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On Degree-Based Topological Indices of Toeplitz Graphs

R. M. K. Iqbal[1], M. Ahmad[2,*], A. Qayyum[3,*], S. S. Supadi[3], M. J. Hussain[1], S. Raza[1]

[1] *Department of Mathematics, Institute of Southern Punjab Multan, Pakistan*

[2] *Department of Mathematics, Khawja Fareed University of Engineering and Information Technology Rahim Yar Khan, Pakistan*

[3] *Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, Malaysia*

[*] *Corresponding authors:* itxmemuktar@gmail.com, dratherqayyum@um.edu.my

Abstract. In this research paper, the determination of the orientation of a linear interpolation in an interconnected graph can be achieved by measuring its distance from a group of sonar stations strategically positioned within the graph. The study utilizes the metric dimension of toeplitz graphs. Several indices play a crucial role in analyzing motivating activities within such complex structures. The indices covered in this study include the general connectivity index of the toeplitz graph, zagreb indices, symmetric division degree index and randic indices, among others.

1. Introduction and Preliminaries

Let $p, b_1, b_2, b_3, ..., b_m$ are different $+ve$ integers, with $0 < b_1 < b_2 < b_3 < ... < b_m < p$. The finite (undirected) toeplitz graph $\tau_p(b_1, b_2, b_3, ..., b_m)$ is a graph with

$$V' = \{0, 1, 2, ..., p-1\}, E' = \{ij : |i-j| \in B\}$$

The toeplitz graph becomes infinite when $|V'|$ is infinite[generalization of toeplitz graph paper se]. The name of that kind of graph class is based on the fact that a toeplitz matrix is its adjacency matrix, i.e. each of its downwards diagonals is constant, from left to right. Toeplitz matrix is also known as constant matrix. Obviously, the first row of, such graph, adjacency matrix determines this graph uniquely, i.e. by a sequence of $0-1$ where the first element is 0. In general, in the toeplitz graph $\tau_p(b_1, b_2, b_3, ..., b_m)$

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No of vertices $= |V'(\tau)| = p$

No of edges of $\tau = |E'(\tau)| = \sum_{i=1}^{m} p - b_i = C_2^p = \frac{p(p-1)}{2}$

A topological index (or descriptor of a molecular structure) is a chemical structure-associated numerical meaning for the association of just a chemical structure with different physical properties, biological activity or chemical reactivity. It is used to evaluate the quantitative structure-activity relationships ($QSARs$) and was proposed by Harlod wiener in 1947. In 1972, Gutman and Trinajstic established the first and second zagreb indices of graph operations, and in 2016, V.R.KULLI described the multiplicative hyper-zagreb indices of graphs. In 1975, milan randic proposed the randic index and Ernesto Estrada described the atom-bond connectivity index. Recently in 2015 Furtula and Gutman established forgotten index or $F-index$ and corresponding polynomial. Shuxian defined two polynomials related to the first zagreb index. There are two types of general connectivity index. The general randic index (or product-connectivity index) was proposed by Bolloba and Erdos and Zhou and Trinajstic obtained the general sum-connectivity index [On the general sum-connectivity index of trees]. Now, in this thesis we derived some results on all these "Degree-Based Topological Indices" for "Teoplitz Graph"

1.1. **Teoplitz Matrix.** The toeplitz matrices are basically those constant matrices that are parallel to the main diagonal throughout all diagonals. Since every diagonal has identical elements, a toeplitz matrix is defined uniquely and therefore easy to memorize from its first row and first column. A matrix of order $p \times p$ is known as teoplitz matrix if $\forall 1 \leq k, l \leq p-1$, such that h_{k+1l+1}. This matrix was introduced by Otto Teoplitz. Below, an example of a toeplitz matrix is shown.

$$\tau = \begin{pmatrix} 0 & 1 & 0 & 2 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -2 \\ 3 & 0 & 1 & 0 & 1 & 0 & 2 \\ -1 & 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 0 & 1 & 0 \end{pmatrix}$$

From its $2p-1$ leading row and column entries, a toeplitz matrix is defined.For a τ toeplitz matrix, there are two ways to obtain a binary matrix from τ. One is by considering a binary matrix where entries are the binary indicator values of the corresponding τ entries, and the other one is whose binary variables are the entries providing the parities of the corresponding τ entries. The binary matrix of the predictor and the binary matrix of parity of this toeplitz matrix can be seen below. Notice that the binary matrix indicator only substitutes 1 for each non-zero entry and the binary parity matrix substitutes its parity for each entry, indicating that if the entry of τ is even then it is substituted by 0 and if entry of τ is odd then it is substituted by 1.

The indicator binary matrix of τ is

$$I = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The parity binary matrix of τ is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The indicator matrix I of τ is symmetric matrix although it is not obtained from a symmetric teoplitz matrix.

Example In this example we see how a teoplitz matrix is shown by an array of length $2p-1$.

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 7 & 2 & 3 & 4 & 5 \\ 8 & 7 & 2 & 3 & 4 \\ 9 & 8 & 7 & 2 & 3 \\ 10 & 9 & 8 & 7 & 2 \end{pmatrix}$$

Let us observe that in this matrix all the elements in the all diagonals are same. The matrix A has elements such as;

$$A[k,l] = A[k+1, l+1].$$

Sufficient elements of matrix A are $2p-1$. A toeplitz matrix A can be determined by its $2p-1$ leading entries of row and column. Now we take an array of size $2p-1$. We use it to represent the teoplitz matrix. According to matrix A the size of array is 9.

$$R = \begin{matrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix}$$

The elements of upper triangular matrix of A are represented in first row and elements of lower triangular matrix of A are represented in first column. For identifying the elements of teoplitz matrix A in array R , we use formula.

