Int. J. Anal. Appl. (2023), 21:95

International Journal of Analysis and Applications

Solving Coupled Impulsive Fractional Differential Equations With Caputo-Hadamard Derivatives in Phase Spaces

Hasanen A. Hammad[1,2,*], **Hassen Aydi**[3,4,*], **Doha A. Kattan**[5]

[1] *Department of Mathematics, Unaizah College of Sciences and Arts, Qassim University, Buraydah 52571, Saudi Arabia*

[2] *Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*

[3] *Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, 4000, Tunisia*

[4] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*

[5] *Department of Mathematics, College of Sciences and Art, King Abdulaziz University, Rabigh, Saudi Arabia*

[*] *Corresponding authors:* hassanein_hamad@science.sohag.edu.eg; hassen.aydi@isima.rnu.tn

Abstract. In this manuscript, we incorporate Caputo-Hadamard derivatives in impulsive fractional differential equations to obtain a new class of impulsive fractional form. Further, the existence of solutions to the proposed problem has been inferred under a state-dependent delay and suitable hypotheses in phase spaces. Finally, the considered problem has been supported by an illustrative application.

1. Introduction

The modeling of several events in numerous branches of science and engineering can greatly benefit from the use of fractional differential equations (FDEs) and fractional integral equations (FIEs). In fact, there are many applications in electrochemistry, control, porous media, viscoelasticity, and other fields. We can find basic mathematics and many applications of fractional calculus in the

Received: Jul. 14, 2023.

2020 *Mathematics Subject Classification.* 26A33, 34A37.

Key words and phrases. impulsive fractional problem; Caputo-Hadamard derivative; state-dependent delay; phase space; fixed point technique.

https://doi.org/10.28924/2291-8639-21-2023-95

ISSN: 2291-8639

monographs [1, 3–9, 14]. FDEs involving the Caputo and Hadamard derivatives have recently received a lot of attention in studies; for examples, see the publications [10–13].

Modeling scientific phenomena has long employed delay differential equations (DEs) or functional DEs with or without impulse. The delay has frequently been thought of as either a fixed constant or as an integral, in which case it is referred to as a distributed delay; for examples, see the books [15–17], and the papers [18, 19].

In the study of both qualitative and quantitative theory for functional DEs, the idea of the phase space (PS) Ω is crucial. A seminormed space that satisfies the appropriate axioms is a usual choice, as described by Hale and Kato [19]. We recommend reading [11, 20, 21] for a more in-depth discussion on this subject.

However, in recent years, modeling has been suggested for complex scenarios where the delay depends on the unidentified functions; for example, see [22, 23] and the references therein. These equations are usually referred to as state-dependent delay equations. Recently, among other things, existence results for functional differential equations were developed when the solution to impulsive situations depended on the delay over a finite interval. Many papers have addressed this purpose, either by introducing the Caputo functional fractional operators or by introducing other fractional operators with state-dependent delays. For more information, see [24–33]. According to our knowledge, there are no works in the literature that discuss Caputo-Hadamard fractional (CHF) order functional DEs with state-dependent delay and impulses. The paper's goal is to get that study started.

In this study, we focus on the existence of solutions to the following initial value problems (IVPs) for coupled Impulsive fractional differential equations (IFDEs):

$$\begin{cases} {}^{CH}D^{\ell}\Im(\tau) = \chi\left(\tau, \Im_{\rho(\tau,\Im_\tau)}, \xi_{\eta(\tau,\xi_\tau)}\right), \text{ for a.e. } \tau \in K = [b, Q],\ b > 0,\ \tau \neq \tau_j,\ j = 1, 2, .., m,\ \ell \in (0, 1], \\ {}^{CH}D^{\sigma}\xi(\tau) = \overline{\chi}\left(\tau, \xi_{\eta(\tau,\xi_\tau)}, \Im_{\rho(\tau,\Im_\tau)}\right), \text{ for a.e. } \tau \in K = [b, Q],\ b > 0,\ \tau \neq \tau_j,\ j = 1, 2, .., m,\ \sigma \in (0, 1], \\ \Delta\Im|_{\tau=\tau_j} = I_j\left(\Im(\tau_j^-)\right),\quad \Delta\xi|_{\tau=\tau_j} = I_j\left(\xi(\tau_j^-)\right),\ j = 1, 2, .., m, \\ \Im(\tau) = \phi(\tau),\ \xi(\tau) = \psi(\tau),\ \tau \in (-\infty, b], \end{cases} \tag{1.1}$$

where ${}^{CH}D^{\ell}$ and ${}^{CH}D^{\sigma}$ are CHF derivatives with order ℓ and σ, respectively, $\chi, \overline{\chi} : K \times A \times A \to \mathbb{R}$ are given functions, $\phi, \psi \in \Omega$, the functions $I_j : \mathbb{R} \to \mathbb{R}$ are continuous, $j = 1, 2, .., m$. Further, $b < \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1=Q}$, $\Delta\Im|_{\tau=\tau_j} = \Im(\tau_j^+) - \Im(\tau_j^-)$, $\Im(\tau_j^+) = \lim_{j\to\epsilon^+} \Im(\tau_j + \epsilon)$, $\Im(\tau_j^-) = \lim_{j\to\epsilon^-} \Im(\tau_j + \epsilon)$, similarly $\Delta\xi|_{\tau=\tau_j}$ and Ω is an abstract PS.

2. Preliminaries

This part is devoted to present some preliminary facts that will be used in the sequel.

Assume that $C(K, \mathbb{R})$ refers to the space of all continuous functions on the interval K. It is a Banach space (BS) under the norm

$$\|\Im\|_\infty = \sup\{|\Im(\tau)| : \tau \in K\}.$$

Assume also $AC(K,\mathbb{R})$ refers to the space of all absolutely continuous functions $\Im : K \to \mathbb{R}$, and $AC^n_\kappa(K,\mathbb{R}) = \{\omega : K \to \mathbb{R} : \kappa^{n-1}\omega(\tau) \in AC(K,\mathbb{R})\}$, where $\kappa = \tau\frac{d}{d\tau}$.

Definition 2.1. *[4] For the function $\omega : [b,c] \to \mathbb{R}$, the HF integral of order ℓ is described as*

$$I^\ell_b\omega(\tau) = \frac{1}{\Gamma(\ell)}\int_b^\tau \left(\log\frac{\tau}{s}\right)^{\ell-1}\frac{\omega(s)}{s}ds, \;\; b,c \geq 0,$$

provided that the integral exists.

Definition 2.2. *[34] Assume that $AC^n_\kappa[b,c] = \{\varpi : [b,c] \to \mathbb{C} : \kappa^{n-1}\varpi(\tau) \in AC[b,c]\}$, $0 < b < c < \infty$, let $\ell \in \mathbb{C}$ such that $Re(\ell) \geq 0$. The CH derivative of fractional order ℓ for the function $\varpi \in AC^n_\kappa[b,c]$ is defined as follows:*

(i) *If $\ell = n \in \mathbb{N}$, then*

$$\left({}^{CH}D^\ell_b\varpi\right)(\tau) = \kappa^n\varpi(\tau).$$

(ii) *If $\ell \notin \mathbb{N}$, and $n = [Re(\ell)] + 1$, then*

$$\left({}^{CH}D^\ell_b\varpi\right)(\tau) = \frac{1}{\Gamma(n-\ell)}\left(\tau\frac{d}{d\tau}\right)^n\int_b^\tau\left(\log\frac{\tau}{s}\right)^{n-\ell-1}\kappa^n\varpi(s)\frac{ds}{s},$$

where $[Re(\ell)]$ is the integer part of the real number $Re(\ell)$ and $\log(.) = \log_e(.)$.

