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Vector General Fuzzy Automaton: A Refining Analyzing

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Abstract. This study aims to investigate the refining relation between two vector general fuzzy automata (VGFA) and prove that refining relation is an equivalence relation. Moreover, Also, we prove that if there exists a refining equivalence between two VGFA, then they have the same language. After that, by considering the notion of refining equivalence, we present the quotient of VGFA. In particular, we show that any quotient of a given VGFA and the VGFA itself have the same language. Furthermore, using the quotient VGFA, we obtain a minimal VGFA with the same language.

1. Introduction

The notion of fuzzy set as a method for representing uncertainty has been introduced by Zadeh in 1965 [36]. Fuzzy set theory has become more and more mature in many fields such as fuzzy relation, fuzzy logic, fuzzy decisionmaking, fuzzy classification, fuzzy pattern recognition, fuzzy control, fuzzy optimization, and fuzzy automata.

Automata have a long history both in theory and application. Automata are the prime example of general computational systems over discrete spaces [6]. Fuzzy finite automata were introduced by Wee [35] and Santos [27, 28] in the late 1960s. Subsequently, the fundamentals of fuzzy language theory were established by Lee and Zadeh [12] and by Thomason and Marinos [34]. Fuzzy finite automata have practical applications in environments where uncertainty is naturally present, including

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fuzzy discrete event systems [24], fault diagnosis, clinical monitoring [26], artificial intelligence , and model checking [15].

Amongst these studies, the most influential works are related to a study by Das [4] which focuses on the fuzzy topological characterization of a fuzzy automaton. In their paper, P. Li and Y.M. Li [13] also studied and investigated the algebraic properties of languages and automata. In another study, Jin and his coworkers [9] carried out the algebraic study of fuzzy automata based on po-monoids. Further, Kim, Kim and Cho [11] developed the algebraic notion of fuzzy automata theory and Mockor [17–19] investigated the use of categorical concepts in the study of fuzzy automata theory. Qiu [20–23] also studied the algebraic, topological and categorical concept of fuzzy automata theory based on residuated lattices. In his study, Li [14] clearly examined a categorical approach to lattice-valued fuzzy automata. Shamsizadeh and her coworkers studied the graph concept for general fuzzy automata [25, 33], and studied and investigated on the minimization of fuzzy multiset finite automaton [31, 32]. Doostfatemeh and Kremer [5] introduced the concept of general fuzzy automata to emphasize the insufficiency of the current literature to deal with the applications which in turn depend on fuzzy automata as a modeling tool, assigning membership values to active states of a fuzzy automaton. In general, general fuzzy automata provide an attractive systematic way for generalizing discrete applications [5]. Moreover, general fuzzy automata are able to create capabilities which are hardly achievable by other tools. On the other hand, the contribution of GFA to neural networks has been considerable, and dynamical fuzzy systems are becoming more and more popular and useful. In 2022, Shamsizadeh and her coworkers [29], present the notion of general fuzzy automata over a field (vector general fuzzy automata). General fuzzy automaton over a field are used for generation of linear codes, detection and correction of errors, construction of testing sequence, and generation of pseudo-random sequences of numbers. They are also used in experiments that require Mote Carlo methods, in the protection of data stored in computer systems and radiolocation.

The minimization concept is a fundamental problem in automaton theory. There are plenty of studies conducted on the minimization problem of fuzzy finite automaton. For example, minimization of the mealy type of fuzzy finite automaton is discussed in [3], minimization of fuzzy finite automaton with crisp final states without outputs is studied in [2], and minimization of deterministic finite automaton with fuzzy (final) states in [16]. For more information see [1, 7, 8, 10, 30].

Refining equivalences have been widely used in many areas of computer science to model equivalence between various systems, and to reduce the number of states of these systems, whereas uniform fuzzy relations have recently been introduced as a means to model the fuzzy equivalence between elements of two possible different sets. Here we use the concept of refining equivalences as a powerful tool in the study of equivalence between fuzzy automata and reduce the number of states of vector general fuzzy automata. The main aim of this paper is to present the concept of refining equivalence to provide a very powerful tool in the study of equivalence between vector general fuzzy automata, as well as between some related structures. And, we want to demonstrate the minimization way of looking at this issue by presenting quotient vector general fuzzy automata.

