\mathcal{A}$ is in the class $S^{\star}_{\Sigma}(\gamma)$ of strongly bi-starlike of order $\gamma\,(0<\gamma\leq 1)$, if $\chi\in\Sigma$,

$$\left|arg\left(\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right)\right|<\frac{\gamma\pi}{2},\qquad(\zeta\in\mathcal{D};0<\gamma\leq 1)$$

and

$$\left|arg\left(\frac{\eta\psi'(\eta)}{\psi(\eta)}\right)\right|<\frac{\gamma\pi}{2},\qquad(\eta\in\mathcal{D};0<\gamma\leq 1),$$

where the function ψ is given by $\psi(\eta)=\eta-r_2\eta^2+(2r_2^2-r_3)\eta^3-(5r_2^3-5r_2r_3+r_4)\eta^4+\ldots$ is the extension of χ^{-1} to $\mathcal{D}$. Similarly, a function $\chi\in\mathcal{A}$ is in the class $\mathcal{K}_{\Sigma}(\gamma)$ of strongly bi-convex functions of order $\gamma\,(0<\gamma\leq 1)$ if $\chi\in\Sigma$,

$$\left|arg\left(1+\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right)\right|<\frac{\gamma\pi}{2},\qquad(\zeta\in\mathcal{D};0<\gamma\leq 1)$$

and

$$\left|arg\left(1+\frac{\eta\psi''(\eta)}{\psi'(\eta)}\right)\right|<\frac{\gamma\pi}{2},\qquad(\eta\in\mathcal{D};0<\gamma\leq 1)$$

where the function ψ is extension of χ^{-1} to $\mathcal{D}$.

The motivation for the current paper is derived from the groundbreaking work conducted by Brannan and Taha [1], which has reignited interest in the exploration of analytic and bi-univalent functions in the field of mathematics in recent years. The success of Brannan and Taha work has served as inspiration and has generated an extensive number of subsequent research papers by other authors ([2], [3], [5], [9], [11], [12]).

The objective of the present paper is to introduce two novel subclasses within the function class Σ and provide estimates for the coefficients $|r_2|$, $|r_3|$ and $|r_4|$ for functions belonging to these newly defined subclasses. To achieve our main results, we need to revisit and state the following lemma.

Lemma 1.1. *[7] Let $\mathcal{H}$ be the family of all functions $\mathfrak{h}$ that are analytic in the open unit disk $\mathcal{D}$ and satisfy $\mathfrak{h}(0) = 1$ and $\Re(\mathfrak{h}(\zeta)) > 0$ for all $\zeta \in \mathcal{D}$. If a function $\mathfrak{h} \in \mathcal{H}$ is given by $\mathfrak{h}(\zeta) = 1 + d_1\zeta + d_2\zeta^2 + \cdots$ for $\zeta \in \mathbb{D}$, then $|d_k| \leq 2$ for all $k \in \mathbb{N}$.*

2. Coefficient bounds for the function class $M_\Sigma(\gamma, b, c, \mu)$

Definition 2.1. *A function $\chi(\zeta)$ given by (1.1) is said to be in the class $M_\Sigma(\gamma, b, c, \mu)$ if the following conditions are satisfied:*
$\chi \in \Sigma$ and

$$\left|arg\left(\frac{1-\gamma}{c}\left[c-1+\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right]+\frac{\gamma}{b}\left[b+\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right]\right)\right| < \frac{\mu\pi}{2}\ \left(0<\mu\leq 1, \gamma\geq 0, b\neq 0, c\neq 0, \zeta\in\mathcal{D}\right) \tag{2.1}$$

and

$$\left|arg\left(\frac{1-\gamma}{c}\left[c-1+\frac{\eta\psi'(\eta)}{\psi(\eta)}\right]+\frac{\gamma}{b}\left[b+\frac{\eta\psi''(\eta)}{\psi'(\eta)}\right]\right)\right| < \frac{\mu\pi}{2}\ \left(0<\mu\leq 1, \gamma\geq 0, b\neq 0, c\neq 0, \eta\in\mathcal{D}\right) \tag{2.2}$$

where the function ψ is given by

$$\psi(\eta) = \eta - r_2\eta^2 + (2r_2^2 - r_3)\eta^3 - (5r_2^3 - 5r_2r_3 + r_4)\eta^4 + \ldots \tag{2.3}$$

By selecting specific parameter values, researchers have identified several significant subclasses that have been extensively studied in previous papers. Here are some examples of these subclasses.