Case 1

if $k \leq l$

index$= l - k$

for $k = 2$ and $l = 4$

$A[2][4] = 4 - 2 = 2 =$ index

Which means that $A[2][4]$ entry is equal to entry of array R at index 2. At index 2 entry is 4. In the same way, we can find all the elements of upper triangular matrix of A.

Case 2

if $k > l$

index$= p + i - j - 1$

for $k = 4$ and $l = 2$

$A[4][2] = 5 + 4 - 2 - 1 = 6$

Which means that $A[4][2]$ entry is equal to entry of array R at index 6. At index 6 entry is 8. In the same way, we can find all the elements of lower triangular matrix of A.

1.2. **Teoplitz graph.** Let H be a toeplitz binary symmetric matrix with all 0 entries in the main diagonal. The column numbers with the leading entry 1 are denoted by $h_1, h_2, ..., h_m$. Then (undirected) teoplitze graph is symbolized as $\tau_p(h_1, h_2, h_3, ..., h_m)$. Which essentially means that the toeplitz graph, which is undirected, $\tau_p(h_1, h_2, h_3, ..., h_m)$ has a vertex set $V'(\tau)$. If $V' = \{1, 2, 3, ..., p\}$, so that a vertex k which is contiguous with the vertices $k + h_l$, $l = 1, 2, 3, ..., m$ for $k + h_l \leq p$. In Figure 1, the toeplitz graph of adjacency matrix I is shown.

Let $p, h_1, h_2, h_3, ..., h_m$ are different $+ve$ integers, with $0 < h_1 < h_2 < h_3 < ... < h_m < p$. The finite (undirected) toeplitz graph $\tau_p(h_1, h_2, h_3, ..., h_m)$ is a graph with

$$V'(\tau) = \{1, 2, 3, ..., p\}$$

$$E'(\tau) = \{kl : |k - l| \in D\}$$

The toeplitz graph becomes infinite when $|V'|$ is infinite.(generalization of toeplitz graph paper se)

The name of that kind of graph class is based on the fact that a toeplitz matrix is its adjacency matrix, i.e. each of its downwards diagonals is constant, from left to right. Toeplitz matrix is also known as constant matrix. Obviously, the first row of, such graph, adjacency matrix determines this graph uniquely, i.e. by a sequence of $0 - 1$ where the first element is 0. In general, in the toeplitz graph $\tau_p(h_1, h_2, h_3, ..., h_m)$,

$$\text{No of edges} = \sum_{k=1}^{m} p - h_k$$

$$\text{No of Vertices} = |V'(\tau)| = p$$

Next we discuss the topological indices

42 Balaban et al. included $M_1(G)$ and $M_2(G)$ among topological indices in a review article and named them "zagreb indices of the group". Some clarification is required with regard to this. First, only a handful of topological indices were recognized in the early 1980s and as many of them as possible were required by the writers of the study. Second, both authors of the paper were representatives of the theoretical chemistry department of the zagreb institute at that time. The term "zagreb group index" was soon abbreviated as "zagreb index" and $M_1G)$ is now known as the "first zagreb index" while $M_2(G)$ is known as the "second zagreb index". In 1972, the first and second zagreb indices were established by Gutman and Trinajstic which are the past degree based topological indices of graph. It is an important molecular descriptor and has been closely correlated with many chemical properties. Thus, it attracted more and more attention from chemists and mathematicians. The first and second zagreb indices of graph are important molecular descriptors and have attracted more attention from chemists and mathematicians.[Degree based topological indices]

The first zagreb index $M_1(G)$ is equal to the sum of the squares of the degrees of the vertices for the (molecular) graph G[1]. It can also be considered as the sum over the edges of G, and $M_1(G)$ is defined as:[The first and second zagreb indices of some graph operations]

$$M_1(G) = \sum_{v'v"\in E'(G)} [deg(v') + deg(v")] \tag{1.1}$$

The second zagreb index $M_2(G)$ is equal to the sum of the products of the degrees of the adjacent vertices for the pair of vertices for the (molecular) graph G, and $M_2(G)$ is defined as resently A. Asghar et.al[14]:[The first and second zagreb indices of some graph operations]

$$M_2(G) = \sum_{v'v"\in E'(G)} [deg(v')deg(v")] \tag{1.2}$$

In 1972, the first zagreb index, a very old topological index, was launched and several variants of the zagreb index were subsequently proposed, e.g. Shirdel et al. described a novel index in 2013 under the title of 'hyper-zagreb index' and then it was identified as[2]: [A note on hyper-zagreb index of graph operations]

$$HM(G) = \sum_{v'v"\in E'(G)} [deg(v') + deg(v")]^2 \tag{1.3}$$

E. Deutshi and S. Klavzar, in 2015, defined a new polynomial, M-polynomial in the following way, based on the degree of the vertex[3]:[COMPUTING HYPER ZAGREB INDEX AND M-POLYNOMIALS]

$$M_1(G, y, z) = \sum_{v'v"\in E'(G)} y^{[deg(v')]} z^{[deg(v")]} \tag{1.4}$$

In Shuxian defined two polynomials related to the first zagreb index as in the form resently Zaib Hassan Niazi et.al[15]:

$$M_1^*(G, y) = \sum_{(v'\in V'(G)} deg(v').y^{deg(v')} \tag{1.5}$$

$$M_0(G,y)=\sum_{(v')\in V'(G)} y^{deg(v')} \tag{1.6}$$

Two zagreb type polynomials are defined as follow resently Mukhtar Ahmad et.al[16]:

$$M_{a,b}(G,x)=\sum_{v'v"\in E'(G)} x^{a(deg(v'))+b(d(v"))} \tag{1.7}$$

$$M'_{a,b}(G,x)=\sum_{v'v"\in E'(G)} x^{(a+deg(v'))(b+deg(v"))} \tag{1.8}$$

Two updated models of the zagreb index, the first multiplicative zagreb index $PM_1(G)$ and the second multiplicative zagreb index PM_2, were introduced, by Todeshine et al. for molecular graph G. Certain characteristics of both the $PM_1(G)$ and $PM_2(G)$ indices of particular chemical structures have been investigated[4].[MULTIPLICATIVE ZAGREB INDICES OF TREES] First multiplicative zagreb index for molecular graph G defined as follows resently Mukhtar Ahmad et.al[17]:

$$PM_1(G)=\prod_{v'v"\in E'(G)} [deg(v')+deg(v")] \tag{1.9}$$

Second multiplicative zagreb index for molecular graph G defined as follows:

$$PM_2(G)=\prod_{v'v"\in E'(G)} [deg(v')deg(v")] \tag{1.10}$$

First multiplicative zagreb polynomial for molecular graph G defined as follows:

$$PM_1(G,y)=\prod_{v'v"\in E'(G)} y^{[deg(v')+deg(v")]} \tag{1.11}$$

Second multiplicative zagreb polynomial for molecular graph G defined as follows:

$$PM_2(G,y)=\prod_{v'v"\in E'(G)} y^{[deg(v')deg(v")]} \tag{1.12}$$

The first degree-based topological index was proposed by Milan Randic in 1975[5]:[Degree-Based Topological Indices]

$$R(G)=\sum_{v'v"\in E'(G)} \frac{1}{\sqrt{deg(v')deg(v")}} \tag{1.13}$$

Atom-bond connectivity index (ABC) is a topological index used in chemistry, environmental sciences and pharmacology[6]: [Estrada, Torres, Rodriguez, and Gutman, 1998]

$$ABC(G)=\sum_{v'v"\in E'(G)} \sqrt{\frac{deg(v')+deg(v")-2}{deg(v')deg(v")}} \tag{1.14}$$

First, second and third reduced zagreb indices[7] are described as follow:

$$MR_1(G)=\sum_{v'v"\in E'(G)} [(deg(v')-1)+(deg(v")-1)] \tag{1.15}$$

$$MR_2(G) = \sum_{v'v"\in E'(G)} [(deg(v') - 1)(deg(v") - 1)] \quad (1.16)$$

$$MR_3(G) = \sum_{v'v"\in E'(G)} |(deg(v') - 1) - (deg(v") - 1)| \quad (1.17)$$

$$RR(G) = \sum_{v'v"\in E'(G)} \frac{1}{\sqrt{(deg(v') - 1)(deg(v") - 1)}} \quad (1.18)$$

The reduced reciprocal randic index[8] is defined as:

$$RRR(G) = \sum_{v'v"\in E'(G)} \sqrt{(deg(v') - 1)(deg(v") - 1)} \quad (1.19)$$

Recently in 2015 Furtula and Gutman [8] introduced another topological index known as forgotten index or $F - index$. For more detail on the $F - index$, we refer to the articles [9]. The forgotten index of a graph G is defined as[10, 11, 12]:

$$F(G) = \sum_{(v'v")\in E(G)} [(deg(v'))^2 + (deg(v")^2] \quad (1.20)$$

The forgotten polynomial of a graph G is defined as:

$$F(G, y) = \sum_{v'v"\in E(G)} y^{[(deg(v'))^2+(deg(v"))^2]} \quad (1.21)$$

The symmetric division degree index of a connected graph G is defined as:

$$SDD(G) = \sum_{v'v"\in E'(G)} \frac{mini(deg(v'), deg(v"))}{max(deg(v'), deg(v"))} + \frac{maxi(deg(v'), deg(v"))}{mini(deg(v'), deg(v"))} \quad (1.22)$$

There are two types of general connectivity index. The general Randic index (or product-connectivity index) was proposed by Bolloba and Erdos and is defined as follows:

$$PCI_\lambda(G) = \sum_{uv\in E'(G)} (deg(v')deg(v"))^\lambda \quad (1.23)$$

where λ is a real number. If $\lambda = -\frac{1}{2}$, then Zhou and Trinajstic modified the randic index to create the general sum-connectivity index:[On the general sum-connectivity index of trees]

$$SCI_\alpha(G) = \sum_{v'v"\in E'(G)} (deg(v') + deg(v"))^\alpha \quad (1.24)$$

where α is a real number. If $\alpha = 1$, then the general sum connectivity index becomes the first zagreb index.

2. Main Results

In this section, we established some results on degree based topological indices of toeplitz graphs.