Lemma 2.1. *[34] Suppose that $\Im \in AC^n_\kappa[b,c]$ and $\ell \in \mathbb{C}$, then*

$$I^\ell_b\left({}^{CH}D^\ell_b\Im\right)(\tau) = \Im(\tau) - \sum_{j=0}^{n-1}\frac{\kappa^j\Im(b)}{j!}\left(\log\frac{\tau}{b}\right)^j.$$

3. Main assumptions

In this study, we will utilize an axiomatic definition of phase Ω that Hale and Kato introduced in [19] and adhere to the language from [35], but we will also add some transformations. As a result, the seminormed linear space of functions $(\Omega, \|.\|_\Omega)$ will map $(-\infty, b]$ into $\mathbb{R}$. Because we want a solution to the problem (1.1) to be continuous on $(\tau_m, \tau_{m+1}]$ and the left hand limit exists for every τ_m, the first two axioms on Ω are necessary. The axioms that Ω must adhere to are as follows:

(S_1) If $\Im, \xi : (-\infty, Q] \to \mathbb{R}$, $Q > 0$, $\Im_0, \xi_0 \in \Omega$, and $\Im(\tau_j^-)$, $\Im(\tau_j^+)$, $\xi(\tau_j^-)$, $\xi(\tau_j^+)$ exist with $\Im(\tau_j^-) = \Im(\tau_j)$ and $\xi(\tau_j^-) = \xi(\tau_j)$, $j = 1,2,..,m$, then for every $\tau \in [b,Q]\backslash\{\tau_1,\tau_2,\cdots,\tau_m\}$ the circumstances below are true:

(1) $\Im_\tau, \xi_\tau \in \Omega$; and $\Im_\tau$ and ξ_τ are continuous functions on $[b,Q]\backslash\{\tau_1,\tau_2,\cdots,\tau_m\}$;

(2) there are constants $X, Y > 0$ such that $|\Im(\tau)| \leq X\|\Im_\tau\|_\Omega$ and $|\xi(\tau)| \leq Y\|\xi_\tau\|_\Omega$;

(3) there are two continuous functions $B(.), C(.) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $\Im$ and two locally bounded functions $D(.), E(.) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of ξ, such that

$$\begin{cases} \|\Im_\tau\|_\Omega \le B(\tau) \sup\{|\Im(s)| : s \in [b, Q]\} + D(\tau)\|\Im_0\|_\Omega, \\ \|\xi_\tau\|_\Omega \le C(\tau) \sup\{|\xi(s)| : s \in [b, Q]\} + E(\tau)\|\xi_0\|_\Omega. \end{cases}$$

Clearly,

$$\|(\Im_\tau, \xi_\tau)\|_{\Omega\times\Omega} \le \Lambda(\tau) \sup\left\{|\Im(s) + \xi(s)| : s \in [b, Q] + \Lambda^*(\tau)\|(\Im_0, \xi_0)\|_{\Omega\times\Omega}\right\},$$

where $\|(\Im_\tau, \xi_\tau)\|_{\Omega\times\Omega} = \|\Im_\tau\|_\Omega + \|\xi_\tau\|_\Omega$, $\Lambda(\tau) = B(\tau) + C(\tau)$, $\Lambda^*(\tau) = D(\tau) + E(\tau)$.

(S_2) The space Ω is complete.

Set

$$\begin{aligned} B_Q &= \sup\{B(\tau) : \tau \in [b, Q]\}, \ C_Q = \sup\{C(\tau) : \tau \in [b, Q]\} \\ D_Q &= \sup\{D(\tau) : \tau \in [b, Q]\}, \ E_Q = \sup\{E(\tau) : \tau \in [b, Q]\}, \\ \Lambda_Q &= \sup\{\Lambda(\tau) : \tau \in [b, Q]\}, \ \Lambda_Q^* = \sup\{\Lambda(\tau) : \tau \in [b, Q]\}. \end{aligned}$$

Consider the following spaces:

$$AC'(K, \mathbb{R}) = \left\{ \begin{array}{l} \Im : K \to \mathbb{R}, \ \Im \in AC_\mu\left((\tau_j, \tau_{j+1}], \mathbb{R}\right), \ j = 1, \cdots, m, \ \text{and there exist} \\ \\ \Im(\tau_j^+) \text{ and } \Im(\tau_j^-), \ j = 1, \cdots, m, \ \text{with } \Im(\tau_j^-) = \Im(\tau_j), \end{array} \right\}$$

and

$$AC''(K, \mathbb{R}) = \left\{ \begin{array}{l} \xi : K \to \mathbb{R}, \ \xi \in AC_\mu\left((\tau_j, \tau_{j+1}], \mathbb{R}\right), \ j = 1, \cdots, m, \ \text{and there exist} \\ \\ \xi(\tau_j^+) \text{ and } \xi(\tau_j^-), \ j = 1, \cdots, m, \ \text{with } \xi(\tau_j^-) = \xi(\tau_j). \end{array} \right\}$$

The above spaces are BSs with norms

$$\|\Im\|_{AC'} = \sup\{\|\Im(\tau)\|_\mathbb{R} : \tau \in [b, Q]\} \ \text{and} \ \|\xi\|_{AC''} = \sup\{\|\xi(\tau)\|_\mathbb{R} : \tau \in [b, Q]\},$$

respectively.

Clearly, the product space $\left(\beth = AC' \times AC'', \mathbb{R}\right)$ is a BS with the norm

$$\|(\Im, \xi)\|_\beth = \|\Im\|_{AC'} + \|\xi\|_{AC''}.$$

Put

$$\Omega_Q = \{\Im, \xi : (-\infty, Q] \to \mathbb{R}\backslash (\Im, \xi) \in \beth(K, \mathbb{R}) \cap \Omega\} \ \text{and} \ K' = K\backslash\{\tau_1, \tau_2, \cdots, \tau_m\}.$$

Definition 3.1. *Suppose that the functions $\Im, \xi \in \Omega_Q$ have ℓ^{th} and σ^{th} derivative on K', respectively. We say that the pair $(\Im, \xi)$ is a solution to Problem (1.1) if $\Im$ and ξ satisfy (1.1).*

The next result will be helpful in the following as a result of Lemma 2.1.

Lemma 3.1. *Assume that $\ell, \sigma \in (0,1]$ and $\rho, \eta \in \beth(K,\mathbb{R})$. The pair $(\Im, \xi)$ is a solution to the FIEs*

$$\Im(\tau) = \begin{cases} \Im_b + \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \frac{\rho(s)}{s} ds, \ \text{if } \tau \in [b, \tau_1], \\ \Im_b + \frac{1}{\Gamma(\ell)} \sum\limits_{j=1}^m \int\limits_{\tau_{j-1}}^{\tau_j} \left(\log \frac{\tau_j}{s}\right)^{\ell-1} \frac{\rho(s)}{s} ds + \frac{1}{\Gamma(\ell)} \int\limits_{\tau_j}^{\tau} \left(\log \frac{\tau}{s}\right)^{\ell-1} \frac{\rho(s)}{s} ds \\ + \sum\limits_{j=1}^m I_j(\Im(\tau_j^-)), \ \text{if } \tau \in [\tau_j, \tau_{j+1}), \ j = 1, \cdots, m, \end{cases}$$

and

$$\xi(\tau) = \begin{cases} \xi_b + \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\eta(s)}{s} ds, \ \text{if } \tau \in [b, \tau_1], \\ \xi_b + \frac{1}{\Gamma(\sigma)} \sum\limits_{j=1}^m \int\limits_{\tau_{j-1}}^{\tau_j} \left(\log \frac{\tau_j}{s}\right)^{\sigma-1} \frac{\eta(s)}{s} ds + \frac{1}{\Gamma(\sigma)} \int\limits_{\tau_j}^{\tau} \left(\log \frac{\tau}{s}\right)^{\sigma-1} \frac{\eta(s)}{s} ds \\ + \sum\limits_{j=1}^m I_j(\xi(\tau_j^-)), \ \text{if } \tau \in [\tau_j, \tau_{j+1}), \ j = 1, \cdots, m, \end{cases}$$

if and only if $\Im$ and ξ are a solution to the fractional IVP

$$\begin{cases} {}^{CH}D_b^\ell \Im(\tau) = \rho(\tau), \ \text{for each } \tau \in K', \\ {}^{CH}D_b^\sigma \xi(\tau) = \eta(\tau), \ \text{for each } \tau \in K', \\ \Delta\Im|_{\tau=\tau_j} = I_j\left(\Im(\tau_j^-)\right), \ \text{and} \ \Delta\xi|_{\tau=\tau_j} = I_j\left(\xi(\tau_j^-)\right), \ j = 1, \cdots, m, \\ \Im(b) = \Im_b, \ \text{and } \xi(b) = \xi_b, \ \tau \in (-\infty, b]. \end{cases}$$

It is necessary to present the following postulates.