(1) For $b = c = 1$, the class $M_\Sigma(\gamma, b, c, \mu)$ reduces to the class $M_\Sigma(\gamma, \lambda)$ studied by Xiao-Fei Li and An-Ping Wang [5].
(2) For $\gamma = 0$, $b = c = 1$, the class $M_\Sigma(\gamma, b, c, \mu)$ reduces to the class $S^\star_\Sigma(\mu)$ studied by Brannan and Taha [1](see also [10]).

Theorem 2.1. *Let $\chi(\zeta)$ given by (1.1) be in the class $M_\Sigma(\gamma, b, c, \mu)$, $0 < \mu \leq 1$,and $\gamma \geq 0, b \neq 0, c \neq 0$. Then*

$$|r_2| \leq \frac{2\mu}{\sqrt{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\left[2\mu-(\mu-1)\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\right]}}, \tag{2.4}$$

,

$$|r_3| \leq \frac{4\mu^2}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2}+\frac{\mu}{\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)} \tag{2.5}$$

and

$$|r_4| \leq \frac{4\mu^3+2\mu}{9\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)}+\frac{16}{3}\frac{\mu^3\left(\frac{1-\gamma}{c}+\frac{5\gamma}{b}\right)}{\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^3}+\frac{5\mu^2}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}. \tag{2.6}$$

Proof: We can express the inequalities stated in equations (2.1) and (2.2) in a simpler form as follows.

$$\frac{1-\gamma}{c}\left[c-1+\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right]+\frac{\gamma}{b}\left[b+\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right]=[m(\zeta)]^{\mu} \tag{2.7}$$

$$\frac{1-\gamma}{c}\left[c-1+\frac{\eta\psi'(\eta)}{\psi(\eta)}\right]+\frac{\gamma}{b}\left[b+\frac{\eta\psi''(\eta)}{\psi'(\eta)}\right]=[n(\eta)]^{\mu} \tag{2.8}$$

respectively, where $m(\zeta)$ and $n(\eta)$ satisfy the following inequalities:
$\Re(m(\zeta))>0$ $(\zeta\in\mathcal{D})$ and $\Re(n(\eta))>0$ $(\eta\in\mathcal{D})$.
Furthermore, the functions $m(\zeta)$ and $n(\eta)$ have the forms

$$m(\zeta)=1+m_1\zeta+m_2\zeta^2+m_3\zeta^3+\ldots \tag{2.9}$$

$$n(\eta)=1+n_1\eta+n_2\eta^2+n_3\eta^3+\ldots \tag{2.10}$$

By comparing the coefficients in equations (2.7) and (2.8), we obtain the following result.

$$r_2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)=m_1\mu \tag{2.11}$$

$$r_3\left(\frac{2(1-\gamma)}{c}+\frac{6\gamma}{b}\right)=\mu m_2+\frac{\mu(\mu-1)m_1^2}{2}+\frac{\mu^2 m_1^2\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2} \tag{2.12}$$

$$\begin{aligned} r_4\left(\frac{3(1-\gamma)}{c}+\frac{12\gamma}{b}\right)&=\mu(\mu-1)m_1m_2+\mu m_3+\frac{\mu(\mu-1)(\mu-2)m_1^3}{6}\\ &\quad+r_2r_3\left(\frac{3(1-\gamma)}{c}+\frac{18\gamma}{b}\right)-r_2^3\left(\frac{1-\gamma}{c}+\frac{8\gamma}{b}\right) \end{aligned} \tag{2.13}$$

and

$$-r_2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)=n_1\mu \tag{2.14}$$

$$-r_3\left(\frac{2(1-\gamma)}{c}+\frac{6\gamma}{b}\right)=\mu n_2+\frac{\mu(\mu-1)n_1^2}{2}-\frac{\mu^2 m_1^2\left(\frac{3(1-\gamma)}{c}+\frac{8\gamma}{b}\right)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2} \tag{2.15}$$