The present paper is organized as folllows: In Section 2, we recall some concepts of vector general fuzzy automaton (VGFA), max-min VGFA and the language of VGFA. In Section 3, we present a refining equivalence on two VGFAs and prove that refining equivalence is an equivalence relation. After that, we prove that the union of all refining equivalences is a refining equivalence, too. Also, by considering the notion of refining equivalence, we give the quotient of VGFA and show that this automaton is a minimal VGFA.

2. Preliminaries

In this section, we review some notions which are needed in the next section.

Definition 2.1. *[37] A fuzzy finite state automaton (FFA) is a six-tuple denoted by $\tilde{F} = (Q, X, R, Z, \delta, \omega)$, where*

- $Q = \{q_1, q_2, ..., q_n\}$ *is a finite set of states,*
- $X = \{a_1, a_2, ..., a_m\}$ *is a finite set of input symbols,*
- R *is the start state of* $\tilde{F}$,
- $Z = \{b_1, b_2, ..., b_k\}$ *is a finite set of output symbols,*
- $\delta : Q \times X \times Q \to [0, 1]$ *is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval* $[0, 1]$,
- $\omega : Q \to Z$ *is the output function.*

In an FFA, as can be seen, associated with each fuzzy transition a membership value in $[0, 1]$. *We call this membership value, the value of the transition.*

As usual X^* denotes the set of all words of elements of X of finite length, including the empty word Λ in X^* and $|x|$ denotes the length of x, for any $x \in X^*$.

Definition 2.2. *[5] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where*

- $Q = \{q_1, q_2, ..., q_n\}$ *is a finite set of states,*
- $X = \{a_1, a_2, ..., a_m\}$ *is a finite set of input symbols,*
- $\tilde{R} \subseteq \tilde{P}(Q)$ *is the set of fuzzy start states, where the fuzzy power set of Q denoted as $\tilde{P}(Q)$.*
- $Z = \{b_1, b_2, ..., b_k\}$ *is a finite set of output symbols,*
- $\tilde{\delta} : (Q \times [0, 1]) \times X \times Q \to [0, 1]$ *is the augmented transition function,*
- $\omega : Q \to Z$ *is the output function,*
- $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ *is called the membership assignment function. The function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value*

of a predecessor and δ is the value of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on an input a_k is represented as

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

- $F_2 : [0,1]^* \to [0,1]$ *is called the multi-membership resolution function. The multi-membership resolution function resolves the multi-membership active states and assigns a single non-membership value to them.*

  $[0,1]^*$ *is the set of elements in [0, 1]. The multi-membership resolution function F_2, is a function which specifies the strategy, that resolves the multi-membership active states and assigns a single mv to them.*

Definition 2.3. *[29] Let F be a field and $n \in N_0$. By F_n we denote the vector space of column vectors of dimension n over F. A vector general fuzzy automaton(VGFA) is an automaton $\tilde{F}_v = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ with the following properties:*

(i) *There exists a field F and integers $k, m, r \in N_0$ such that*

1. $Q = F_k$ *is a nonempty finite set of states,* $Q = \{u_1, u_2, u_3 ...\}$, *where* $u_1 = (u_1^{(1)}, u_1^{(2)}, ..., u_1^{(k)}) \in F_k$,
2. $X = F_m$ *is a finite set of input symbols,* $X = \{a_1, a_2, a_3 ...\}$, *where* $a_1 = (a_1^{(1)}, a_1^{(2)}, ..., a_1^{(m)}) \in F_m$,
3. $\tilde{R} \subseteq P(\tilde{Q})$ *is the set of L-fuzzy starts symbols, where the fuzzy power set of Q denoted as $P(\tilde{Q})$,*
4. $Z = F_r$ *is a finite set of output symbols,* $Z = \{z_1, z_2, z_3 ...\}$, *where* $z_1 = (z_1^{(1)}, z_1^{(2)}, ..., z_1^{(r)}) \in F_r$,

(ii) *There exist a $k \times k$ matrix A, a $k \times m$ matrix B, and a $r \times k$ matrix C, all over F such that*

