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Representation of ABA's by Sections of Sheaves

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Abstract. An Almost Boolean Algebra $(A, \wedge, \vee, 0)$ (abbreviated as ABA) is an Almost Distributive Lattice (ADL) with a maximal element in which for any $x \in A$, there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y$ is a maximal element in A. If (S, Π, X) is a sheaf of nontrivial discrete ADL's over a Boolean space such that for any global section f, support of f is open, then it is proved that the set $\Gamma(X, S)$ of all global sections is an ABA. Conversely, it is proved that every ABA is isomorphic to the ADL of global sections of a suitable sheaf of discrete ADL's over a Boolean space.

1. Introduction

The axiomatization of George Boole's two valued propositional calculus led to the concept of Boolean algebra (ring). M. H. Stone [3] established a strong duality between Boolean algebras (rings) and Compact Hausdorff topological spaces in which closed and open subsets form a base for the topology. After the notion of Boolean algebra (ring) came to light, several generalizations have come up in which the ring theoretic generalizations like regular rings, p-rings, biregular rings etc and the lattice theoretic generalizations like distributive lattices, Hayting algebras, post algebras, pseudo-complemented distributive lattices, Stone lattices etc. U. M. Swamy and G. C. Rao [6] have introduced a common abstraction of these two streams in the form of an Almost Distributive Lattice (ADL) as an algebra

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$(A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ which satisfies all the axioms of a distributive lattice with 0 except the commutativity of the binary operations and, in this case, the commutativity of either of the binary operation is equivalent to that of the other. An ADL A with a maximal element m is said to be an ABA if for each $x \in A$, there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y$ is maximal (equivalently, for each $x \in A$, the interval $[0, x]$ is a Boolean algebra under the induced operations $\wedge$ and $\vee$).

In this paper, first we obtain an ABA from a given sheaf over a Boolean spaces. Also, we have considered the prime spectrum of a nontrivial Almost Bololean Algebra which forms a compact, Hausdorff and totally disconnected topological space and it is well known that such spaces are called Boolean spaces. With this as the base space, we have constructed a sheaf of discrete ADL's whose ADL of continuous sections is isomorphic to the given Almost Boolean Algebra.

2. Preliminaries

In this section, we collect certain definitions and properties of ADLs from [6–8] that are required in the main text of this paper.

Definition 2.1. *An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following identities.*

(1) $0 \wedge a = 0$
(2) $a \vee 0 = a$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(6) $(a \vee b) \wedge b = b$.

Example 2.1. *Every non-empty set X can be regarded as an ADL as follows. Let $a_0 \in X$. Define the binary operations $\wedge, \vee$ on X by*

$$a \wedge b = \begin{cases} b & \text{if } a \neq a_0 \\ a_0 & \text{if } a = a_0 \end{cases} \qquad a \vee b = \begin{cases} a & \text{if } a \neq a_0 \\ b & \text{if } a = a_0. \end{cases}$$

Then $(X, \wedge, \vee, a_0)$ is an ADL with a_0 as the zero element.

Definition 2.2. *Let $(A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define*

$$a \leq b \text{ if } a = a \wedge b \text{ (equivalently } a \vee b = b).$$

Then $\leq$ is a partial order on A.

Theorem 2.1. *If $(A, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in A$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* A

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a,\ (a \wedge b) \vee b = b$
and $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$

(12) *If* $a \leq c,\ b \leq c$ *then*
$a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a$.

Definition 2.3. *A homomorphism between ADLs* $(A, \vee, \wedge, 0)$ *into an ADL* L'*, we mean, a mapping* $f : A \to A'$ *satisfying the following:*

(1) $f(a \vee b) = f(a) \vee f(b)$

(2) $f(a \wedge b) = f(a) \wedge f(b)$

(3) $f(0) = 0$.