(P_1) The functions $\tau \to \phi_\tau$ and $\tau \to \psi_\tau$ are continuous from

$$z(\rho^-, \eta^-) = \{(\rho(s,\phi), \eta(s,\psi)) : (s,\phi), (s,\psi) \in K \times \Omega, \ \rho(s,\phi) \le 0 \text{ and } \eta(s,\psi) \le 0\}$$

into Ω and there is a bounded and continuous functions $M^\phi, M^\psi : z(\rho^-, \eta^-) \to (0,\infty)$ so that

$$\|\phi(\tau)\|_\Omega \le M^\phi(\tau) \|\phi\|_\Omega, \ \text{and} \ \|\psi(\tau)\|_\Omega \le M^\psi(\tau) \|\psi\|_\Omega \ \text{for every } \tau \in z(\rho^-, \eta^-).$$

(P_2) The functions $\chi, \overline{\chi} : K \times A \times A \to \mathbb{R}$ are continuous.

(P_3) There exist $u, v \in C(K, \mathbb{R}_+)$ and a continuous and nondecreasing function $\Phi : [0,\infty) \times [0,\infty) \to (0,\infty)$ such that

$$\begin{cases} |\chi(\tau, \Im, \xi)| \le u(\tau)\Phi(|\Im|, |\xi|), \ \text{for each } \tau \in K, \ \Im, \xi \in \Omega \text{ with } \left\|I^\ell u\right\|_\infty < \infty, \\ |\overline{\chi}(\tau, \xi, \Im)| \le v(\tau)\Phi(|\xi|, |\Im|), \ \text{for each } \tau \in K, \ \Im, \xi \in \Omega \text{ with } \|I^\sigma v\|_\infty < \infty. \end{cases}$$

(P_4) There are positive constants $L_1, L_1^*, \cdots, L_m, L_m^*$ so that

$$\begin{aligned} \frac{L_j}{\frac{1}{\Gamma(\ell+1)} \Phi\left(L_j, L_j^*\right) (\log(\tau))^\ell \|u\|_\infty} &> 1, \\ \text{and } \frac{L_j^*}{\frac{1}{\Gamma(\sigma+1)} \Phi\left(L_j^*, L_j\right) (\log(\tau))^\sigma \|v\|_\infty} &> 1, \ j = 1, \cdots, m. \end{aligned} \tag{3.1}$$

4. Main theorem

Now, we are able to formulate our basic theory in this part as follows:

Theorem 4.1. *According to the hypotheses $(P_1)-(P_4)$, the problem (1.1) has at least one solution on $(-\infty, Q]$.*

Proof. We split the proof into the following steps:

Step 1: Consider the problem below

$$\begin{cases} {}^{CH}D^{\ell}\Im(\tau) = \chi\left(\tau, \Im_{\rho(\tau,\Im_\tau)}, \xi_{\eta(\tau,\xi_\tau)}\right), \text{ for a.e. } \tau \in K = [b, \tau_1], \\ {}^{CH}D^{\sigma}\xi(\tau) = \overline{\chi}\left(\tau, \xi_{\eta(\tau,\xi_\tau)}, \Im_{\rho(\tau,\Im_\tau)}\right), \text{ for a.e. } \tau \in K = [b, \tau_1], \\ \Im(\tau) = \phi(\tau), \ \xi(\tau) = \psi(\tau), \ \tau \in (-\infty, b]. \end{cases}$$

Describe the operator $W : \Omega_{\tau_1} \times \Omega_{\tau_1} \to \Omega_{\tau_1}$ as

$$W(\Im, \xi)(\tau) = \begin{cases} \phi(\tau), \text{ if } \tau \in (-\infty, b], \\ \phi(b) + \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(\tau, \Im_{\rho(s,\Im_s)}, \xi_{\eta(s,\xi_s)}\right) \frac{ds}{s}, \text{ if } \tau \in [b, \tau_1], \end{cases}$$

and

$$W(\xi, \Im)(\tau) = \begin{cases} \psi(\tau), \text{ if } \tau \in (-\infty, b], \\ \psi(b) + \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \xi_{\eta(s,\xi_s)}, \Im_{\rho(s,\Im_s)}\right) \frac{ds}{s}, \text{ if } \tau \in [b, \tau_1], \end{cases}$$

Assume that $r(.), q(.) : (-\infty, \tau_1] \to \mathbb{R}$ are functions given by

$$r(\tau) = \begin{cases} \phi(\tau), \text{ if } \tau \in (-\infty, b], \\ \phi(b), \text{ if } \tau \in [b, \tau_1], \end{cases} \quad \text{and } q(\tau) = \begin{cases} \psi(\tau), \text{ if } \tau \in (-\infty, b], \\ \psi(b), \text{ if } \tau \in [b, \tau_1]. \end{cases}$$

Then $r_0 = \phi$ and $q_0 = \psi$. For each $h, \widehat{h} \in \Omega_{\tau_1}$ with $h_0 = 0 = \widehat{h}_0$, we denote by $v, \widehat{v}$ the functions defined by

$$v(\tau) = \begin{cases} 0, \text{ if } \tau \in (-\infty, b], \\ h(\tau), \text{ if } \tau \in [b, \tau_1], \end{cases} \quad \text{and } \widehat{v}(\tau) = \begin{cases} 0, \text{ if } \tau \in (-\infty, b], \\ \widehat{h}(\tau), \text{ if } \tau \in [b, \tau_1]. \end{cases}$$

If $\Im(.)$ and $\xi(.)$ fulfill the integral equations

$$\begin{cases} \Im(\tau) = \phi(b) + \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(\tau, \Im_{\rho(s,\Im_s)}, \xi_{\eta(s,\xi_s)}\right) \frac{ds}{s}, \\ \xi(\tau) = \psi(b) + \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \xi_{\eta(s,\xi_s)}, \Im_{\rho(s,\Im_s)}\right) \frac{ds}{s}, \end{cases}$$

we can split up $\Im(.)$ and $\xi(.)$ into $\Im(\tau) = v(\tau) + r(\tau)$ and $\xi(\tau) = \widehat{v}(\tau) + q(\tau)$, $\tau \in [b, \tau_1]$, which yields $\Im_\tau = v_\tau + r_\tau$ and $\xi_\tau = \widehat{v}_\tau + q_\tau$, for each $\tau \in [b, \tau_1]$ and the functions $h(.)$ and $\widehat{h}(.)$ satisfy

$$\begin{cases} h(\tau) = \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \frac{ds}{s}, \\ \widehat{h}(\tau) = \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right) \frac{ds}{s}. \end{cases}$$

Set the product

$$R_0 \times R_0 = \left\{ \left(h, \widehat{h}\right) \in \Omega_{\tau_1} \times \Omega_{\tau_1} : h_0 = 0 = \widehat{h}_0 \right\}$$

under the norm

$$\left\| \left(h, \widehat{h}\right) \right\|_0 = \left\| \left(h, \widehat{h}\right) \right\|_{\Omega\times\Omega} + \sup\left\{ \left| h(s) + \widehat{h}(s) \right| : s \in [b, \tau_1] \right\},$$

where $\left\| \left(h, \widehat{h}\right) \right\|_0 = \|(h)\|_0 + \left\|\widehat{h}\right\|_0$ and $\left\| \left(h, \widehat{h}\right) \right\|_{\Omega\times\Omega} = \|h\|_\Omega + \left\|\widehat{h}\right\|_\Omega$.

Define the operator $B : R_0 \times R_0 \to R_0$ by

$$\begin{cases} B\left(h, \widehat{h}\right)(\tau) = \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \frac{ds}{s}, \\ B\left(\widehat{h}, h\right)(\tau) = \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right) \frac{ds}{s}. \end{cases}$$

It is obvious that the operator B possessing a coupled fixed point (CFP) is equivalent to the operator W having a CFP. So, we must demonstrate that B has a CFP, which is the solution to problem (1.1). For this regards, the Leray-Schauder alternate will be employed as the following claims:

Claim (i): Show that B is continuous. Assume that $\{h_n\}$ and $\{\widehat{h}_n\}$ are two sequences such that $h_n \to h$ and $\widehat{h}_n \to \widehat{h}$ in R_0. Then

$$\begin{aligned} & \left| B\left(h_n, \widehat{h}_n\right)(\tau) - B\left(h, \widehat{h}\right)(\tau) \right| \\ \leq\ & \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \left| \chi\left(s, v_{\rho(s,v_{ns}+r_s)} + r_{\rho(s,v_{ns}+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_{ns}+q_s)} + q_{\eta(s,\widehat{v}_{ns}+q_s)}\right) \right. \\ & \left. - \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \right| \frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} & \left| B\left(\widehat{h}_n, h_n\right)(\tau) - B\left(\widehat{h}, h\right)(\tau) \right| \\ \leq\ & \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \left| \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_{ns}+q_s)} + q_{\eta(s,\widehat{v}_{ns}+q_s)}, v_{\rho(s,v_{ns}+r_s)} + r_{\rho(s,v_{ns}+r_s)}\right) \right. \\ & \left. - \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right) \right| \frac{ds}{s}. \end{aligned}$$