1. $\tilde{\delta} : (Q \times [0,1]) \times X \times Q \to [0,1]$ *is the augmented transition function, where $\delta(u, a, Au + Ba) \in \Delta$, Since A is a $k \times k$ matrix, B is a $k \times m$ matrix, $u \in F^k$ and $a \in F^m$.*
2. $\tilde{\omega} : (Q \times [0,1]) \times Z \to [0,1]$ *is the augmented output function.*
3. $F_1 : [0,1] \times [0,1] \to [0,1]$ *is called the membership assignment function. The function $F_1(\mu, \delta)$, as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the value of a transition. In this definition, the process that takes place upon the transition from state u_i to u_j on an input a_k is represented as*

$$\mu^{t+1}(u_j) = \tilde{\delta}((u_i, \mu^t(u_i)), a_k, u_j) = F_1(\mu^t(u_i), \delta(u_i, a_k, u_j)).$$

Which means that membership value (MV) of the state u_j at time $t+1$ is computed by function F_1 using both the membership value of u_i at time t and the membership value of the transition. There are many options which can be used for the function $F_1(\mu, \delta)$. For example, it can be $\max\{\mu, \delta\}, \min\{\mu, \delta\}, \frac{\mu+\delta}{2}$ or any other applicable mathematical function.

4. $F_2 : [0,1] \times [0,1] \to [0,1]$ *is called the membership assignment output function.* $F_2(\mu, \omega)$ *as is seen, is motivated by two parameters* μ *and* ω*, where* μ *is the membership value of present state and* ω *is the membership value of an output function. Then*

$$\tilde{\omega}((u, \mu^{t_i}(u)), z) = F_2(\mu^{t_i}(u), \omega(u, z)),$$

Notice that, $\omega(u, z) > 0$ *if and only if* $z = Cu$.

5. $F_3 : [0,1]^* \to [0,1]$ *is called the multi-membership resolution function. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.*

6. $F_4 : [0,1]^* \to [0,1]$ *is called the multi-membership resolution output function. The multi-membership resolution output function resolves the output multi-membership active state and assigns a single output membership value to it*

We let the set of all transitions of $\tilde{F}_v$ *is denoted by* Δ*. Now, suppose that* $Q_{act}(t_i)$ *be the set of all active states at time* t_i*, for all* $i \geq 0$*. We have* $Q_{act}(t_0) = \tilde{R}$ *and*

$$Q_{act}(t_i) = \{(Au + Ba, \mu^{t_i}(Au + Ba)) \big| \exists u \in Q_{act}(t_{i-1}), \exists a \in X, \delta(u, a, Au + Ba) \in \Delta\},$$

where $\Delta = \{\delta(u, a, Au + Ba) | u \in Q, a \in X\}$ *for every* $i \geq 1$*. Since* $Q_{act}(t_i)$ *is a fuzzy set, to show that a state* u *belongs to* $Q_{act}(t_i)$ *and* T *is a subset of* $Q_{act}(t_i)$*, we write* $u \in Domain(Q_{act}(t_i))$*. Hereafter, we denote these notations by*

$$u \in Q_{act}(t_i) \;\; \text{and}\; T \subseteq Q_{act}(t_i).$$

Definition 2.4. *[29] Let* $\tilde{F}_v = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ *be a VGFA. We define the max-min vector general fuzzy automaton* $\tilde{F}^*_v = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ *such that* $\tilde{\delta}^* : Q_{act} \times X^* \times Q \to [0,1] \times [0,1]$*, where* $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), ...\}$ *and for every* $i \geq 0$,

$$\tilde{\delta}^*((u_1, \mu^{t_i}(u_1)), \Lambda, u_2) = \begin{cases} 1 & \text{if } u_1 = u_2 \\ 0 & \text{otherwise} \end{cases}. \tag{2.1}$$

Also for every $i \geq 0$, $\tilde{\delta}^*((u_1, \mu^{t_i}(u_1)), a_{i+1}, u_2) = \tilde{\delta}((u_1, \mu^{t_i}(u_1)), a_{i+1}, u_2)$ *and recursively,*

$$\begin{aligned} \tilde{\delta}^*((u_1, \mu^{t_0}(u_1)), a_1a_2...a_{n-1}, u_n) = \vee\{&\tilde{\delta}((u_1, \mu^{t_0}(u_1)), a_1, u_2) \wedge \tilde{\delta}((u_2, \mu^{t_1}(u_2)), a_2, u_3) \wedge ... \\ &\wedge \tilde{\delta}((u_{n-1}, \mu^{t_{n-2}}(u_{n-1}), a_{n-1}, u_n) \big| u_2 \in Q_{act}(t_1), ..., u_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{2.2}$$

in which $a_i \in X$ *for every* $1 \leq i \leq n-1$*, and assume that* a_{i+1} *is the entered input at time* t_i*, for every* $0 \leq i \leq n-2$.