A nonempty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ whenever $x, y \in I$ and $a \in L$. For any $X \subseteq A$, the ideal generated by X is $(X] = \big\{ \big(\bigvee_{i=1}^{n} a_i \big) \wedge x : a_i \in X, x \in A, n \in \mathbb{Z}^+ \big\}$. If $X = \{x\}$, then we write $(x]$ for $(X]$ and this is called a principal ideal generated by x. The set of all principal ideals of A is a distributive lattice and it is denoted by $P\mathfrak{I}(A)$. A proper ideal P of A is called prime if for any $x, y \in A, x \wedge y \in P$ then $x \in P$ or $y \in P$. For any $x, y \in A$ with $x \leq y, [x, y] = \{t \in A : x \leq t \leq y\}$ is a bounded distributive lattice with respect to the operations induced from those on A. An element m is maximal in $(A, \leq)$ if and only if $m \wedge x = x$ for all $x \in X$. An ADL A is said to be discrete if every nonzero element is maximal. If m is maximal element in A, then $x \vee m$ is maximal in A, for each $x \in A$. The ADL given in the example 2.2 is a discrete ADL. For any $X \subseteq A,\ X^* = \big\{a \in A :\ x \wedge a = 0 \ \forall\ x \in X\big\}$ is an ideal of A and X^* is called the annihilator of X. We don't know, so far, whether $\vee$ is associative in an ADL or not. In this paper A denotes an ADL $(A, \wedge, \vee, 0)$ in which $\vee$ is associative.

Lemma 2.1. *Let* A *be an ADL and* I *is an ideal of* A*. Then, for any* $a, b \in A$*, we have the following:*

(1) $(a] = \big\{a \wedge x : x \in A\big\}$

(2) $a \in (b] \Leftrightarrow b \wedge a = a$

(3) $a \wedge b \in I \Leftrightarrow b \wedge a \in I$

(4) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$

(5) $(a] \vee (b] = (a \vee b] = (b \vee a]$

(6) $(a] = A \Longleftrightarrow a$ *is maximal.*

Lemma 2.2. *Let A be an ADL and $x, y \in A$. Then the following statements hold:*

(1) $\{x \vee y\}^* = \{x\}^* \cap \{y\}^*$

(2) $\{x \wedge y\}^* = \{y \wedge x\}^*$

(3) $\{x\}^{***} = \{x\}^*$

(4) $x \leq y \Rightarrow \{y\}^* \subseteq \{x\}^*$

(5) $\{x \wedge y\}^{**} = \{x\}^{**} \cap \{y\}^{**}$

Definition 2.4. *An ADL $(A, \wedge, \vee, 0)$ is said to be relatively complemented if every interval in A is a Boolean algebra.*

Theorem 2.2. *Let A be an ADL. Then the following are equivalent to each other.*

(1) *For any $a, b \in A$ there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$*

(2) *For any $a \leq b$ in A, $[a, b]$ is a complemented lattice.*

(3) *For any $a \in A$, $[0, a]$ is complemented lattice*

Definition 2.5. *A nontrivial ADL A is called an Almost Boolean algebra (ABA) if it has a maximal element and satisfies one, and hence all of the equivalent conditions given in the above theorem.*

Theorem 2.3. *Let A be an ADL with a maximal element. Then the following are equivalent to each other.*

(1) *A is an Almost Boolean algebra*

(2) *For any $a \in A$, there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal.*

(3) *$[0, m]$ is a Boolean algebra for all maximal elements m*

(4) *There esists a maximal element m such that $[0, m]$ is a Boolean algebra.*

Theorem 2.4. *Let A be an ADL and m and n be maximal elements in A. Then the lattices $[0, m]$ and $[0, n]$ are isomorphic to each other. Moreover, the Boolean algebras $[0, m]$ and $[0, n]$ are isomorphic when A is Almost Boolean algebra.*

Theorem 2.5. *Let $(A, \wedge, \vee, 0)$ be an ABA. Then for any a and b in A there exists a unique $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$.*

Definition 2.6. *A nontrivial ADL A is called dense if $a \wedge b \neq 0$ for all $a \neq 0$ and $b \neq 0$ (equivalently, $\{a\}^* = \{0\}$, for any $0 \neq a \in A$).*

3. Sheaf of ADL's

Definition 3.1. *A triple (S, π, X) is called a sheaf over X if S and X are topological spaces and $\pi : S \to X$ is a local homeomorphism of S onto X; that is π is a surjective map and, for any $s \in S$,* *there are open sets G and U containing s and $\pi(s)$ in S and X respectively such that the restriction of π to G is a homeomorphism of G onto U.*

Let (S, π, X) be a sheaf over X. Then $\pi : S \to X$ is continuous and open map. Here S is called the sheaf space, X the base space and π the projection. For any $Y \subseteq X$, a continuous map $f : Y \to S$ such that $\pi \circ f$ is identity on Y is called a section on Y. Sections on the whole space X are called global sections. For any $x \in X$, the set $S_x = \{s \in S : \pi(s) = x\} = \pi^{-1}(\{x\})$ is called the stalk at x. The set of all global sections will be denoted by $\Gamma(X, S)$.