The continuity of χ and $\overline{\chi}$ implies that

$$\left\|B\left(h_n,\widehat{h}_n\right)-B\left(h,\widehat{h}\right)\right\|\to 0,\ \text{and}\ \left\|B\left(\widehat{h}_n,h_n\right)-B\left(\widehat{h},h\right)\right\|\to 0,\ \text{as}\ n\to\infty.$$

Claim (ii): Prove that B maps bounded sets into bounded sets in R_0. In fact, it suffices to demonstrate that for any $\lambda>0$, there exists a positive constant Θ such that $h,\widehat{h}\in\Omega_\lambda=\left\{h,\widehat{h}\in R_0:\left\|\left(h,\widehat{h}\right)\right\|_{R_0\times R_0}\leq\lambda\right\}$,

$$\left\|B\left(h,\widehat{h}\right)\right\|\leq\Theta\ \text{and}\ \left\|B\left(\widehat{h},h\right)\right\|\leq\Theta.$$

To achieve this, using hypothesis (P_3), we get

$$\begin{aligned}
&\left|B\left(h,\widehat{h}\right)(\tau)\right|\\
\leq\ &\frac{1}{\Gamma(\ell)}\int_b^\tau\left(\log\frac{\tau}{s}\right)^{\ell-1}\left|\chi\left(s,v_{\rho(s,v_s+r_s)}+r_{\rho(s,v_s+r_s)},\widehat{v}_{\eta(s,\widehat{v}_s+q_s)}+q_{\eta(s,\widehat{v}_s+q_s)}\right)\right|\frac{ds}{s}\\
\leq\ &\frac{1}{\Gamma(\ell)}\int_b^\tau\left(\log\frac{\tau}{s}\right)^{\ell-1}\left[u(s)+\widehat{u}(s)\left\|(v_{\rho(s,v_s+r_s)}+r_{\rho(s,v_s+r_s)},\widehat{v}_{\eta(s,\widehat{v}_s+q_s)}+q_{\eta(s,\widehat{v}_s+q_s)}\right\|_{\Omega\times\Omega})\right]\frac{ds}{s}\\
\leq\ &\frac{1}{\Gamma(\ell)}\int_b^\tau\left(\log\frac{\tau}{s}\right)^{\ell-1}u(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\ell)}\int_b^\tau\left(\log\frac{\tau}{s}\right)^{\ell-1}\widehat{u}(s)\left\|(v_{\rho(s,v_s+r_s)}+r_{\rho(s,v_s+r_s)},\widehat{v}_{\eta(s,\widehat{v}_s+q_s)}+q_{\eta(s,\widehat{v}_s+q_s)}\right\|_{\Omega\times\Omega})\frac{ds}{s}\\
\leq\ &\frac{(\log Q)^\ell}{\Gamma(\ell+1)}\|u\|_\infty+\frac{(\log Q)^\ell}{\Gamma(\ell+1)}\|\widehat{u}(s)\|_\infty\left(\left\|\Im_{\rho(s,v_s+r_s)},\xi_{\rho(s,v_s+r_s)}\right\|_{\Omega\times\Omega}\right)\frac{ds}{s}\\
\leq\ &\frac{(\log Q)^\ell}{\Gamma(\ell+1)}\|u\|_\infty+\frac{(\log Q)^\ell}{\Gamma(\ell+1)}\|\widehat{u}(s)\|_\infty\left(\Lambda_Q\|(\Im,\xi)\|+\Lambda_Q|\phi(b)+\psi(b)|+\Lambda_Q^*\|\phi,\psi\|_{\Omega\times\Omega}\right)\\
\leq\ &\frac{(\log Q)^\ell}{\Gamma(\ell+1)}\|u\|_\infty+\frac{(\log Q)^\ell}{\Gamma(\ell+1)}\|\widehat{u}(s)\|_\infty\left(\Lambda_Q\lambda+\Lambda_Q|\phi(b)+\psi(b)|+\Lambda_Q^*\|\phi,\psi\|_{\Omega\times\Omega}\right)=\Theta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|B\left(\widehat{h},h\right)(\tau)\right|\\
\leq\ &\frac{(\log Q)^\sigma}{\Gamma(\rho+1)}\|v\|_\infty+\frac{(\log Q)^\sigma}{\Gamma(\sigma+1)}\|\widehat{v}(s)\|_\infty\left(\Lambda_Q\lambda+\Lambda_Q|\phi(b)+\psi(b)|+\Lambda_Q^*\|\psi,\phi\|_{\Omega\times\Omega}\right)=\Theta,
\end{aligned}$$

where $\left\|I^\ell u\right\|_\infty<\infty$, $\left\|I^\ell\widehat{u}\right\|_\infty<\infty$, $\left\|I^\ell v\right\|_\infty<\infty$, and $\left\|I^\ell\widehat{v}\right\|_\infty<\infty$. The above inequalities imply that

$$\left\|B\left(h,\widehat{h}\right)\right\|\leq\Theta\ \text{and}\ \left\|B\left(\widehat{h},h\right)\right\|\leq\Theta.$$

Claim (iii): Show that B maps bounded sets into equicontinuous sets in R_0. For this, assume that $\epsilon_1, \epsilon_2 \in [b, \tau_1]$ with $\epsilon_1 < \epsilon_2$ and Ω_λ be bounded set of $R_0 \times R_0$ as in Claim (ii). Then for $h, \widehat{h} \in \Omega_\lambda$, we can write

$$
\begin{aligned}
& \left| B\left(h, \widehat{h}\right)(\epsilon_2) - B\left(h, \widehat{h}\right)(\epsilon_1) \right| \\
\leq\ & \frac{1}{\Gamma(\ell)} \int_{\epsilon_1}^{\epsilon_2} \left(\log \frac{\epsilon_2}{s}\right)^{\ell-1} \left| \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \right| \frac{ds}{s} \\
& + \frac{1}{\Gamma(\ell)} \int_{0}^{\epsilon_1} \left(\left(\log \frac{\epsilon_2}{s}\right)^{\ell-1} - \left(\log \frac{\epsilon_1}{s}\right)^{\ell-1} \right) \\
& \times \left| \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \right| \frac{ds}{s} \\
\leq\ & \frac{\|u\|_\infty}{\Gamma(\ell)} \left| \int_{\epsilon_1}^{\epsilon_2} \left(\log \frac{\epsilon_2}{s}\right)^{\ell-1} \frac{ds}{s} + \int_{0}^{\epsilon_1} \left(\log \frac{\epsilon_2}{s}\right)^{\ell-1} \frac{ds}{s} \right| \\
& + \frac{\|\widehat{u}\|_\infty \left(\Lambda_Q \lambda + \Lambda_Q |\phi(b) + \psi(b)| + \Lambda_Q^* \|\psi, \phi\|\right)}{\Gamma(\ell)} \\
& \times \left| \int_{\epsilon_1}^{\epsilon_2} \left(\log \frac{\epsilon_2}{s}\right)^{\ell-1} \frac{ds}{s} + \int_{0}^{\epsilon_1} \left(\log \frac{\epsilon_2}{s}\right)^{\ell-1} \frac{ds}{s} \right| \\
\leq\ & \frac{2\|u\|_\infty}{\Gamma(\ell+1)} \left(\log(\epsilon_2) - \log(\epsilon_1)\right)^\ell \\
& + \frac{2\|\widehat{u}\|_\infty \left(\Lambda_Q \lambda + \Lambda_Q |\phi(b) + \psi(b)| + \Lambda_Q^* \|\psi, \phi\|\right)}{\Gamma(\ell+1)} \left(\log(\epsilon_2) - \log(\epsilon_1)\right)^\ell .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \left| B\left(\widehat{h}, h\right)(\epsilon_2) - B\left(\widehat{h}, h\right)(\epsilon_1) \right| \\
\leq\ & \frac{2\|v\|_\infty}{\Gamma(\sigma+1)} \left(\log(\epsilon_2) - \log(\epsilon_1)\right)^\sigma \\
& + \frac{2\|\widehat{v}\|_\infty \left(\Lambda_Q \lambda + \Lambda_Q |\phi(b) + \psi(b)| + \Lambda_Q^* \|\phi, \psi\|\right)}{\Gamma(\sigma+1)} \left(\log(\epsilon_2) - \log(\epsilon_1)\right)^\sigma .
\end{aligned}
$$

The right-hand side of the above inequalities tends to zero as $\epsilon_1 \to \epsilon_2$. Hence, B is continuous and completely continuous (CC) as a result of Claims (i) to (iii) and the Arzela-Ascoli theorem.