Theorem 2.1. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs. Then, for $p \geq 2$, zagreb indices are
$M_1(\tau) = p^3 - 2p^2 + p$
$M_2(\tau) = \frac{p(p-1)^3}{2}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graph $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
Now, by using equations $(4.1) - (4.5)$, we have

$M_1(G) = \sum_{v'v'' \in E'(G)} [deg(v') + deg(v'')]$
$M_1(\tau) = \sum_{v'v'' \in E'(\tau)} [deg(v') + deg(v'')]$
$M_1(\tau) = |E'(\tau)|(p-1+p-1)$
$M_1(\tau) = \frac{p(p-1)}{2}(2p-2)$
$M_1(\tau) = \frac{p(p-1)}{2}(2(p-1))$
$M_1(\tau) = p(p-1)^2$
$M_1(\tau) = p(p^2 - 2p + 1)$
$M_1(\tau) = p^3 - 2p^2 + p$

$M_2(G) = \sum_{v'v'' \in E'(G)} (deg(v') deg(v''))$
$M_2(\tau) = \sum_{v'v'' \in E'(\tau)} (deg(v') deg(v''))$
$M_2(\tau) = |E'(\tau)|(p-1)(p-1))$
$M_2(\tau) = \frac{p(p-1)}{2}(p-1)^2$
$M_2(\tau) = \frac{p(p-1)^3}{2}$.

Theorem 2.2. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs, for $p \geq 2$. Then
$HM(\tau) = 2p(p-1)^3$
$HM(\tau, y) = \frac{p(p-1)}{2} y^{4(p-1)^2}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$
$deg(v') = p-1, \forall v' \in V'(\tau)$

$|E'(\overline{\tau})| = C_2^p - r$, where $r = |E'(\tau)| = C_2^p$
$|E'(\overline{\tau})| = C_2^p - C_2^p$
$|E'(\overline{\tau})| = 0$
Now, from equation (4.) – (4.), we have $HM(\tau) = \sum_{v'v"\in E'(\tau)}[deg(v') + deg(v")]^2$

$HM(\tau) = |E'(\tau)|(p-1+p-1)^2$
$HM(\tau) = \frac{p(p-1)}{2}(2p-2)^2$
$HM(\tau) = \frac{p(p-1)}{2}(4(p-1)^2)$
$HM(\tau) = 2p(p-1)^3$

$HM(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{[deg(v')+deg(v")]^2}$
$HM(\tau, y) = |E'(\tau)|y^{(p-1+p-1)^2}$
$HM(\tau, y) = \frac{p(p-1)}{2}y^{(2p-2)^2}$
$HM(\tau, y) = \frac{p(p-1)}{2}y^{4(p-1)^2}$.

Theorem 2.3. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs. Then, for $p \geq 2$ zagreb polynomials of Toeplitz graphs are
$M_1(\tau, y) = \frac{p(p-1)}{2}y^{2(p-1)}$
$M_2(\tau, y) = \frac{p(p-1)}{2}y^{(p-1)^2}$
$M_3(\tau, y) = \frac{p(p-1)}{2}$
$M_4(\tau, y) = \frac{p(p-1)}{2}y^{2p^2-4p+2}$
$M_5(\tau, y) = \frac{p(p-1)}{2}y^{2(p-1)^2}$
$M_1(\tau, y, z) = \frac{p(p-1)}{2}y^(p-1)z^{(p-1)}$
$M_1^*(\tau, y) = p(p-1)y^{(p-1)}$
$M_0(\tau, y) = py^{(p-1)}$
$M_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{[(a+b)(p-1)]}$
$M'_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{(ab+(p-1)(a+b+p-1)}$.

Proof: The toeplitz graph $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e. ,
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
Now, by using equations (4.) – (4.), we have

$M_1(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{deg(v')+deg(v")}$
$M_1(\tau, y) = |E'(\tau)|y^{p-1+p-1}$
$M_1(\tau, y) = \frac{p(p-1)}{2}y^{(2p-2)}$
$M_1(\tau, y) = \frac{p(p-1)}{2}y^{2(p-1)}$

$M_2(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{deg(v')deg(v")}$
$M_2(\tau, y) = |E'(\tau)|y^{(p-1)(p-1)}$
$M_2(\tau, y) = |E'(\tau)|y^{(p-1)^2)}$
$M_2(\tau, y) = \frac{p(p-1)}{2}y^{(p^2-2p+1)}$

$M_3(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{|deg(v')-deg(v")|}$
$M_3(\tau, y) = |E'(\tau)|y^{|(p-1)-(p-1)|}$
$M_3(\tau, y) = \frac{p(p-1)}{2}y^0$
$M_3(\tau, y) = \frac{p(p-1)}{2}$

$M_4(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{deg(v')[deg(v')+deg(v")]}$
$M_4(\tau, y) = |E'(\tau)|y^{(p-1)[p-1+p-1]}$
$M_4(\tau, y) = \frac{p(p-1)}{2}y^{(p-1)(2p-2)}$
$M_4(\tau, y) = \frac{p(p-1)}{2}y^{2(p-1)^2}$
$M_4(\tau, y) = \frac{p(p-1)}{2}y^{2(p^2-2p+1)}$
$M_4(\tau, y) = \frac{p(p-1)}{2}y^{2p^2-4p+2)}$

$M_5(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{deg(v")[deg(v')+deg(v")]}$
$M_5(\tau, y) = |E'(\tau)|y^{(p-1)[p-1+p-1]}$
$M_5(\tau, y) = \frac{p(p-1)}{2}y^{(p-1)(2p-2)}$
$M_5(\tau, y) = \frac{p(p-1)}{2}y^{2(p-1)^2}$

$M_1(\tau, y, z) = \sum_{v'v"\in E'(\tau)} y^{[deg(v')]}z^{[deg(v")]}$
$M_1(\tau, y, z) = |E'(\tau)|y^(p-1)z^{(p-1)}$
$M_1(\tau, y, z) = \frac{p(p-1)}{2}y^(p-1)z^{(p-1)}$