Definition 3.2. *A sheaf (S, π, X) is called a sheaf of (discrete) ADLs if for each $x \in X$, the stalk S_x is an (discrete) ADL $(S_x, \wedge, \vee, 0_x)$ satisfies the following.*

(1) *The operations $\wedge$ and $\vee$ are continuous mappings of $S \otimes S$ into S, where* $S \otimes S = \{(s, t) \in S \times S : \pi(s) = \pi(t)\}$.
(2) *The map $x \mapsto 0_x$ is a continuous map of X into S, where 0_x is the zero element in the stalk S_x.*

In the following, we obtain an ABA from a given sheaf over a Boolean space.

Theorem 3.1. *Let (S, π, X) be a sheaf of nontrivial discrete ADLs over a Boolean space X such that, for any global section f, support $|f| = \{x \in X : f(x) \neq 0_x\}$ is open in X. Then $\Gamma(X, S)$ is an ABA.*

Proof. Since $x \mapsto 0_x$ is continuous, it is a section on X and we denote it by $\overline{0}$. Therefore $\overline{0} \in \Gamma(X, S)$ and hence $\Gamma(X, S) \neq \emptyset$. For any $f, g \in \Gamma(X, S)$, we define $f \wedge g$ and $f \vee g$ by $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in X$.

The $\wedge$ and $\vee$ on the right-hand side are those in the stalk S_x which is an ADL. It can be easily verified that $\left(\Gamma(X, S), \wedge, \vee, \overline{0}\right)$ an ADL. It is easily verified that, for any global sections f and g, the set

$$\langle f, g\rangle = \{x \in X : f(x) = g(x)\}$$

is a closed set in X and by hypothesis $\langle f, \overline{0}\rangle (= |f|\ )$ is open also. Thus $|f|$ is a closed set in X for all global sections f. For each $x \in X$, we can choose $s (\neq 0_x) \in S_x$ and a global section f_x such that $f_x(x) = s \in S_x$ and hence $x \in |f_x|$, the support of f_x. Since to each $x \in X$, $|f_x|$ is clopen and X is compact, there exists clopen sets $U_1, U_2, \ldots U_n$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and $U_1 \cup U_2 \cup \cdots \cup U_n = X$ and there is global section f_i such that $f_i(y) \neq o_y$ for all $y \in U_i$.
Let $f = f_1 \vee f_2 \vee \cdots \vee f_n$. Then f is a global section and $|f| = X$. Let $g \in \Gamma(X, S)$. Then $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$ (since each S_x is discrete and hence every nonzero element in S_x is maximal) and hence $f \wedge g = g$ for all $g \in \Gamma(X, S)$. Therefore f is a maximal element in the ADL $\Gamma(X, S)$. Further, let $g \in \Gamma(X, S)$ and f be any maximal element in $\Gamma(X, S)$. Then $f(x) \neq 0_x$

for all $x \in X$. Define $h : S \to X$ by

$$h(x) = \begin{cases} 0_x & \text{if} \quad x \in |g| \\ f(x) & \text{if} \quad x \notin |g|. \end{cases}$$

Then h is continuous (since $x \mapsto 0_x$ and f are continuous and $|g|$ is clopen). Therefore $h \in \Gamma(X, S)$. Also $(g \wedge h)(x) = g(x) \wedge h(x) = 0_x$ for all $x \in X$ and hence $g \wedge h = \overline{0}$. Further $(g \vee h)(x) \neq 0_x$ for all $x \in X$ and hence $g \vee h$ is maximal in $\Gamma(X, S)$. Thus $\Gamma(X, S)$ is an ABA. □

4. Representation Theorem

In this section, we represent a given ABA by a sheaf over a Boolean space. For this, we obtain a sheaf (S, π, X) of discrete ADL's from a given ABA A and also prove that A is isomorphic to $\Gamma(X, S)$. Let us recall the following from [6].