Claim (iv): A priori bounds. Assume that $\left(h, \widehat{h}\right)$ is a solution of the equations $h = \delta B\left(h, \widehat{h}\right)$ and $\widehat{h} = \delta B\left(\widehat{h}, h\right)$, for some $\delta \in (0, 1)$. Then, for each $\tau \in [b, \tau_1]$, one has

$$
\begin{cases}
B\left(h, \widehat{h}\right)(\tau) = \frac{1}{\Gamma(\ell)} \int_{b}^{\tau} \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \frac{ds}{s}, \\
B\left(\widehat{h}, h\right)(\tau) = \frac{1}{\Gamma(\sigma)} \int_{b}^{\tau} \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right) \frac{ds}{s},
\end{cases}
$$

which yields from (P_3) and (P_4) that

$$
\begin{aligned}
|h(\tau)| &\leq \frac{1}{\Gamma(\ell)} \int_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \left|\chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right)\right| \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\ell)} \int_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} u(s)\Phi\left(\left|v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right|, \left|\widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right|\right) \frac{ds}{s} \\
&\leq \frac{(\log Q)^\ell \|u\|_\infty}{\Gamma(\ell)} \Phi\left(\left\|v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right\|, \left\|\widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right\|\right) \\
&= \frac{(\log Q)^\ell \|u\|_\infty}{\Gamma(\ell+1)} \Phi(\Delta, \Delta^*).
\end{aligned}
$$

Similarly

$$
\left|\widehat{h}(\tau)\right| \leq \frac{(\log Q)^\sigma \|v\|_\infty}{\Gamma(\sigma+1)} \Phi(\Delta^*, \Delta).
$$

But, using $(S_1)_{(3)}$, we have

$$
\begin{aligned}
&\left\|v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right\|_\Omega \\
\leq\ & \left\|v_{\rho(s,v_s+r_s)}\right\|_\Omega + \left\|r_{\rho(s,v_s+r_s)}\right\|_\Omega \\
\leq\ & B_Q \sup_{s\in[b,\tau]} \{h(s)\} + D_Q \|h_0\|_\Omega + B_Q \sup_{s\in[b,\tau]} \{r(s)\} + D_Q \|r_0\|_\Omega \\
\leq\ & B_Q \sup\{h(s) : s \in [b,\tau]\} + D_Q \|\phi\|_\Omega + B_Q \|\phi(b)\| = \Delta
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right\|_\Omega \\
\leq\ & \left\|\widehat{v}_{\eta(s,\widehat{v}_s+q_s)}\right\|_\Omega + \left\|q_{\eta(s,\widehat{v}_s+q_s)}\right\|_\Omega \\
\leq\ & C_Q \sup_{s\in[b,\tau]} \left\{\widehat{h}(s)\right\} + E_Q \left\|\widehat{h}_0\right\|_\Omega + C_Q \sup_{s\in[b,\tau]} \{q(s)\} + E_Q \|q_0\|_\Omega \\
\leq\ & C_Q \sup\left\{\widehat{h}(s) : s \in [b,\tau]\right\} + E_Q \|\psi\|_\Omega + C_Q \|\psi(b)\| = \Delta^*.
\end{aligned}
$$

This implies that

$$
\frac{\|h\|_\infty}{\frac{1}{\Gamma(\ell+1)} (\log Q)^\ell \|u\|_\infty \Phi(\Delta, \Delta^*)} \leq 1 \text{ and } \frac{\left\|\widehat{h}\right\|_\infty}{\frac{1}{\Gamma(\sigma+1)} (\log Q)^\sigma \|v\|_\infty \Phi(\Delta^*, \Delta)} \leq 1.
$$

Then by axiom (3.1), there exist $L_1 > 0$, $L_1^* > 0$ such that $\|h\|_\infty \neq L_1$ and $\left\|\widehat{h}\right\|_\infty \neq L_1^*$. Let $V_1 = \left\{h, \widehat{h} \in R_0 : \|h\|_\infty < L_1,\ \left\|\widehat{h}\right\|_\infty < L_1^*\right\}$. The operator $B : V_1 \times V_1 \to R_0$ is CC. From the choice of V_1, there is no h and $\widehat{h}$ such that $h = \delta B\left(h, \widehat{h}\right)$ and $\widehat{h} = \delta B\left(\widehat{h}, h\right)$, $\delta \in (0,1)$. We conclude that B has a CFP $\left(h, \widehat{h}\right) \in V_1 \times V_1$ as a result of the nonlinear Leray-Shauder alternative, which is a solution to Problem (1.1).

Step 2: Consider the following problem:

$$\begin{cases} {}^{CH}D^{\ell}\Im(\tau) = \chi\left(\tau, \Im_{\rho(\tau,\Im_\tau)}, \xi_{\eta(\tau,\xi_\tau)}\right), \text{ for a.e. } \tau \in K = [\tau_1, \tau_2], \\ {}^{CH}D^{\sigma}\xi(\tau) = \overline{\chi}\left(\tau, \xi_{\eta(\tau,\xi_\tau)}, \Im_{\rho(\tau,\Im_\tau)}\right), \text{ for a.e. } \tau \in K = [\tau_1, \tau_2], \\ \Im(\tau_1^+) - \Im(\tau_1^-) = I_1\left(\Im_b(\tau_1^-)\right), \ \xi(\tau_1^+) - \xi(\tau_1^-) = I_1\left(\xi_b(\tau_1^-)\right), \\ \Im(b) = \Im_b(\tau), \ \xi(b) = \xi_b(\tau), \ \tau \in (-\infty, \tau_1]. \end{cases}$$

Let $R_1 = \left\{\Im, \xi \in \Omega_{\tau_2} : \Im(\tau_1^+) \text{ and } \xi(\tau_1^+) \text{ exist}\right\}$. Describe the operator $W_1 : R_1 \times R_1 \to R_1$ as

$$W_1(\Im,\xi)(\tau) = \begin{cases} \Im_b(\tau), \text{ if } \tau \in (-\infty, \tau_1], \\ \Im_b(\tau_1^-) + I_b\left(\Im_b(\tau_1^-)\right) \\ +\frac{1}{\Gamma(\ell)}\int\limits_{\tau_1}^{\tau}\left(\log\frac{\tau}{s}\right)^{\ell-1}\chi\left(\tau, \Im_{\rho(s,\Im_s)}, \xi_{\eta(s,\xi_s)}\right)\frac{ds}{s}, \text{ if } \tau \in [\tau_1, \tau_2], \end{cases}$$

and

$$W_1(\xi,\Im)(\tau) = \begin{cases} \xi_b(\tau), \text{ if } \tau \in (-\infty, \tau_1], \\ \xi_b(\tau_1^-) + I_b\left(\xi_b(\tau_1^-)\right) \\ +\frac{1}{\Gamma(\sigma)}\int\limits_{\tau_1}^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\overline{\chi}\left(\tau, \xi_{\eta(\tau,\xi_\tau)}, \Im_{\rho(\tau,\Im_\tau)}\right)\frac{ds}{s}, \text{ if } \tau \in [\tau_1, \tau_2], \end{cases}$$

Let $r(.), q(.) : (-\infty, \tau_1] \to \mathbb{R}$ be functions defined by

$$r(\tau) = \begin{cases} \Im_b(\tau), \text{ if } \tau \in (-\infty, \tau_1], \\ \Im_b(\tau_1^-) + I_b\left(\Im_b(\tau_1^-)\right), \text{ if } \tau \in [\tau_1, \tau_2], \end{cases}$$

$$\text{and } q(\tau) = \begin{cases} \xi_b(\tau), \text{ if } \tau \in (-\infty, \tau_1], \\ \xi_b(\tau_1^-) + I_b\left(\xi_b(\tau_1^-)\right), \text{ if } \tau \in [\tau_1, \tau_2]. \end{cases}$$

Then $r_{\tau_1} = \Im_b$ and $q_{\tau_1} = \xi_b$. For each $h, \widehat{h} \in R_1$ with $h_{\tau_1} = 0 = \widehat{h}_{\tau_1}$, we denote by $v, \widehat{v}$ the functions described as

$$v(\tau) = \begin{cases} 0, \text{ if } \tau \in (-\infty, \tau_1], \\ h(\tau), \text{ if } \tau \in [\tau_1, \tau_2], \end{cases} \quad \text{and } \widehat{v}(\tau) = \begin{cases} 0, \text{ if } \tau \in (-\infty, \tau_1], \\ \widehat{h}(\tau), \text{ if } \tau \in [\tau_1, \tau_2]. \end{cases}$$