$M_1^*(\tau, y) = \sum_{(v'\in V'(\tau)} deg(v').y^{deg(v')}$
$M_1^*(\tau, y) = |V'(\tau)|(p-1)y^{(p-1)}$
$M_1^*(\tau, y) = p(p-1)y^{(p-1)}$

$M_0(\tau, y) = \sum_{(v')\in V'(\tau)} y^{deg(v')}$
$M_0(\tau, y) = |V'(\tau)|y^{(p-1)}$
$M_0(\tau, y) = py^{(p-1)}$

$M_{a,b}(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{a(deg(v'))+b(d(v"))}$
$M_{a,b}(\tau, y) = |E'(\tau)|y^{[a(p-1)+b(p-1)]}$
$M_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{[(a+b)(p-1)]}$
$M_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{[(a+b)(p-1)]}$
$M_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{[(a+b)(p-1)]}$

$M'_{a,b}(\tau, y) = \sum_{v'v"\in E'(\tau)} y^{(a+deg(v'))(b+deg(v"))}$
$M'_{a,b}(\tau, y) = |E'(\tau)|y^{(a+p-1)(b+p-1)}$
$M'_{a,b}(\tau, y) = |E'(\tau)|y^{(ab+ap-a+bp+p^2-p-b-p+1)}$

$M'_{a,b}(\tau, y) = |E'(\tau)|y^{(ab+ap+bp-a-b+p^2-2p+1)}$
$M'_{a,b}(\tau, y) = |E'(\tau)|y^{(ab+p(a+b)-(a+b)+(p-1)^2)}$
$M'_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{(ab+(a+b)(p-1)+(p-1)^2)}$
$M'_{a,b}(\tau, y) = \frac{n(n-1)}{2}y^{(ab+(p-1)(a+b+p-1)}$.

Theorem 2.4. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs, for $p \geq 2$. Then
$PM_1(\tau) = [2(p-1)]^{\frac{p(p-1)}{2}}$
$PM_2(\tau) = [p-1]^{p(p-1)}$
$PM_1(\tau, y) = y^{p(p-1)^2}$
$PM_2(\tau, y) = y^{[\frac{p(p-1)^3}{2}}$
$M_2(\tau, y, z) = y^{\frac{p(p-1)^3}{2}}z^{\frac{p(p-1)^3}{2}}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
$|E'(\overline{\tau})| = C_2^p - r$, where $r = |E'(\tau)| = C_2^p$
$|E'(\overline{\tau})| = C_2^p - C_2^p$
$|E'(\overline{\tau})| = 0$
Now, by using equations (4.) − (4.), we have

$PM_1(\tau) = \prod_{v'v"\in E'(\tau)}[deg(v') + deg(v")]$
$PM_1(\tau) = [p-1+p-1]^{|E'(\tau)|}$
$PM_1(\tau) = [2p-2]^{\frac{p(p-1)}{2}}$
$PM_1(\tau) = [2(p-1)]^{\frac{p(p-1)}{2}}$

$PM_2(\tau) = \prod_{v'v"\in E'(\tau)}[deg(v')deg(v")]$
$PM_2(\tau) = [(p-1)(p-1)]^{|E'(\tau)|}$
$PM_2(\tau) = [(p-1)^2]^{\frac{p(p-1)}{2}}$
$PM_2(\tau) = [p-1]^{p(p-1)}$

$PM_1(\tau, y) = \prod_{v'v"\in E'(\tau)} y^{[deg(v')+deg(v")]}$
$PM_1(\tau, y) = y^{[p-1+p-1]^{|E'(\tau)|}}$
$PM_1(\tau, y) = y^{(2p-2)\times\frac{p(p-1)}{2}}$
$PM_1(\tau, y) = y^{p(p-1)^2}$

$PM_2(\tau, y) = \prod_{v'v"\in E'(\tau)} y^{[deg(v')deg(v")]}$
$PM_2(\tau, y) = y^{[(p-1)(p-1)]^{|E'(\tau)|}}$

$PM_2(\tau, y) = y^{[(p-1)^2]\times \frac{p(p-1)}{2}}$
$PM_2(\tau, y) = y^{[\frac{p(p-1)^3}{2}}$

$M_2(\tau, y, z) = \prod_{v'v"\in E'(\tau)} y^{[deg(v')]} z^{[deg(v")]}$
$M_2(\tau, y, z) = y^{[p-1]^{|E'(\tau)|}} z^{[p-1]^{|E'(\tau)|}}$
$M_2(\tau, y, z) = y^{[p-1]\times \frac{p(p-1)}{2}} z^{[p-1]\times \frac{p(p-1)}{2}}$
$M_2(\tau, y, z) = y^{\frac{p(p-1)^3}{2}} z^{\frac{p(p-1)^3}{2}}$.

Theorem 2.5. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs, for $p \geq 2$. Then
$ABC(\tau) = p\sqrt{\frac{p-2}{2}}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
Now, by using equation (4.), we have

$ABC(\tau) = \sum_{v'v"\in E'(\tau)} \sqrt{\frac{deg(v')+deg(v")-2}{deg(v')deg(v")}}$
$ABC(\tau) = |E'(\tau)|\sqrt{\frac{(p-1)+(p-1)-2}{(p-1)(p-1)}}$
$ABC(\tau) = \frac{p(p-1)}{2}\sqrt{\frac{(2p-4)}{(p-1)^2}}$
$ABC(\tau) = \frac{p(p-1)}{2}\frac{\sqrt{2(p-2)}}{(p-1)^2}$
$ABC(\tau) = \frac{p\sqrt{2(p-2)}}{2}$
$ABC(\tau) = p \times \sqrt{\frac{p-2}{2}}$.