$$\begin{aligned} -r_4\left(\frac{3(1-\gamma)}{c}+\frac{12\gamma}{b}\right)&=\mu(\mu-1)n_1n_2+\mu n_3+\frac{\mu(\mu-1)(\mu-2)n_1^3}{6}\\ &\quad-r_2r_3\left(\frac{12(1-\gamma)}{c}+\frac{42\gamma}{b}\right)+r_2^3\left(\frac{10(1-\gamma)}{c}+\frac{32\gamma}{b}\right). \end{aligned} \tag{2.16}$$

From (2.11) and (2.14), we find that

$$r_2 = \frac{m_1\mu}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)} = \frac{-n_1\mu}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)} \tag{2.17}$$

which implies

$$m_1 = -n_1 \tag{2.18}$$

and

$$2r_2^2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2 = \mu^2(m_1^2+n_1^2). \tag{2.19}$$

Adding (2.12) and (2.15), we obtain

$$r_2^2\left(\frac{2(1-\gamma)}{c}+\frac{4\gamma}{b}\right) = \mu(m_2+n_2)+\frac{\mu(\mu-1)}{2}(m_1^2+n_1^2). \tag{2.20}$$

Now from (2.19) in (2.20), we obtain

$$r_2^2 = \frac{\mu^2(m_2+n_2)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\left[2\mu-(\mu-1)\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\right]}. \tag{2.21}$$

By utilizing Lemma 1.1 for the coefficients m_2 and n_2, we can directly derive the following

$$|r_2| \le \frac{2\mu}{\sqrt{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\left[2\mu-(\mu-1)\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\right]}}.$$

This gives the bound on $|r_2|$ as asserted in (2.4).

To determine the bound for r_3, we can subtract equation (2.15) from equation (2.12), we get

$$2r_3\left(\frac{2(1-\gamma)}{c}+\frac{6\gamma}{b}\right) - r_2^2\left(\frac{4(1-\gamma)}{c}+\frac{12\gamma}{b}\right) = \mu(m_2-n_2)+\frac{\mu(\mu-1)}{2}(m_1^2-n_1^2). \tag{2.22}$$

By substituting the value of r_2^2 from equation (2.19) and noticing that $m_1^2 = n_1^2$, we get the following

$$r_3 = r_2^2 + \frac{\mu(m_2-n_2)}{4\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)} \tag{2.23}$$

$$r_3 = \frac{\mu^2(m_1^2+n_1^2)}{2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2} + \frac{\mu(m_2-n_2)}{4\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}. \tag{2.24}$$

By utilizing Lemma 1.1 for the coefficients m_1, m_2, n_1 and n_2, we readily get

$$|r_3| \le \frac{4\mu^2}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2} + \frac{\mu}{\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}.$$

To determine the bound on $|r_4|$, subtracting (2.16) from (2.13) with $m_1 = -n_1$ gives

$$r_4\left(\frac{6(1-\gamma)}{c}+\frac{24\gamma}{b}\right) = \mu(\mu-1)(m_1m_2+m_1n_2)+\mu(m_3-n_3)+\frac{\mu(\mu-1)(\mu-2)}{6}2\mu^3 + r_2r_3\left(\frac{15(1-\gamma)}{c}+\frac{60\gamma}{b}\right)-r_2^3\left(\frac{11(1-\gamma)}{c}+\frac{40\gamma}{b}\right) \tag{2.25}$$

substitute the values of r_2 and r_3 from (2.17) and (2.24) in (2.25).

By utilizing Lemma 1.1 for the coefficients m_1, m_2 and n_2, we get

$$|r_4| \le \frac{4\mu^3+2\mu}{9\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)}+\frac{16}{3}\frac{\mu^3\left(\frac{1-\gamma}{c}+\frac{5\gamma}{b}\right)}{\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^3}+\frac{5\mu^2}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}$$

By setting $b = c = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let $\chi(\zeta)$ given by (1.1) be in the class $M_\Sigma(\mu,\gamma)$ $(0<\mu\le 1, \gamma\ge 0)$. Then*

$$|r_2| \le \frac{2\mu}{\sqrt{(1+\gamma)(\mu+1+\gamma-\gamma\mu)}} \tag{2.26}$$

and

$$|r_3| \le \frac{4\mu^2}{(1+\gamma)^2}+\frac{\mu}{(1+2\gamma)}. \tag{2.27}$$

Putting $\gamma = 0$, in Corollary 2.1, we get the following corollary.