Actually, the fact that the vector GFA acts in discrete time we will also use the notation

$$u_{t+1} = Au_t + Ba_t, \tag{2.3}$$

$$\omega_{u_t} = Cu_t, \tag{2.4}$$

where $u_t \in Q_{act}(t), a_t \in X, \omega_{q_t} \in Z$ *for* $t \in N_0$.

Since field F and matrices A, B and C entirely characterize the vector general fuzzy automaton(VGFA), we shall also denote automaton by 13-tuple machine $(F, Q, X, A, B, C, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$.

Definition 2.5. *[29] Let* $\tilde{F} = (F, Q, X, A, B, C, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ *be a max-min VGFA. The language with threshold* α, $\alpha \in [0,1]$, *recognized by* $\tilde{F}$ *is a subset of* F_m^* *defined by*

$$\mathcal{L}^{\alpha}(\tilde{F}) = \{x \in F_m^* | \tilde{\delta}^*((u, \mu^{t_0}(u)), x, v) \wedge \tilde{\omega}((v, \mu^{t_0+|x|}(v)), z) > \alpha, \text{ for some } u, v \in F_k, z \in Z, \text{ for } i = 1, ..., n\}.$$

3. Refining equivalence on vector general fuzzy automata

Definition 3.1. *Let* $\tilde{F}_{vi} = (F^i, Q_i, X, A_i, B_i, C_i, \tilde{R}_i = (u_i, 1), Z_i, \tilde{\delta}_i, \tilde{\omega}_i, F_1, F_2, F_3, F_4)$ *be two VGFAs over field* F^i, *where* $i = 1, 2$. *Let* $\cong$ *be a binary relation on* $Q_1 \times Q_2$. *For* $D \subseteq Q_1$, *define*

$$S_{\cong}(D) = \{u \in Q_2 \big| \exists u' \in D, u \cong u'\}.$$

The relation $\cong$ can be extended to subsets of Q_1 and Q_2. For $D \subseteq Q_1$ and $E \subseteq Q_2$ we have

$$\begin{aligned} D \cong E &\Leftrightarrow D \subseteq S_{\cong}(E) \text{ and } E \subseteq S_{\cong}(D) \\ &\Leftrightarrow (\forall u \in Q_1)\ (\exists u' \in Q_2) u \cong u' \text{ and } (\forall u' \in Q_2)(\exists u \in Q_1) u \cong u'. \end{aligned}$$

Definition 3.2. *Let* $\tilde{F}_{vi} = (F^i, Q_i, X, A_i, B_i, C_i, \tilde{R}_i = (u_i, 1), Z_i, \tilde{\delta}_i, \tilde{\omega}_i, F_1, F_2, F_3, F_4)$ *be two VGFAs. Then relation* $\cong$ *is called a refining relation if the following hold:*

1. $u_0^1 \cong u_0^2$,
2. $u_1 \cong u_2$ *implies that if there exists* $u_1' \in Q_1$ *such that* $\tilde{\delta}_1((u_1, \mu^{t_i}(u_1)), a, u_1') > 0$, *then there exists* $u_2' \in Q_2$ *such that* $\tilde{\delta}_2((u_2, \mu^{t_i}(u_2)), a, u_2') > 0$ *and* $u_1' \cong u_2'$, *where* $a \in X$,
3. $u_1 \cong u_2$ *implies that if there exists* $z_1 \in Z_1$ *such that* $z_1 = C_1 u_1$, *then there exists* $z_2 \in Z_2$ *such that* $z_2 = C_2 u_2$ *and vice versa.*