Theorem 2.6. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ be the toeplitz graphs, for $p \geq 2$. Then
$R(\tau) = \frac{p}{2}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
Now, from equation (4.), we have

$R(\tau) = \sum_{v'v"\in E'(\tau)} \frac{1}{\sqrt{deg(v')deg(v")}}$
$R(\tau) = |E'(\tau)|\frac{1}{\sqrt{(p-1)(p-1)}}$
$R(\tau) = \frac{p(p-1)}{2} \times \frac{1}{\sqrt{(p-1)^2}}$

$R(\tau) = \frac{p(p-1)}{2} \times \frac{1}{(p-1)}$
$R(\tau) = \frac{p}{2}$.

Theorem 2.7. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs. Then, for $p \geq 2$,
$MR_1(\tau) = p^3 - 3p^2 + 2p$
$MR_2(\tau) = \frac{p(p-1)(p-2)^2}{2}$
$MR_3(\tau) = 0$
$RR(\tau) = \frac{p(p-1)}{2(p-2)}$
$RRR(\tau) = \frac{p(p-1)(p-2)}{2}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
Now, by using equations (4.) − (4.), we have

$MR_1(\tau) = \sum_{v'v'' \in E'(\tau)} [(deg(v') - 1) + (deg(v'') - 1)]$
$MR_1(\tau) = |E'(\tau)|[(p-1-1) + (p-1-1)]$
$MR_1(\tau) = \frac{p(p-1)}{2} \times (2p-4)$
$MR_1(\tau) = \frac{p(p-1)}{2} \times (2(p-2))$
$MR_1(\tau) = p(p-1)(p-2)$
$MR_1(\tau) = p(p^2 - 2p - p + 2)$
$MR_1(\tau) = p^3 - 3p^2 + 2p$

$MR_2(\tau) = \sum_{v'v'' \in E'(\tau)} (deg(v') - 1)(deg(v'') - 1)$
$MR_2(\tau) = |E'(\tau)|(p-1-1)(p-1-1))$
$MR_2(\tau) = \frac{p(p-1)}{2} \times (p-2)^2$
$MR_2(\tau) = \frac{p(p-1)(p-2)^2}{2}$

$MR_3(\tau) = \sum_{v'v'' \in E'(\tau)} |(deg(v') - 1) - (deg(v'') - 1)|$
$MR_3(\tau) = |E'(\tau)||(p-1-1) - (p-1-1)|$
$MR_3(\tau) = \frac{p(p-1)}{2}(0)$
$MR_3(\tau) = 0$

$RR(\tau) = \sum_{v'v'' \in E'(\tau)} \frac{1}{\sqrt{(deg(v')-1)(deg(v'')-1)}}$
$RR(\tau) = |E'(\tau)| \frac{1}{\sqrt{(p-1-1)(p-1-1)}}$
$RR(\tau) = \frac{p(p-1)}{2} \times \frac{1}{\sqrt{(p-2)^2}}$
$RR(\tau) = \frac{p(p-1)}{2} \times \frac{1}{(p-2)}$
$RR(\tau) = \frac{p(p-1)}{2(p-2)}$

$RRR(G) = \sum_{v'v"\in E'(G)} \sqrt{(deg(v') - 1)(deg(v") - 1)}$
$RRR(\tau) = |E'(\tau)|\sqrt{(p-1-1)(p-1-1)}$
$RRR(\tau) = \frac{p(p-1)}{2} \times \sqrt{(p-2)^2}$
$RRR(\tau) = \frac{p(p-1)(p-2)}{2}$.

Theorem 2.8. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ be the toeplitz graphs, for $p \geq 2$. Then
$F(\tau) = p(p-1)^3$
$F(\tau, y) = \frac{p(p-1)}{2} y^{[2(p-1)^2]}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$
Now, by using equation (4.), we have

$F(\tau) = \sum_{v'v"\in E(\tau)}[(deg(v'))^2 + (ded(v"))^2]$
$F(\tau) = |E(\tau)|[(p-1)^2 + (p-1)^2]$
$F(\tau) = \frac{p(p-1)}{2}[p^2 + 1 - 2p + p^2 + 1 - 2p]$
$F(\tau) = \frac{p(p-1)}{2}[2p^2 - 4p + 2]$
$F(\tau) = \frac{p(p-1)}{2}[2(p^2 - 2p + 1)]$
$F(\tau) = p(p-1)(p-1)^2$
$F(\tau) = p(p-1)^3$

$F(\tau, y) = \sum_{v'v"\in E(\tau)} y^{[(deg(v'))^2+(ded(v"))^2]}$
$F(\tau, y) = |E(\tau)|y^{[(p-1)^2+(p-1)^2]}$
$F(\tau, y) = |E(\tau)|y^{[p^2+1-2p+p^2+1-2p]}$
$F(\tau, y) = |E(\tau)|y^{[2p^2-4p+2]}$
$F(\tau, y) = \frac{p(p-1)}{2} y^{[2(p^2-2p+1)]}$
$F(\tau, y) = \frac{p(p-1)}{2} y^{[2(p-1)^2]}$.

Theorem 2.9. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs, for $p \geq 2$. Then
$SDD(\tau) = p(p-1)$.