Definition 4.1. *[6] For any element a in an ADL A, define*

$$\theta_a = \big\{(x, y) \in A \times A : a \wedge x = a \wedge y\big\}.$$

Then θ_a is a congruence relation on A.

Lemma 4.1. *Let A be an ADL and P be a prime ideal of A. Let*

$$\theta_P = \big\{(x, y) \in A \times A : a \wedge x = a \wedge y\big\}$$

for some $a \in A - P$. Then θ_P is a congruence relation on A and the Quotient A/θ_P is a nontrivial discrete ABA.

Proof. Clearly $\theta_P = \bigcup\limits_{a \in A-P} \theta_a$ and it is easy to verified that θ_P is a congruence relation on A. Also, for any $x \in A$,

$$x \in P \Longleftrightarrow (x, 0) \in \theta_P.$$

Since P is a proper ideal of A, there exists $x \in A - P$ and hence $(x, 0) \notin \theta_P$ so that $x/\theta_P \neq 0/\theta_P$. Therefore A/θ_P is a nontrivial ADL. Now, if $x/\theta_P \neq 0/\theta_P$, then $x \notin P$ and $(x \wedge a, a) \in \theta_P$ so that $x/\theta_P \wedge a/\theta_P = (x \wedge a)/\theta_P = a/\theta_P$ for all $a \in A$ and hence x/θ_P is maximal in A/θ_P. Therefore, A/θ_P is discrete ADL. □

Let us recall from [6] that, for any ABA $A = (A, \wedge, \vee, 0, m)$, $Spec(A)$ denotes the space of all prime ideals of A together with the hull-kernal topology for which $\{X_a : a \in A\}$ is a base, where $X_a = \{P \in Spec(A) : a \notin P\}$ and that $Spec(A)$ is a Boolean space (a Compact Hausdorff and totally disconnected topological space).

Lemma 4.2. *Let A be an ABA and $a \in A$. Then*
$\theta_a = \bigcap \big\{\theta_P : P \in Spec(A) \text{ and } a \notin P\big\} = \bigcap \big\{\theta_P : P \in X_a\big\}$.

Proof. First, it can be easily verified $x, y \in A$, the set $\langle x, y\rangle = \{b \in A : b \wedge x = b \wedge y\}$ is an ideal of A. Let $a \in A$. we have that $\theta_P = \bigcup_{b \in A-P} \theta_b$. Therefore

$$P \in X_a \Rightarrow a \notin P \Rightarrow \theta_a \subseteq \theta_P.$$

Hence $\theta_a \subseteq \bigcap_{P \in X_a} \theta_P$. On the otherhand, suppose that $(x, y) \notin \theta_a$. Then $a \wedge x \neq a \wedge y$ and hence $a \notin \langle x, y\rangle$. Since $\langle x, y\rangle$ is an ideal, there exists a prime ideal P of A such that $\langle x, y\rangle \subseteq P$ and $a \notin P$. Now $P \in X_a$ and $(x, y) \notin \theta_P$. Therefore $\bigcap_{P \in X_a} \theta_P \subseteq \theta_a$. Thus $\bigcap_{P \in X_a} \theta_P = \theta_a$. □

Corollary 4.1. *If A is an ABA, then $\bigcap \{\theta_P : P \in Spec(A)\} = \Delta_A$, the diagonal relation on A.*

Proof. It follows by the fact that, for any maximal element m in A, $X_m = X$ and, $\theta_m = \Delta_A$. □

Theorem 4.1. *Any ABA is isomorphic to $\Gamma(X, S)$ for a suitable sheaf (S, π, X) of discrete ADLs over a Boolean space X.*