Let $\tilde{F}_{vi} = (F^i, Q_i, X, A_i, B_i, C_i, \tilde{R}_i = (u_i, 1), Z_i, \tilde{\delta}_i, \tilde{\omega}_i, F_1, F_2, F_3, F_4), i = 1, 2, 3,$ be three VGFAs. Then

1. Clearly, if $\cong$ is a refining relation between $\tilde{F}_{v1}$ and $\tilde{F}_{v2}$, then its reverse is a refining relation between $\tilde{F}_{v1}$ and $\tilde{F}_{v2}$. So, relation $\cong$ is a symmetric relation between Q_1 and Q_2.
2. if $\cong_1$ is a refining relation between $\tilde{F}_{v1}$ and $\tilde{F}_{v2}$ and $\cong_2$ is a refining relation between $\tilde{F}_{v2}$ and $\tilde{F}_{v3}$, then their composition

$$\cong = \cong_1 \circ \cong_2 = \{(u, v) \big| \exists w, u \cong w \text{ and } w \cong v\},$$

is a refining relation between $\tilde{F}_{v1}$ and $\tilde{F}_{v3}$ and it implies that the relation $\cong$ is a transitive relation between Q_1 and Q_3.

Theorem 3.1. *Let $\tilde{F}_{vi} = (F^i, Q_i, X, A_i, B_i, C_i, \tilde{R}_i = (u_i, 1), Z_i, \tilde{\delta}_i, \tilde{\omega}_i, F_1, F_2, F_3, F_4), i = 1, 2,$ be two VGFAs and $\cong$ be a refining relation between $\tilde{F}_{v1}$ and $\tilde{F}_{v2}$. Let $u \cong u'$. If there exists $u_1 \in Q_1$ such that $\tilde{\delta}_1((u, \mu^{t_i}(u)), x, u_1) > 0$, then there exists $u'_1 \in Q_2$ such that $\tilde{\delta}_2((u', \mu^{t_i}(u')), x, u'_1) > 0$ and $u_1 \cong u'_1$, where $x \in X^*$.*

Proof. We prove the claim by induction on $|x| = n$. Let $u \cong u'$ and $n = 0$. Then we have $x = \Lambda$. So, by Definition 2.4, $\tilde{\delta}_1((u, \mu^{t_i}(u)), \Lambda, u) > 0$ and $\tilde{\delta}_2((u', \mu^{t_i}(u')), \Lambda, u') > 0$ and $u \cong u'$. Now, let the claim holds, for every $y \in X^*$, such that $|y| = n - 1$ and $n \geq 2$. Let $u \cong u'$ and $x = ya$, where $y \in X^*, a \in X$ and $|x| = n$. Then by induction hypothesis if there exists $u_1 \in Q_1$ such that $\tilde{\delta}_1((u, \mu^{t_i}(u)), y, u_1) > 0$, then there exists $u'_1 \in Q_2$ such that $\tilde{\delta}_2((u', \mu^{t_i}(u')), y, u'_1) > 0$ and $u_1 \cong u'_1$. Since $u_1 \cong u'_1$ and by Definition 3.2, if there exists $u_2 \in Q_1$ such that $\tilde{\delta}_1((u_1, \mu^{t_{i+1}}(u_1)), a, u_2) > 0$, then there exists $u'_2 \in Q_2$ such that $\tilde{\delta}_2((u'_1, \mu^{t_{i+1}}(u'_1)), a, u'_2) > 0$ and $u_2 \cong u'_2$. Therefore, we can show that if there exists $u_2 \in Q_1$ such that $\tilde{\delta}_1((u, \mu^{t_i}(u)), ya, u_2) > 0$, then there exists $u'_2 \in Q_2$ such that $\tilde{\delta}_2((u', \mu^{t_i}(u')), ya, u'_2) > 0$ and $u_2 \cong u'_2$. □

Theorem 3.2. *Let $\tilde{F}_{vi} = (F^i, Q_i, X, A_i, B_i, C_i, \tilde{R}_i = (u_0^i, 1), Z_i, \tilde{\delta}_i, \tilde{\omega}_i, F_1, F_2, F_3, F_4), i = 1, 2,$ be two VGFAs and $\cong$ be a refining relation between $\tilde{F}_{v1}$ and $\tilde{F}_{v2}$. Then they are equivalent.*