Proof: The Toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e.
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and

$deg(v') = p-1, \forall v' \in V'(\tau)$

Now, by using equation (4.), we have

$SDD(\tau) = \sum_{v'v'' \in E'(\tau)} \frac{mini(deg(v'),deg(v''))}{max(deg(v'),deg(v''))} + \frac{maxi(deg(v'),deg(v''))}{mini(deg(v'),deg(v''))}$
$SDD(\tau) = |E'(\tau)| \frac{mini(p-1,p-1)}{max(p-1,p-1)} + \frac{p-1}{p-1}$
$SDD(\tau) = |E'(\tau)| \frac{p-1}{p-1} + \frac{maxi(p-1,p-1)}{mini(p-1,p-1)}$
$SDD(\tau) = \frac{p(p-1)}{2}(2)$
$SDD(\tau) = p(p-1)$.

Theorem 2.10. Let $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$, be the toeplitz graphs, for $p \geq 2$. Then
$PCI_\lambda(\tau) = \frac{p(p-1)2\lambda+1}{2}$
$SCI_\alpha(\tau) = \frac{p(p-1)^{(1+\alpha)}}{2^{(1-\alpha)}}$.

Proof: The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ appears in figure(graph). The toeplitz graphs $\tau = \tau_p(b_1, b_2, b_3, ..., b_m)$ contains p no of vertices and C_2^p no of edges. The degree of each vertex in $V'(\tau)$ is $p-1$, i.e. ,
$|V'(\tau)| = p$,
$|E'(\tau)| = C_2^p = \frac{p(p-1)}{2}$ and
$deg(v') = p-1, \forall v' \in V'(\tau)$

Now, by using equation (4.), we have

$PCI_\lambda(\tau) = \sum_{uv \in E'(\tau)} (deg(v')deg(v''))^\lambda$
$PCI_\lambda(\tau) = |E'(\tau)|[(p-1)(p-1)]^\lambda$
$PCI_\lambda(\tau) = \frac{p(p-1)}{2}(p-1)^{2\lambda}$
$PCI_\lambda(\tau) = \frac{p(p-1)2\lambda+1}{2}$

$SCI_\alpha(\tau) = \sum_{v'v'' \in E'(\tau)} (deg(v') + deg(v''))^\alpha$
$SCI_\alpha(\tau) = |E'(\tau)|[p-1+p-1]^\alpha$
$SCI_\alpha(\tau) = \frac{p(p-1)}{2}(2p-2)^\alpha$
$SCI_\alpha(\tau) = \frac{p(p-1)^{(1+\alpha)}}{2^{(1-\alpha)}}$

3. Numerical Examples

Let H be a toeplitz binary symmetric matrix with all 0 entries in the main diagonal. The column numbers with the leading entry 1 are denoted by $h_1, h_2, ..., h_m$. Then (undirected) teoplitze graphs is symbolized as $\tau_p(h_1, h_2, h_3, ..., h_m)$. Which essentially means that the toeplitz graphs, which is undirected, $\tau_p(h_1, h_2, h_3, ..., h_m)$ has a vertex set $V'(\tau)$. If $V' = \{1, 2, 3, ..., p\}$, so that a vertex k which is contiguous with the vertices $k + h_l$, $l = 1, 2, 3, ..., m$ for $k + h_l \leq p$. In Figure 1, the toeplitz graphs of adjacency matrix I is shown.

Let $p, h_1, h_2, h_3, ..., h_m$ are different $+ve$ integers, with $0 < h_1 < h_2 < h_3 < ... < h_m < p$. The finite (undirected) toeplitz graphs $\tau_p(h_1, h_2, h_3, ..., h_m)$ is a graph with

$$V'(\tau) = \{1, 2, 3, ..., p\},$$
$$E'(\tau) = \{kl : |k - l| \in D\}$$

The toeplitz graphs becomes infinite when $|V'|$ is infinite.

A toeplitz matrix is an adjacency matrix that determines the first row of a graph uniquely i.e. by a sequence of $0 - 1$ where the first element is 0. In general, in the toeplitz graphs $\tau_p(h_1, h_2, h_3, ..., h_m)$,

$$\text{No of vertices} = |V'(\tau)| = p$$
$$\text{No of edges} = \sum_{k=1}^{m} p - h_k$$

Example 3.1. For the teoplitz graphs $\tau_p(h_1, h_2, h_3, ..., h_m)$, let $p = 3$, then
$|V'(\tau)| = 3$
$|E'(\tau)| = \sum_{k=1}^{2}(p - h_k)$
$|E'(\tau)| = p - h_1 + p - h_2$
$|E'(\tau)| = 3 - 1 + 3 - 2$
$|E'(\tau)| = 2 + 1$
$|E'(\tau)| = 3$
The adjacency symmetric teoplitz matrix of $\tau_3(1, 2)$ is given below

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Here, the degree of each vertex is "2".