If $\Im(.)$ and $\xi(.)$ satisfy the integral equations

$$\begin{cases} \Im(\tau) = \Im_b(\tau_1^-) + I_b\left(\Im_b(\tau_1^-)\right) + \frac{1}{\Gamma(\ell)}\int\limits_{\tau_1}^{\tau}\left(\log\frac{\tau}{s}\right)^{\ell-1}\chi\left(\tau, \Im_{\rho(s,\Im_s)}, \xi_{\eta(s,\xi_s)}\right)\frac{ds}{s} \\ \xi(\tau) = \xi_b(\tau_1^-) + I_b\left(\xi_b(\tau_1^-)\right) + \frac{1}{\Gamma(\sigma)}\int\limits_{\tau_1}^{\tau}\left(\log\frac{\tau}{s}\right)^{\sigma-1}\overline{\chi}\left(\tau, \xi_{\eta(\tau,\xi_\tau)}, \Im_{\rho(\tau,\Im_\tau)}\right)\frac{ds}{s}, \end{cases}$$

we can decompose up $\Im(.)$ and $\xi(.)$ into $\Im(\tau) = v(\tau) + r(\tau)$ and $\xi(\tau) = \widehat{v}(\tau) + q(\tau)$, $\tau \in [\tau_1, \tau_2]$, which lead to $\Im_\tau = v_\tau + r_\tau$ and $\xi_\tau = \widehat{v}_\tau + q_\tau$, for each $\tau \in [\tau_1, \tau_2]$ and the functions $h(.)$ and $\widehat{h}(.)$ justify

$$\begin{cases} h(\tau) = \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \frac{ds}{s}, \\ \widehat{h}(\tau) = \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right) \frac{ds}{s}. \end{cases}$$

Set the product

$$R_{\tau_1} \times R_{\tau_1} = \left\{\left(h, \widehat{h}\right) \in R_1 \times R_1 : h_{\tau_1} = 0 = \widehat{h}_{\tau_1}\right\}.$$

Define the operator $B_1 : R_{\tau_1} \times R_{\tau_1} \to R_{\tau_1}$ by

$$\begin{cases} B_1\left(h, \widehat{h}\right)(\tau) = \frac{1}{\Gamma(\ell)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\ell-1} \chi\left(s, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}\right) \frac{ds}{s}, \\ B_1\left(\widehat{h}, h\right)(\tau) = \frac{1}{\Gamma(\sigma)} \int\limits_b^\tau \left(\log \frac{\tau}{s}\right)^{\sigma-1} \overline{\chi}\left(s, \widehat{v}_{\eta(s,\widehat{v}_s+q_s)} + q_{\eta(s,\widehat{v}_s+q_s)}, v_{\rho(s,v_s+r_s)} + r_{\rho(s,v_s+r_s)}\right) \frac{ds}{s}. \end{cases}$$

By the same method used in Step 1, we can prove that B_1 is continuous and CC, and if the pair $\left(h, \widehat{h}\right)$ is a possible solution of the equations $h = \delta B_1\left(h, \widehat{h}\right)$ and $\widehat{h} = \delta B_1\left(\widehat{h}, h\right)$, for some $\delta \in (0,1)$, then there exist $L_2 > 0$, $L_2^* > 0$ such that $\|h\|_\infty \neq L_2$ and $\left\|\widehat{h}\right\|_\infty \neq L_2^*$. Put $V_2 = \left\{h, \widehat{h} \in R_{\tau_1} : \|h\|_\infty < L_2, \ \left\|\widehat{h}\right\|_\infty < L_2^*\right\}$. From the choice of V_2, there is no h and $\widehat{h}$ such that $h = \delta B_1\left(h, \widehat{h}\right)$ and $\widehat{h} = \delta B_1\left(\widehat{h}, h\right)$, $\delta \in (0,1)$. According to nonlinear Leray-Shauder alternative, we conclude that B_1 has a CFP $\left(h, \widehat{h}\right) \in V_1 \times V_1$, which is a solution to Problem (1.1).

Step 3: We carry on with this method while keeping in mind that $\Im_m = \Im|_{[m,Q]}$ and $\xi_m = \xi|_{[m,Q]}$ are solutions to the problem

$$\begin{cases} {}^{CH}D^\ell \Im(\tau) = \chi\left(\tau, \Im_{\rho(\tau,\Im_\tau)}, \xi_{\eta(\tau,\xi_\tau)}\right), \text{ for a.e. } \tau \in K = [\tau_m, Q], \\ {}^{CH}D^\sigma \xi(\tau) = \overline{\chi}\left(\tau, \xi_{\eta(\tau,\xi_\tau)}, \Im_{\rho(\tau,\Im_\tau)}\right), \text{ for a.e. } \tau \in K = [\tau_m, Q], \\ \Im(\tau_m^+) - \Im(\tau_{m-1}^-) = I_m\left(\Im_{m-1}(\tau_m^-)\right), \ \xi(\tau_m^+) - \xi(\tau_{m-}^-) = I_m\left(\xi_{m-1}(\tau_m^-)\right), \\ \Im(\tau) = \Im_{m-1}(\tau), \ \xi(\tau) = \xi_{m-1}(\tau), \ \tau \in (-\infty, \tau_{m-1}]. \end{cases}$$

Following that, the solution to Problem (1.1) is defined by

$$\Im(\tau) = \begin{cases} \Im_b(\tau), \text{ if } \tau \in (-\infty, \tau_1], \\ \Im_1(\tau), \text{ if } \tau \in [\tau_1, \tau_2], \\ \dots \\ \Im_m(\tau), \ \tau \in (\tau_m, Q], \end{cases} \quad \text{and } \xi(\tau) = \begin{cases} \xi_b(\tau), \text{ if } \tau \in (-\infty, \tau_1], \\ \xi_1(\tau), \text{ if } \tau \in [\tau_1, \tau_2], \\ \dots \\ \xi_m(\tau), \ \tau \in (\tau_m, Q]. \end{cases}$$

□

5. An application

To strengthen and support our findings, in this section, we apply Theorem 4.1 to discuss the existence of the solution to the following coupled IFDEs:

$$\begin{cases} {}^{CH}D^{\ell}\Im(\tau) = \frac{e^{-\tau}\Im(\tau-\varrho(\Im(\tau)))\xi(\tau-\zeta(\xi(\tau)))}{1+\Im^2(\tau-\varrho(\Im(\tau)))\xi^2(\tau-\zeta(\xi(\tau)))}, \text{ for a.e. } \tau\in K=[1,e],\ \ell\in(0,1], \\ {}^{CH}D^{\sigma}\xi(\tau) = \frac{e^{-2\tau}\xi(\tau-\zeta(\xi(\tau)))\Im(\tau-\varrho(\Im(\tau)))}{1+\Im^3(\tau-\varrho(\Im(\tau)))\xi^3(\tau-\zeta(\xi(\tau)))}, \text{ for a.e. } \tau\in K=[1,e],\ \sigma\in(0,1], \\ I_j\left(\Im(\tau_j)\right) = \int_{-\infty}^{\tau_j}(\tau_j-s)\Im(s)ds,\ I_j\left(\xi(\tau_j)\right) = \int_{-\infty}^{\tau_j}(\tau_j-s)\xi(s)ds,\ j=1,2,..,m, \\ \Im(\tau)=\phi(\tau),\ \xi(\tau)=\psi(\tau),\ \tau\in(-\infty,1], \end{cases} \tag{5.1}$$

where ${}^{CH}D^{\ell}$ and ${}^{CH}D^{\sigma}$ are CHF derivatives with order ℓ and σ, respectively, $\varrho,\zeta\in C(\mathbb{R},[0,\infty))$, $(\phi,\psi)\in\Omega_{\varrho\times\zeta}$, and

$$\Omega=\left\{\Im,\xi\in C((-\infty,1,\mathbb{R}):\lim_{\theta\to-\infty}e^{\varrho\theta}\Im(\theta)\text{ and }\lim_{\theta\to-\infty}e^{\zeta\theta}\xi(\theta)\text{ exist in }\mathbb{R}\right\}.$$

The norm of $\Omega_{\varrho\times\zeta}$ is described as

$$\|\Im,\xi\|_{\varrho\times\zeta}=\sup_{\theta\in(-\infty,0]}\left\{e^{(\varrho+\zeta)\theta}\left|\Im(\theta)+\xi(\theta)\right|\right\}.$$