Proof. Let $x \in L(\tilde{F}_{v1})$. Then there exist $u_1 \in Q_1$ and $z \in Z_1$ such that

$$\tilde{\delta}_1^*((u_0^1, \mu^{t_0}(u_0^1)), x, u_1) \wedge \tilde{\omega}_1((u_1, \mu^{t_0+|x|}(u_1)), z) > 0.$$

So, $\tilde{\delta}_1^*((u_0^1, \mu^{t_0}(u_0^1)), x, u_1) > 0$ and $\tilde{\omega}_1((u_1, \mu^{t_0+|x|}(u_1)), z) > 0$. Therefore, $\mu^{t_0}(u_0^1) > 0$ and $u_0^1 \cong u_0^2$, so by Theorem 3.1, there exists $u'_1 \in Q_2$ such that $\tilde{\delta}_2^*((u_0^2, \mu^{t_0}(u_0^2)), x, u'_1) > 0$ and $u_1 \cong u'_1$. Since $\omega_1(u_1, z) > 0$, then $z = Cu_1$. By Definition 3.2,there exists $z' \in Z_2$ such that $z' = Cu'_1$ and $\omega_2(u'_1, z') > 0$. Therefore, $x \in L(\tilde{F}_{v2})$. So, $L(\tilde{F}_{v1}) \subseteq L(\tilde{F}_{v2})$. Similarly, $L(\tilde{F}_{v2}) \subseteq L(\tilde{F}_{v1})$. Hence, the claim holds. □

Suppose that $|\tilde{F}_v|$ denote the cardinality of states of VGFA $\tilde{F}_v = (F, Q, X, A, B, C, \tilde{R} = (u_0, 1), Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$.

Definition 3.3. *Let $\tilde{F}_v$ be a VGFA. If $|\tilde{F}_v| \leq |\tilde{F}'_v|$, for every VGFA $\tilde{F}'_v$, then $\tilde{F}_v$ is called a minimal VGFA.*

A refining equivalence between a VGFA and itself is called a refining equivalence on VGFA.

Theorem 3.3. *Let $\tilde{F}_v$ be a VGFA and $\mathcal{M}$ be the set of all refining equivalences on $\tilde{F}_v$. Then the union of all refining equivalences in $\mathcal{M}$ is a refining equivalence on $\tilde{F}_v$ and also it is a equivalence relation on Q.*

Proof. Let $\{\cong_i \,\big|\, i \in I\}$ be a nonempty set of refining equivalence between $\tilde{F}_{v1}$ and $\tilde{F}_{v2}$. Consider $\cong = \cup_{i \in I} \cong_i$. $u_0^1 \cong_i u_0^2$ becouse of I is nonempty. Then $u_0^1 \cong u_0^2$. Let $u \cong u'$. Then $u \cong_i u'$, for some

$i \in I$. If there exists $u_1 \in Q_1$ such that $\tilde{\delta}_1^*((u, \mu^{t_i}(u)), x, u_1) > 0$, then there exists $u_1' \in Q_2$ such that $\tilde{\delta}_2^*((u', \mu^{t_i}(u')), x, u_1') > 0$ and $u_1 \cong_i u_1'$, for some $i \in I$. So, $u_1 \cong u_1'$. Also, Let $u \cong u'$. Then $u \cong_i u'$, for some $i \in I$. If $z_1 \in Z_1$ such that $z_1 = C_1 u$, then there exists $z_2 \in Z_2$ such that $z_2 = C_2 u'$ and vice versa. Then it is proved that the union of all refining equivalences in $\mathcal{M}$ is a refining equivalence on $\tilde{F}_v$. On the other hand it is symmetric and transitive. Therefore, the union of all refining equivalences is an equivalence relation on Q. $\square$

Let $\approx$ be the union of all refining equivalences on $\tilde{F}_v$ and $M \subseteq Q$. Define

(1) $[E] = \{F \big| E \approx F\}$,
(2) $\sim = \{(E, [E]) \big| E \in Q\}$,
(3) $M' = \{[E] \big| E \in M\}$.