Proof. Let A ba an ABA and $X = Spec(A)$. Then X is a Boolean space. For any $P \in X$, let $S_P = A/\theta_P$ and S be the disjoint union of $A/{\theta_P}'^s$, $P \in X$. For any $x \in A$, define $\hat{x} : X \to S$ by $\hat{x}(P) = x/\theta_P$. Define $\pi : S \to X$ by $\pi(s) = P$ if $s \in S_P$. Let S be equipped with the largest topology with respect to which each $\hat{x}$, $x \in A$, is continuous. Now we shall prove that (S, π, X) is a sheaf of discrete ADLs over the Boolean space X and finally we proved that $A \cong \Gamma(X, S)$. First, we prove that the class $\{\hat{x}(X_a) : x, a \in A\}$ forms a base for the topology on S. Clearly $\bigcup_{x,a \in A} \hat{x}(X_a) = S$. For any $x, y \in A$ and $a, b \in A$, we have
$s \in \hat{x}(X_a) \cap \hat{y}(X_b) \Rightarrow s = x/\theta_P = y/\theta_Q$ for some $P \in X_a, Q \in X_b$
$\Rightarrow P = \pi(s) = Q$
$\Rightarrow P = Q \in X_a \cap X_b$
and $c \wedge x = c \wedge y$ for some $c \in A - P$
$\Rightarrow P \in X_{a \wedge b \wedge c}$ and $a \wedge b \wedge c \wedge x = a \wedge b \wedge c \wedge y$
$\Rightarrow P \in X_{a \wedge b \wedge c}$ and $s = \hat{x}(P) = \hat{y}(Q)$
$\Rightarrow s \in \hat{x}(X_{a \wedge b \wedge c}) \subseteq \hat{x}(X_a) \cap \hat{y}(X_b)$.
Therefore the class mentioned is a base for a topology, say τ on S. Also, for any x, y and $a \in A, P \in \hat{x}^{-1}(\hat{y}(X_a))$
$\Rightarrow x/\theta_P = y/\theta_P$ and $P \in X_a$
$\Rightarrow b \wedge x = b \wedge y$ for some $b \in A - P$
$\Rightarrow P \in X_{a \wedge b} \subseteq \hat{x}^{-1}(\hat{y}(X_a))$.
Therefore each $\hat{x}$, $x \in A$ is continuous with respect to the topology τ and it can be easily proved that τ is the largest topology on S with respect to which each $\hat{x}$ is continuous. Since $\pi(s) = P$ for all $s \in S_P = A/\theta_P, P \in X$, $\pi : S \to X$ is a bijection mapping. Let $s \in S$. Then $s = x/\theta_P = \hat{x}(P)$ for some $P \in X$ and $x \in A$. If $\pi(s) = P$, choose $a \in A - P$, then $P \in X_a$ and $s \in \hat{x}(X_a)$. Since $\hat{x}(X_a)$

is open in S and $\hat{x}(X_a)$ is a neighbourhood of s in S and X_a is open neighbourhood of $\pi(s)$ in X and hence it can ba easily verified that any restriction of π to $\hat{x}(X_a)$is a homeomorphism of $\hat{x}(X_a)$ onto X_a. Therefore π is a local homeomorphism of S onto X and hence (S, π, X) is a sheaf over Boolean space X. Since θ_P is a congruence on A, it follows that the continuity of ADL operations $\wedge$, $\vee$ and 0. And, for any $x, y \in A$, $\{P \in X : (x, y) \in \theta_P\}$ is an open subset of X. Also, for each $P \in X$, A/θ_P is a nontrivial discrete ADL (by lemma 4.2). Thus (S, π, X) is a sheaf of nontrivial discrete ADLs over the Boolean space X. Since the operations are point-wise and each stalk A/θ_P is an ADL, it follows that $\Gamma(X, S)$ is also an ADL. Finally define $\alpha : A \to \Gamma(X, S)$ by $\alpha(x) = \hat{x}$ for all $x \in A$. Since $\widehat{x \wedge y} = \hat{x} \wedge \hat{y}$ and $\widehat{x \vee y} = \hat{x} \vee \hat{y}$, it follows that α is a homomorphism of ADLs. Also, for any $x, y \in A$,

$\alpha(x) = \alpha(y) \Rightarrow \hat{x} = \hat{y}$

$\Rightarrow \hat{x}(P) = \hat{y}(P)$ for all $P \in X$

$\Rightarrow x/\theta_P = y/\theta_P$ for all $P \in X$

$\Rightarrow (x, y) \in \bigcap\limits_{P \in X} \theta_P = \Delta_A$

$\Rightarrow x = y$.