Corollary 2.2. *Let $\chi(\zeta)$ given by (1.1) and in the class $S^\star_\Sigma(\mu)$ $\left(0<\mu\le 1\right)$. Then*

$$|r_2| \le \frac{2\mu}{\sqrt{\mu+1}} \tag{2.28}$$

and

$$|r_3| \le 4\mu^2+\mu. \tag{2.29}$$

3. Coefficient bounds for the function class $B_\Sigma(\gamma, b, c, \rho)$

A function $\chi(\zeta)$ given by (1.1) is said to be in the class $B_\Sigma(\gamma, b, c, \rho)$ if the following conditions are satisfied: $\chi\in\Sigma$ and

$$\Re\left(\frac{1-\gamma}{c}\left[c-1+\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right]+\frac{\gamma}{b}\left[b+\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right]\right) > \rho\ \left(0\le\rho<1, \gamma\ge 0, b\ne 0, c\ne 0, \zeta\in\mathcal{D}\right) \tag{3.1}$$

and

$$\Re\left(\frac{1-\gamma}{c}\left[c-1+\frac{\eta\psi'(\eta)}{\psi(\eta)}\right]+\frac{\gamma}{b}\left[b+\frac{\eta\psi''(\eta)}{\psi'(\eta)}\right]\right) > \rho\ \left(0\le\rho<1, \gamma\ge 0, b\ne 0, c\ne 0, \eta\in\mathcal{D}\right) \tag{3.2}$$

where the function ψ is given by (2.3).

By selecting specific parameter values, researchers have identified several significant subclasses that have been extensively studied in previous papers. Here are some examples of these subclasses.

(1) For $b = c = 1$, the class $B_\Sigma(\gamma, b, c, \rho)$ reduces to the class $B_\Sigma(\mu, \rho)$ studied by Xiao-Fei Li and An-Ping Wang [5].

(2) For $\gamma = 0$, $b = c = 1$, the class $B_\Sigma(\gamma, b, c, \rho)$ reduces to the class $S^\star_\Sigma(\rho)$ studied by Brannan and Taha [1](see also [10]).

Next, we find the estimates on the cofficients $|r_2|, |r_3|$ and $|r_4|$ for function in the class $B_\Sigma(\gamma, b, c, \rho)$.

Theorem 3.1. *Let $\chi(\zeta)$ given by (1.1) be in the class $B_\Sigma(\gamma, b, c, \rho)$, $0 \le \rho < 1$ and $\gamma \ge 0$. Then*

$$|r_2| \le \sqrt{\frac{2(1-\rho)}{\left(\frac{1-\gamma}{c} + \frac{2\gamma}{b}\right)}} \tag{3.3}$$

$$|r_3| \le \frac{4(1-\rho)^2}{\left(\frac{1-\gamma}{c} + \frac{2\gamma}{b}\right)} + \frac{(1-\rho)}{\left(\frac{1-\gamma}{c} + \frac{3\gamma}{b}\right)} \tag{3.4}$$

and

$$|r_4| \le \frac{2}{3}\frac{(1-\rho)}{\left(\frac{1-\gamma}{c} + \frac{4\gamma}{b}\right)} + \frac{16}{3}\frac{(1-\rho)^3\left(\frac{1-\gamma}{c} + \frac{5\gamma}{b}\right)}{\left(\frac{1-\gamma}{c} + \frac{4\gamma}{b}\right)\left(\frac{1-\gamma}{c} + \frac{2\gamma}{b}\right)^3} + \frac{5(1-\rho)^2}{\left(\frac{1-\gamma}{c} + \frac{2\gamma}{b}\right)\left(\frac{1-\gamma}{c} + \frac{3\gamma}{b}\right)}. \tag{3.5}$$