Clearly,

$$\|\Im\|_{\varrho}=\sup_{\theta\in(-\infty,0]}\left\{e^{\varrho\theta}\left|\Im(\theta)\right|\right\}\text{ and }\|\xi\|_{\zeta}=\sup_{\theta\in(-\infty,0]}\left\{e^{\zeta\theta}\left|\xi(\theta)\right|\right\}.$$

We claim that $\Omega_{\varrho\times\zeta}$ is a PS. For this, we realize the following two steps:

(I) For $(\Im_\tau,\xi_\tau)\in\Omega_\varrho\times\Omega_\zeta=\Omega_{\varrho\times\zeta}$, let $\Im,\xi:(-\infty,e]\to\mathbb{R}$ such that $\Im_0\in\Omega_\varrho$ and $\xi_0\in\Omega_\zeta$. Then

$$\begin{aligned}\lim_{\theta\to-\infty}e^{\varrho\theta}\Im_\tau(\theta) &= \lim_{\theta\to-\infty}e^{\varrho\theta}\Im(\tau+\theta)=\lim_{\theta\to-\infty}e^{\varrho(\theta-\tau)}\Im(\theta)=e^{\varrho}\lim_{\theta\to-\infty}e^{\varrho\theta}\Im_0(\theta)<\infty,\\ \lim_{\theta\to-\infty}e^{\zeta\theta}\xi_\tau(\theta) &= \lim_{\theta\to-\infty}e^{\zeta\theta}\xi_\tau(\tau+\theta)=\lim_{\theta\to-\infty}e^{\zeta(\theta-\tau)}\xi_\tau(\theta)=e^{\zeta}\lim_{\theta\to-\infty}e^{\zeta\theta}\xi_0(\theta)<\infty.\end{aligned}$$

(II) A priori bounds. We show that

$$\|(\Im_\tau,\xi_\tau)\|_{\varrho\times\zeta}\le\Lambda(\tau)\sup\left\{|\Im(s)+\xi(s)|:s\in[1,e]+\Lambda^*(\tau)\|(\Im_0,\xi_0)\|_{\varrho\times\zeta}\right\}.$$

Taking $X=Y=\Lambda=\Lambda^*=1$, it follows from $(S_1)_{(2),(3)}$ that $|\Im(\tau)|\le\|\Im_\tau\|_\varrho$, $|\xi(\tau)|\le\|\xi_\tau\|_\zeta$ and

$$\|(\Im_\tau,\xi_\tau)\|_{\varrho\times\zeta}\le\sup\{|\Im(s)+\xi(s)|:s\in[1,e]\}+\|(\Im_0,\xi_0)\|_{\varrho\times\zeta}.$$

Hence

$$|\Im_\tau(\theta)|=|\Im(\theta+\tau)|\text{ and }|\Im_\tau(\theta)|=|\xi(\theta+\tau)|.$$

If $\theta+\tau\le1$, we have

$$|\Im_\tau(\theta)|\le\sup\{|\Im(s)|:-\infty<s\le1\}\text{ and }|\xi_\tau(\theta)|\le\sup\{|\xi(s)|:-\infty<s\le1\},$$

If $\theta+\tau>1$, we get

$$|\Im_\tau(\theta)|\le\sup\{|\Im(s)|:1<s\le e\}\text{ and }|\xi_\tau(\theta)|\le\sup\{|\xi(s)|:1<s\le e\}.$$

For $\theta+\tau\in[1,e]$, we can write

$$\begin{aligned}|\Im_\tau(\theta)| &\leq \sup\{|\Im(s)|:-\infty<s\leq 1\}+\sup\{|\Im(s)|:1<s\leq e\},\\ |\xi_\tau(\theta)| &\leq \sup\{|\xi(s)|:-\infty<s\leq 1\}+\sup\{|\xi(s)|:1<s\leq e\}.\end{aligned}$$

Thus, we have

$$\|\Im_\tau\|_\varrho \leq \sup\{|\Im(s)|:1\leq s\leq e\}+\|\Im_0\|_\varrho, \tag{5.2}$$

and

$$\|\xi_\tau\|_\zeta \leq \sup\{|\xi(s)|:1\leq s\leq e\}+\|\xi_0\|_\zeta. \tag{5.3}$$

Combining (5.2) and (5.3), one has

$$\|(\Im_\tau,\xi_\tau)\|_{\varrho\times\zeta}\leq \sup\{|\Im(s)+\xi(s)|:s\in[1,e]\}+\|(\Im_0,\xi_0)\|_{\varrho\times\zeta}.$$

It is obvious that $\left(\Omega_{\varrho\times\zeta},\|.\|_{\varrho\times\zeta}\right)$ is a BS. Thus, $\Omega_{\varrho\times\zeta}$ is a PS.

Next, set

$$\rho(\tau,\phi)=\tau-\varrho(\phi(\tau)),\ \eta(\tau,\psi)=\tau-\zeta(\psi(\tau)),\ \text{for a.e. } (\tau,\phi),(\tau,\psi)\in K\times\Omega,$$

$$\chi(\tau,\phi,\psi)=\frac{e^{-\tau}\rho(\tau,\phi)\eta(\tau,\psi)}{1+\rho^2(\tau,\phi)\eta^2(\tau,\psi)},\ \overline{\chi}(\tau,\phi,\psi)=\frac{e^{-2\tau}\rho(\tau,\phi)\eta(\tau,\psi)}{1+\rho^3(\tau,\phi)\eta^3(\tau,\psi)},$$

for a.e. $(\tau,\phi),(\tau,\psi)\in K\times\Omega_{\varrho\times\zeta}$,

$$I_j\left(\Im(\tau_j)\right)=\int_{-\infty}^{\tau_j}(\tau_j-s)\Im(s)ds,\ \text{and } I_j\left(\xi(\tau_j)\right)=\int_{-\infty}^{\tau_j}(\tau_j-s)\xi(s)ds, j=1,2,..,m.$$

Then there exists a continuous and nondecreasing function $\Phi:[0,\infty)\times[0,\infty)\to(0,\infty)$ such that

$$\begin{aligned}|\chi(\tau,\phi,\psi)| &\leq e^{-\tau}\frac{|\rho(\tau,\phi)|\,|\eta(\tau,\psi)|}{1+|\rho^2(\tau,\phi)|\,|\eta^2(\tau,\psi)|}=u(\tau)\Phi(|\phi|,|\psi|),\\ |\overline{\chi}(\tau,\phi,\psi)| &\leq e^{-2\tau}\frac{|\rho(\tau,\phi)|\,|\eta(\tau,\psi)|}{1+|\rho^3(\tau,\phi)|\,|\eta^3(\tau,\psi)|}=v(\tau)\Phi(|\psi|,|\phi|),\end{aligned}$$

where $u(\tau)=e^{-\tau}$ and $v(\tau)=e^{-2\tau}$.

Since

$$\begin{aligned}\Phi(|\phi|,|\psi|) &= \frac{|\rho(\tau,\phi)|\,|\eta(\tau,\psi)|}{1+|\rho^2(\tau,\phi)|\,|\eta^2(\tau,\psi)|}\leq|\rho(\tau,\phi)|\,|\eta(\tau,\psi)|,\\ \Phi(|\psi|,|\phi|) &= \frac{|\rho(\tau,\phi)|\,|\eta(\tau,\psi)|}{1+|\rho^3(\tau,\phi)|\,|\eta^3(\tau,\psi)|}\leq|\rho(\tau,\phi)|\,|\eta(\tau,\psi)|,\end{aligned}$$

then for each $\tau\in[1,e]$, $\ell,\sigma\in(0,1]$, there exist $L_1,L_1^*,\cdots,L_m,L_m^*>0$ such that

$$\begin{aligned}\frac{\Gamma(\ell+1)L_j}{\Phi\left(L_j,L_j^*\right)(\log(\tau))^\ell\|u\|_\infty} &\geq \frac{\Gamma(\ell+1)L_j}{L_jL_j^*e^{-\tau}}=\frac{e^\tau\Gamma(\ell+1)}{L_j^*}>1,\ ,\ j=1,\cdots,m,\\ \text{and } \frac{\Gamma(\sigma+1)L_j^*}{\Phi\left(L_j^*,L_j\right)(\log(\tau))^\sigma\|v\|_\infty} &= \frac{\Gamma(\sigma+1)L_j^*}{L_jL_j^*e^{-2\tau}}=\frac{e^{2\tau}\Gamma(\sigma+1)}{L_j}>1,\ j=1,\cdots,m.\end{aligned}$$

Hence, the hypotheses $(P_1)-(P_4)$ of Theorem 4.1 are fulfilled. Therefore, the system (5.1) has at least one solution on $(-\infty,e]$.