The graph is given below:

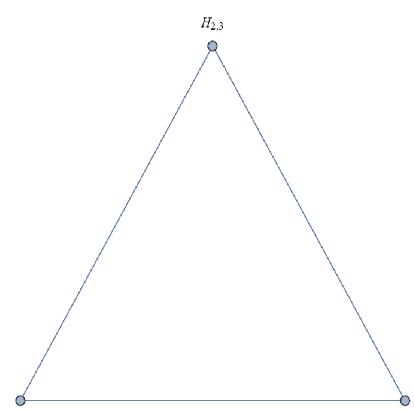

Figure 1. Graph

Example 3.2. For the teoplitz graph $\tau_p(h_1, h_2, h_3, ..., h_m)$, let $p = 5$, then
$|V'(\tau)| = 5$

$|E'(\tau)| = \sum_{k=1}^{4}(p - h_k)$

$|E'(\tau)| = p - h_1 + p - h_2 + p - h_3 + p - h_4$

$|E'(\tau)| = 5 - 1 + 5 - 2 + 5 - 3 + 5 - 4$

$|E'(\tau)| = 4 + 3 + 2 + 1$

$|E'(\tau)| = 10$

The adjacency symmetric teoplitz matrix of $\tau_5(1, 2, 3, 4)$ is given below

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Here, the degree of each vertex is "4". The graph is given below:

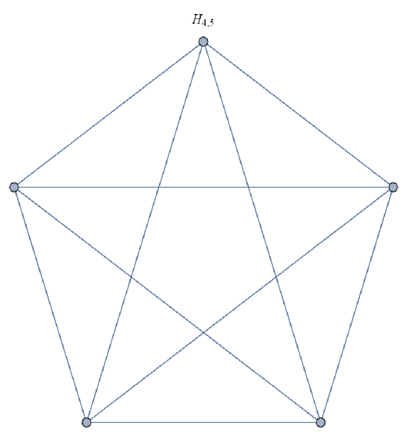

Figure 2. Graph

Example 3.3. For the teoplitz graph $\tau_p(h_1, h_2, h_3, ..., h_m)$, let $p = 10$, then

$|V'(\tau)| = 10$

$|E'(\tau)| = \sum_{k=1}^{9}(p - h_k)$

$|E'(\tau)| = p - h_1 + p - h_2 + p - h_3 + p - h_4 + p - h_6 + p - h_7 + p - h_8 + p - h_9$

$|E'(\tau)| = 10 - 1 + 10 - 2 + 10 - 3 + 10 - 4 + 10 - 5 + 10 - 6 + 10 - 7 + 10 - 8 + 10 - 9$

$|E'(\tau)| = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$

$|E'(\tau)| = 45$

The adjacency symmetric teoplitz matrix of $\tau_{10}(1, 2, 3, 4, 5, 6, 7, 8, 9)$ is given below

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Here, the degree of each vertex is "9".

The graph is given below:

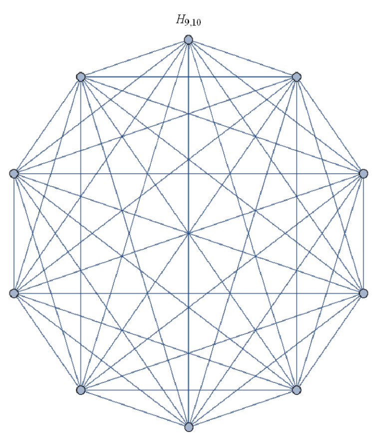

Figure 3. Graph

Example 3.4. For the teoplitz graphs $\tau_p(h_1, h_2, h_3, ..., h_m)$, let $p = 25$, then

$|V'(\tau)| = 25$

$|E'(\tau)| = \sum_{k=1}^{24}(p - h_k)$

$|E'(\tau)| = p-h_1+p-h_2+p-h_3+p-h_4+p-h_6+p-h_7+p-h_8+p-h_9+p-h_{10}+p-h_{11}+p-h_{12}+p-h_{13}+p-h_{14}+p-h_{15}+p-h_{16}+p-h_{17}+p-h_{18}+p-h_{19}+p-h_{20}+p-h_{21}+p-h_{22}+p-h_{23}+p-h_{24}$

$|E'(\tau)| = 25 - 1 + 25 - 2 + 25 - 3 + 25 - 4 + 25 - 5 + 25 - 6 + 25 - 7 + 25 - 8 + 25 - 9 + 25 - 10 + 25 - 11 + 25 - 12 + 25 - 13 + 25 - 14 + 25 - 15 + 25 - 16 + 25 - 17 + 25 - 18 + 25 - 19 + 25 - 20 + 25 - 21 + 25 - 22 + 25 - 23 + 25 - 24$

$|E'(\tau)| = 24+23+22+21+20+19+18+17+16+15+14+13+12+11+10+9+8+7+6+5+4+3+2+1$

$|E'(\tau)| = 300$

The adjacency symmetric teoplitz matrix of τ_{25} $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24)$ is given below

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

Here, the degree of each vertex is "24".

The graph is given below:

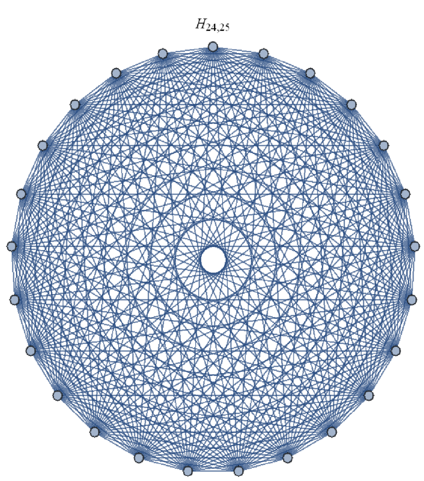

Figure 4. Graph

4. Conclusion

The metric dimension of the crystal cubic carbon structure has been investigated and a formula for its metric dimension has been derived. The metric dimension of specific families of toeplitz graphs has been examined as well and it has been determined that these graphs exhibit a constant metric dimension.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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