Proof: It fallows from (3.1) and (3.2) that there exist $m(\zeta)$ and $n(\eta) \in \mathcal{H}$ such that

$$\frac{1-\gamma}{c}\left[c - 1 + \frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right] + \frac{\gamma}{b}\left[b + \frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right] = \rho + (1-\rho)p(\zeta) \tag{3.6}$$

and

$$\frac{1-\gamma}{c}\left[c - 1 + \frac{\eta\psi'(\eta)}{\psi(\eta)}\right] + \frac{\gamma}{b}\left[b + \frac{\eta\psi''(\eta)}{\psi'(\eta)}\right] = \rho + (1-\rho)q(\eta) \tag{3.7}$$

where $m(\zeta)$ and $n(\eta)$ have the forms (2.9) and (2.10), respectively. Equating cofficients in (3.6) and (3.7), we get

$$r_2\left(\frac{1-\gamma}{c} + \frac{2\gamma}{b}\right) = m_1(1-\rho) \tag{3.8}$$

$$r_3\left(\frac{2(1-\gamma)}{c} + \frac{6\gamma}{b}\right) = m_2(1-\rho) + \frac{(1-\rho)^2 m_1^2\left(\frac{1-\gamma}{c} + \frac{4\gamma}{b}\right)}{\left(\frac{1-\gamma}{c} + \frac{2\gamma}{b}\right)^2} \tag{3.9}$$

$$r_4\left(\frac{3(1-\gamma)}{c}+\frac{12\gamma}{b}\right) = m_3(1-\rho)+r_2r_3\left(\frac{3(1-\gamma)}{c}+\frac{18\gamma}{b}\right)-r_2^3\left(\frac{1-\gamma}{c}+\frac{8\gamma}{b}\right) \tag{3.10}$$

and

$$-r_2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)=n_1(1-\rho) \tag{3.11}$$

$$-r_3\left(\frac{2(1-\gamma)}{c}+\frac{6\gamma}{b}\right)=n_2(1-\rho)-\frac{(1-\rho)^2m_1^2\left(\frac{3(1-\gamma)}{c}+\frac{8\gamma}{b}\right)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2} \tag{3.12}$$

$$-r_4\left(\frac{3(1-\gamma)}{c}+\frac{12\gamma}{b}\right)=n_3(1-\rho)-r_2r_3\left(\frac{12(1-\gamma)}{c}+\frac{42\gamma}{b}\right)+r_2^3\left(\frac{10(1-\gamma)}{c}+\frac{32\gamma}{b}\right). \tag{3.13}$$

From (3.8) and (3.11), we find that

$$r_2=\frac{m_1(1-\rho)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)}=\frac{-n_1(1-\rho)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)} \tag{3.14}$$

which implies

$$m_1=-n_1 \tag{3.15}$$

and

$$2r_2^2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2=(1-\rho)^2(m_1^2+n_1^2). \tag{3.16}$$

Adding (3.9) and (3.12), we obtain

$$r_2^2=\frac{(m_2+n_2)(1-\rho)}{2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)}. \tag{3.17}$$

By utilizing Lemma 1.1 for the cofficients m_2 and n_2, we immediately have

$$|r_2|\leq\sqrt{\frac{2(1-\rho)}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)}}. \tag{3.18}$$

To determine the bound for r_3, we can subtract equation (3.12) from equation (3.9) and substitute the value of r_2^2 from equation (3.16), resulting in the following expression.

$$r_3=\frac{(1-\rho)^2(m_1^2+n_1^2)}{2\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^2}+\frac{(1-\rho)(m_2-n_2)}{4\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}. \tag{3.19}$$

By utilizing Lemma 1.1 for the coefficients m_1, m_2, n_1 and n_2, we get

$$|r_3|\leq\frac{4(1-\rho)^2}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)}+\frac{(1-\rho)}{\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}. \tag{3.20}$$