6. Conclusion

Fractional DEs are considered a fruitful branch of nonlinear analysis. Impulsive DEs appear when, at certain moments, they change their state quickly and have variant applications in medicine, engineering, physics, dynamics, economics, pharmacology, etc. On the other hand, functional DEs via state-dependent delay typically arise in applications as models of equations. As a consequence, work on these types of equations has gained a lot of attention in the last few years by using several kinds of fractional derivatives. In this work, by using a fixed point technique, we ensured the existence of solutions to an IVP for a coupled FDE involving Caputo-Hadamard derivatives in PSs with a state-dependent delay and an impulsive. Moreover, we illustrated the obtained results with a concrete application where the suitable conditions are well applicable.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Abbas, M. Benchohra, J.R. Graef, J. Henderson, Implicit Fractional Differential and Integral Equations: Existence and Stability, De Gruyter, Berlin, (2018).

[2] M.I. Abbas, M. Alessandra Ragusa, Solvability of Langevin Equations With Two Hadamard Fractional Derivatives via Mittag-Leffler Functions, Appl. Anal. 101 (2021), 3231–3245. https://doi.org/10.1080/00036811.2020.1839645.

[3] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, (2000).

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier, Amsterdam, (2006).

[5] S. Muthaiah, D. Baleanu, N. Gopal Thangaraj, Existence and Hyers-Ulam Type Stability Results for Nonlinear Coupled System of Caputo-Hadamard Type Fractional Differential Equations, AIMS Math. 6 (2021), 168–194. https://doi.org/10.3934/math.2021012.

[6] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, (1999).

[7] Humaira, H.A. Hammad, M. Sarwar, M. De la Sen, Existence Theorem for a Unique Solution to a Coupled System of Impulsive Fractional Differential Equations in Complex-Valued Fuzzy Metric Spaces, Adv. Diff. Equ. 2021 (2021), 242. https://doi.org/10.1186/s13662-021-03401-0.

[8] H.A. Hammad, M. De la Sen, Analytical Solution of Urysohn Integral Equations by Fixed Point Technique in Complex Valued Metric Spaces, Mathematics. 7 (2019), 852. https://doi.org/10.3390/math7090852.

[9] H.A. Hammad, M. Zayed, Solving Systems of Coupled Nonlinear Atangana–Baleanu-Type Fractional Differential Equations, Bound Value Probl. 2022 (2022), 101. https://doi.org/10.1186/s13661-022-01684-0.

[10] R.P. Agarwal, M. Benchohra, S. Hamani, A Survey on Existence Results for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions, Acta Appl. Math. 109 (2008), 973–1033. https://doi.org/10.1007/s10440-008-9356-6.

[11] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Existence Results for Fractional Order Functional Differential Equations With Infinite Delay, J. Math. Anal. Appl. 338 (2008), 1340–1350. https://doi.org/10.1016/j.jmaa.2007.06.021.

[12] H.A. Hammad, M. De la Sen, Stability and Controllability Study for Mixed Integral Fractional Delay Dynamic Systems Endowed with Impulsive Effects on Time Scales, Fractal Fract. 7 (2023), 92. https://doi.org/10.3390/fractalfract7010092.

[13] H.A. Hammad, H. Aydi, H. Işık, M. De la Sen, Existence and Stability Results for a Coupled System of Impulsive Fractional Differential Equations With Hadamard Fractional Derivatives, AIMS Math. 8 (2023), 6913–6941. https://doi.org/10.3934/math.2023350.

[14] M.I. Abbas, M. Alessandra Ragusa, Solvability of Langevin Equations With Two Hadamard Fractional Derivatives via Mittag-leffler Functions, Appl. Anal. 101 (2021), 3231–3245. https://doi.org/10.1080/00036811.2020.1839645.

[15] J.K. Hale and S.M.V. Lunel, Introduction to Functional Differential Equations, Applied Mathematical Sciences, 99, Springer-Verlag, New York, (1993).

[16] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, (1999).

[17] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differntial Equations, Worlds Scientific, Singapore, (1989).

[18] C. Corduneanu, V. Lakshmikantham, Equations With Unbounded Delay: A Survey, Nonlinear Anal.: Theory Methods Appl. 4 (1980), 831–877. https://doi.org/10.1016/0362-546x(80)90001-2.

[19] J. Hale, J. Kato, Phase Space for Retarded Equations With Infinite Delay, Funkcial. Ekvac. 21 (1978), 11–41.

[20] F. Kappel, W. Schappacher, Some Considerations to the Fundamental Theory of Infinite Delay Equations, J. Diff. Equ. 37 (1980), 141–183.

[21] K. Schumacher, Existence and Continuous Dependence for Functional-Differential Equations With Unbounded Delay, Arch. Rational Mech. Anal. 67 (1978), 315–335. https://doi.org/10.1007/bf00247662.

[22] A.V. Rezounenko, J. Wu, A Non-Local Pde Model for Population Dynamics With State-Selective Delay: Local Theory and Global Attractors, J. Comput. Appl. Math. 190 (2006), 99–113. https://doi.org/10.1016/j.cam.2005.01.047.

[23] D.R. Willé, C.T.H. Baker, Stepsize Control and Continuity Consistency for State-Dependent Delay-Differential Equations, J. Comput. Appl. Math. 53 (1994), 163-170. https://doi.org/10.1016/0377-0427(94)90043-4.

[24] N. Abada, R.P. Agarwal, M. Benchohra, H. Hammouche, Existence Results for Nondensely Defined Impulsive Semilinear Functional Differential Equations With State-Dependent Delay, Asian-Eur. J. Math. 01 (2008), 449–468. https://doi.org/10.1142/s1793557108000382.

[25] E.H.A. Dads, K. Ezzinbi, Boundedness and Almost Periodicity for Some State-Dependent Delay Differential Equations, Electron. J. Diff. Equ. 2002 (2002), 67.

[26] A. Anguraj, M.M. Arjunaj and E.M. Hernandez, Existence Results for an Impulsive Neutral Functional Differential Equation With State-Dependent Delay, Appl. Anal. 86 (2007), 861-872. https://doi.org/10.1080/00036810701354995.

[27] E. Hernandez, A. Prokopczyk, L. Ladeira, A Note on Partial Functional Differential Equations With State-Dependent Delay, Nonlinear Anal.: Real World Appl. 7 (2006), 510–519. https://doi.org/10.1016/j.nonrwa.2005.03.014.

[28] E. Hernandez, R. Sakthivel, S.T. Aki, Existence Results for Impulsive Evolution Differential Equations With State-Dependent Delay, Electron. J. Diff. Equ. 2008 (2008), 28.

[29] H.A. Hammad, M.D. la Sen, H. Aydi, Generalized Dynamic Process for an Extended Multi-Valued F-Contraction in Metric-Like Spaces With Applications, Alexandria Eng. J. 59 (2020), 3817–3825. https://doi.org/10.1016/j.aej.2020.06.037.

[30] H.A. Hammad, H. Aydi, M. Zayed, Involvement of the Topological Degree Theory for Solving a Tripled System of Multi-Point Boundary Value Problems, AIMS Math. 8 (2022), 2257–2271. https://doi.org/10.3934/math.2023117.

[31] H.A. Hammad, M.F. Bota, L. Guran, Wardowski's Contraction and Fixed Point Technique for Solving Systems of Functional and Integral Equations, J. Funct. Spaces. 2021 (2021), 7017046. https://doi.org/10.1155/2021/7017046.

[32] M.A. Darwish, S.K. Ntouyas, Semilinear Functional Differential Equations of Fractional Order With State-Dependent Delay, Electron. J. Diff. Equ. 2009 (2009), 38.

[33] M. Benchohra, F. Berhoun, Impulsive Fractional Differential Equations With State-Dependent Delay, Commun. Appl. Anal. 14 (2010), 213–224.

[34] F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-Type Modification of the Hadamard Fractional Derivatives, Adv. Diff. Equ. 2012 (2012), 142. https://doi.org/10.1186/1687-1847-2012-142.

[35] Y. Hino, S. Murakami, T. Naito, Functional Differential Equations with Unbounded Delay, Springer-Verlag, Berlin, (1991).