Definition 3.4. *Let $\tilde{F}_v = (F, Q, X, A, B, C, \tilde{R} = (u_0, 1), Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a VGFA and $\approx$ be the union of all refining equivalences on $\tilde{F}_v$. Define $\tilde{F}_v' = (F, Q', X, A, B, C, \tilde{R}', Z, \tilde{\delta}', \tilde{\omega}', F_1, F_2, F_3, F_4)$ the quotient of VGFA $\tilde{F}_v$, where $Q' = \{[E] | E \in Q\}$, $R' = [u_0]$, $\mu^{t_0}([u_0]) = 1$, $\delta' : Q' \times X \times Q' \to [0, 1]$ by $\delta'([E], a, [F]) = \vee\{\delta(E', a, F') \big| E \approx E', F \approx F'\}$, $\omega'([E], z) = \vee\{\omega(E', z') \big| E \approx E',$ for some $z' \in Z\}$. Clearly, δ' and ω' are well-defined.*

Example 3.1. *Let $\tilde{F}_v = (F, Q, X, A, B, C, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a VGFA defined over field $F = Q_2$ of integers modulo 2 such that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $Q = \{u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ $\tilde{R} = (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1)$, $X = \{a = [0], b = [1]\}$, $Z = \{\begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}\}$ and δ and ω are as follows:*

$$\delta(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0.4, \qquad \delta(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 0.6,$$

$$\delta(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0.6, \qquad \delta(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 0.7,$$

$$\delta(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 0.7, \qquad \delta(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0.9,$$

$$\delta(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 0.8, \qquad \delta(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0.4,$$

$$\omega(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\end{bmatrix}) = 0.4, \qquad \omega(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\end{bmatrix}) = 0.7,$$

$$\omega(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\end{bmatrix}) = 0.2, \qquad \omega(\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\end{bmatrix}) = 0.5.$$

Then we have the quotient of VGFA $\tilde{F}_v$ as follows: $Q' = \{[\begin{bmatrix}0\\0\end{bmatrix}], [\begin{bmatrix}1\\0\end{bmatrix}]\}$, $\tilde{R}' = ([\begin{bmatrix}0\\0\end{bmatrix}], 1)$, *and δ' and ω' are as follows:*

$$\delta(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}) = 0.8, \qquad \delta(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}) = 0.6,$$

$$\delta(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}) = 0.7, \qquad \delta(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}) = 0.9,$$

$$\omega(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\end{bmatrix}) = 0.5, \qquad \omega(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\end{bmatrix}) = 0.7.$$

Theorem 3.4. *Let $\tilde{F}_v$ be a VGFA as Definition 3.4, and $\tilde{F}'_v$ be the quotient VGFA of $\tilde{F}_v$. Then $\sim$ is a refining equivalence between $\tilde{F}_v$ and $\tilde{F}'_v$.*

Proof. Clearly, $u_0 \sim [u_0]$. Let $u_1 \sim [u_2]$. Then $u_1 \approx u_2$. So, if there exists $u'_1 \in Q_1$ such that $\tilde{\delta}_1((u_1, \mu^{t_i}(u_1)), a,$
$u_1) > 0$, then there exists $u'_2 \in Q_1$ such that $\tilde{\delta}_1((u_2, \mu^{t_i}(u_2)), a, u'_2) > 0$. Therefore, $\tilde{\delta}'_1(([u_2], \mu^{t_i}([u_2])), a, [u'_2]) > 0$, such that $u'_1 \approx u'_2 \sim [u'_2]$. On the other hand, if there exists $[u'_2] \in Q'$ such that $\tilde{\delta}'_1(([u_2], \mu^{t_i}([u_2])), a, [u'_2]) > 0$, then by Definition 3.4, there exists $u \approx u_2$ and $u' \approx u'_2$, such that $\tilde{\delta}_1((u, \mu^{t_i}(u)), a, u') > 0$. Since, $u_1 \approx u_2 \approx u$, then there exists $u'_1 \in Q_1$, such that $\tilde{\delta}_1((u_1, \mu^{t_i}(u_1)), a, u'_1) > 0$, where $u'_1 \approx u' \approx u'_2$. So, $u'_1 \sim [u'_2]$.