To determine the bound on $|r_4|$, subtracting (3.13) from (3.10) and substitute the values of r_2 and r_3 from (3.14) and (3.19).
After Applying Lemma 1.1 for the cofficients m_1, m_2 and n_2, we get

$$|r_4| \leq \frac{2}{3}\frac{(1-\rho)}{\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)}+\frac{16}{3}\frac{(1-\rho)^3\left(\frac{1-\gamma}{c}+\frac{5\gamma}{b}\right)}{\left(\frac{1-\gamma}{c}+\frac{4\gamma}{b}\right)\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)^3}+\frac{5(1-\rho)^2}{\left(\frac{1-\gamma}{c}+\frac{2\gamma}{b}\right)\left(\frac{1-\gamma}{c}+\frac{3\gamma}{b}\right)}.$$

By setting $b = c = 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let $\chi(\zeta)$ given by (1.1) be in the class $B_\Sigma(\rho,\gamma)$ $(0 \leq \rho < 1, \gamma \geq 0)$. Then*

$$|r_2| \leq \sqrt{\frac{2(1-\rho)}{1+\gamma}} \tag{3.21}$$

and

$$|r_3| \leq \frac{4(1-\rho)^2}{(1+\gamma)^2}+\frac{(1-\rho)}{(1+\gamma)}. \tag{3.22}$$

Putting $\gamma = 0$, in Corollary 3.1, we get the following corollary.

Corollary 3.2. *Let $\chi(\zeta)$ given by (1.1) and in the class $\mathcal{K}_\Sigma(\rho)$, $(0 \leq \rho < 1)$. Then*

$$|r_2| \leq \sqrt{2(1-\rho)} \tag{3.23}$$

and

$$|r_3| \leq 4(1-\rho)^2 + (1-\rho). \tag{3.24}$$

4. Conclusions

The primary focus of this paper is to introduce novel subclasses of bi-univalent functions in open unit disc $\mathcal{D}$. Furthermore, we present upper limits for the coefficients $|r_2|$, $|r_3|$ and $|r_4|$ for functions belonging to these new subclasses and their respective subclasses.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] D.A. Brannan, T.S. Taha, On Some Classes of Bi-Univalent Functions, in: Mathematical Analysis and Its Applications, Elsevier, 1988: pp. 53–60. https://doi.org/10.1016/B978-0-08-031636-9.50012-7.
[2] B.A. Frasin, M.K. Aouf, New Subclasses of Bi-Univalent Functions, Appl. Math. Lett. 24 (2011), 1569–1573. https://doi.org/10.1016/j.aml.2011.03.048.
[3] T. Hayami, S. Owa, Coefficient Bounds for Bi-Univalent Functions, Pan Amer. Math. J. 22 (2012), 15–26.
[4] M. Lewin, On a Coefficient Problem for Bi-Univalent Functions, Proc. Amer. Math. Soc. 18 (1967), 63–68.
[5] X.F. Li, A.P. Wang, Two New Subclasses of Bi-Univalent Functions, Int. Math. Forum, 7 (2012), 1495–1504.

[6] E. Netanyahu, The Minimal Distance of the Image Boundary From the Origin and the Second Coefficient of a Univalent Function in $|z| < 1$, Arch. Rational Mech. Anal. 32 (1969), 100–112. https://doi.org/10.1007/bf00247676.

[7] C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, (1975).

[8] N. Ravikumar, S. Latha, On Analytic Functions With Generalized Bounded Variation, Stud. Univ. Babes-Bolyai Math. 57 (2012), 43–52.

[9] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain Subclasses of Analytic and Bi-Univalent Functions, Appl. Math. Lett. 23 (2010), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009.

[10] T.S. Taha, Topics in Univalent Function Theory, Ph.D. Thesis, University of London, (1981).

[11] Q.H. Xu, Y.C. Gui, H.M. Srivastava, Coefficient Estimates for a Certain Subclass of Analytic and Bi-Univalent Functions, Appl. Math. Lett. 25 (2012), 990–994. https://doi.org/10.1016/j.aml.2011.11.013.

[12] Q.H. Xu, H.G. Xiao, H.M. Srivastava, A Certain General Subclass of Analytic and Bi-Univalent Functions and Associated Coefficient Estimate Problems, Appl. Math. Comput. 218 (2012), 11461–11465. https://doi.org/10.1016/j.amc.2012.05.034.