Therefore α is an injection. Lastly, let $f \in \Gamma(X, S)$. For each $P \in X$, there exists $x_P \in A$ such that $f(P) = x_P/\theta_P = \hat{x_P}(P)$. Since f and $\hat{x_P}$ are sections on X, we can choose open sets G and W in S and X respectively such that $f(P)(= \hat{x_P}(P)) \in G$, $P(= \pi(f(P))) \in W$ and $\pi/G : G \to W$ is a homeomorphism, $f^{-1}(G) \cap \hat{x_P}^{-1}(G)$ is open in X and hence $f^{-1}(G) \cap \hat{x_P}^{-1}(G) = X_{a_P}$ for some $a_P \in A$ (since X_a's, $a \in A$ form a base for open sets) and $P \in X_{a_P}$. Also, for any $Q \in X_{a_P}$, $f(Q)$ and $\hat{x}_P(Q) \in G$ and $\pi(f(Q)) = \pi(\hat{x_P}(Q))$ and implies that $f(Q) = \hat{x_P}(Q)$ (since π is an injection on G). By using the compactness of X and the fact that X in a Boolean space, there exists $a_1, a_2, \cdots a_n \in A$ and $x_1, x_2, \cdots x_n \in A$ such that $\bigcup\limits_{i=1}^{n} x_i = X$ and $f(Q) = \hat{x}_i(Q)$ for all $Q \in X_{a_i}$. Since each X_{a_i} is clopen, we choose $b_1, b_2, \cdots b_n \in A$ such that

$X_{b_1} = X_{a_1}$

$X_{b_2} = x_{a_2} \cap (X - X_{a_1})$

$X_{b_3} = X_{a_3} \cap (X - X_{a_1}) \cap (X - X_{a_2})$

$\vdots$

$X_{b_n} = X_{a_n} \cap \bigcap\limits_{i=1}^{n-1} (X - X_{a_i})$.

Then $X_{b_i \wedge b_j} = X_{b_i} \cap X_{b_j} = \emptyset$ for all $i \neq j$, $\bigcup\limits_{i=1}^{n} X_{b_i} = X$ and $f(Q) = \hat{x}_i(Q)$ for all $Q \in X_{b_i}$. Put $x = \bigvee\limits_{i=1}^{n} (b_i \wedge x_i)$. Then, for any $1 \leq i \leq n$, $(b_i \wedge x_i, x_i) \in \theta_{b_i}$ and hence $(b_i \wedge x_i) \in \theta_Q$ for all $Q \in X_{b_i}$ and it implies that $(b_i \wedge x_i)/\theta_Q = x_i/\theta_Q$ so that $\widehat{b_i \wedge x_i}(Q) = \hat{x}_i(Q) = f(Q)$ for all $Q \in X_{b_i}$ and for $j \neq i$, $\widehat{b_j \wedge x_j}(Q) = \hat{b}_j(Q) \wedge \hat{x}_j(Q) = \hat{0}(Q)$ for all $Q \in X_{b_j}$ (since $b_j \in Q \Rightarrow \hat{b}_j(Q) = \hat{0}(Q)$). Therefore, for any $Q \in X$, there exists unique i such that $Q \in X_{b_i}$ and $Q \notin X_{b_j}$ for all $j \neq i$ and hence $\hat{x}(Q) = \bigvee_{i=1}^{n} \widehat{(b_i \wedge x_i)}(Q) = \hat{x}_i(Q) = f(Q)$. Thus $f = \hat{x}$. This implies that $\alpha : A \to \Gamma(X, S)$ is an isomorphism of A onto $\Gamma(X, S)$. □

Note that in the above sheaf (S, π, X) the support of any global section is a closed set, since any global section f is of the form $\widehat{x}$ for some $x \in A$ and $|f| = |\widehat{x}| = X_x$ which is clopen in X.

5. Conclusion

It is yet to be investigated whether the correspondences $A \mapsto (S, \pi, X)$ and $(S, \pi, X) \mapsto \Gamma(X, S)$ are inverses to each other (up to isomorphism) and give us a duality between the class of ABA's and the class of sheaves of nontrivial discrete ADL's over Boolean spaces in which the support of any global section is a clopen set in the base space.

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