Now, let $u_1 \sim [u_2]$. Then $u_1 \approx u_2$. If there exists $z_1 \in Z$ such that $z_1 = Cu_1$, then there exists $z_2 \in Z$ such that $z_2 = Cu_2$. Therefore, $\omega(u_2, z_2) > 0$. So, by Definition 3.4, there exists $z_3 \in Z$ such that $\omega'([u_2], z_3) > 0$ and $z_3 = C[u_2]$. On the other hand if there exists $z_3 \in Z$ such that $z_3 = C[u_2]$, then $\omega'([u_2], z_3) > 0$. So, there is $u \in Q$ and $z_2 \in Z$ such that $u \approx u_2$ and $\omega(u, z_2) > 0$, where $u \approx u_2$. Since $u \approx u_2 \approx u_1$, then there exists $z \in Z$ in which $\omega(u_1, z) > 0$ and the claim hold. □

Theorem 3.5. *Let $\tilde{F}_v$ be a VGFA and $\tilde{F}'_v$ be the quotient of $\tilde{F}_v$ defined in Definition 3.4. Then they have the same behavior.*

Proof. By Theorems 3.2 and 3.4, clearly, $\tilde{F}_v$ and $\tilde{F}'_v$ have the same behavior. □

Theorem 3.6. *The only refining equivalence on the quotient VGFA $\tilde{F}'_v$ is the identity relation.*

Proof. Let $\equiv$ be a refining equivalence on $\tilde{F}'_v$ and let $[u] \equiv [u']$, but $[u] \neq [u']$. Now, let $\sim \circ \equiv \circ (\sim)^{-1}$, where $(\sim)^{-1}$ is the inverse of $\sim$. Then $u \sim [u] \equiv [u'] \sim u'$. So, $u \approx u'$, that is a contradiction. Hence $\equiv$ is the identity relation. □

Definition 3.5. *Let $\tilde{F}_v$ be a VGFA. Then $u \in Q$ is called accessible if there exists $x \in X^*$ such that* $\tilde{\delta}^*((u_0, \mu^{t_0}(u_0)),$
$x, u) > 0$.

Theorem 3.7. *Let $\tilde{F}_v$ be a VGFA with no inaccessible sates and $\equiv$ be the greatest refining equivalence on Q. Then the quotient VGFA $\tilde{F}'_v$ is the minimal VGFA.*

Proof. It is suffice to show that for every VGFA $\tilde{F}_{v2}$ with no inaccessible states that there is a refining equivalence between $\tilde{F}_v$ and $\tilde{F}_{v2}$, any refining equivalence between $\tilde{F}'_v$ and $\tilde{F}'_{v2}$ gives a one-to-one correspondence between the states of $\tilde{F}'_v$ and $\tilde{F}'_{v2}$, where $\tilde{F}'_{v2}$ is the quotient VGFA $\tilde{F}_{v2}$ according to greatest refining equivalence $\equiv$. Let $\approx$ be a refining equivalence between $\tilde{F}'_v$ and $\tilde{F}'_{v2}$. Suppose that every states $\tilde{F}'_{v2}$ is related to at least one state of $\tilde{F}'_v$ and every state $\tilde{F}'_v$ is related to at most one states $\tilde{F}_{v2}$. So, the composition $\approx$ with its inverse would not be the identity of $\tilde{F}'_v$, which is a contradiction with Theorem 3.6. Therefore, $\approx$ gives a one-to-one correspondece between the states of $\tilde{F}'_v$ and $\tilde{F}'_{v2}$. □

4. Conclusion

In fact, general fuzzy automaton over a field processing has appeared frequently in various areas of mathematics. General fuzzy automaton over a field are used for generation of linear codes, detection and correction of errors, construction of testing sequence, and generation of pseudo-random sequences of numbers. They are also used in experiments that require Mote Carlo methods, in the protection of data stored in computer systems and radiolocation.

The current study aimed at investigating the refining equivalence between two VGFAs. Moreover, it was shown that the union of all refining equivalences is a refining equivalence, too. A connection between the refining equivalence and the quotient VGFA was then introduced and it was shown that any quotient of a given VGFA and the VGFA itself have the same language. By using the equivalence classes, a minimal VGFA was then introduced and studied. In particular, it was shown that any quotient of a given VGFA and the VGFA itself have the same language.

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